Characterizations of non-Seymour graphs

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Outline

1 Motivation
2 Definitions: joins, complete packing of cuts
3 Seymour graphs
4 Characterizations of non-Seymour graphs
5 Ingredients from Matching Theory
6 Equivalent forms
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9 Open problem
Motivation

Edge-disjoint paths problem

Given a graph $H = (V, E)$ and $k$ pairs of vertices $\{s_i, t_i\}$, decide whether there exist $k$ edge-disjoint paths connecting the $k$ pairs $s_i, t_i$. 
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Applications:

1. Real-time communication,
2. VLSI design,
3. Transportation networks,
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Reformulation by adding the set $F$ of edges $s_it_i$. 

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**Complete packing of cycles**

Given a graph $H' = (V, E + F)$, decide whether there exist $|F|$ edge-disjoint cycles in $H'$, each containing exactly one edge of $F$. 

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Suppose $H'$ is planar. The problem in the dual:
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Given a graph $H' = (V, E + F)$, decide whether there exist $|F|$ edge-disjoint cycles in $H'$, each containing exactly one edge of $F$.

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**Complete packing of cuts**

Given a graph $G = (V', E' + F')$, decide whether there exist $|F'|$ edge-disjoint cuts in $G$, each containing exactly one edge of $F'$.

An example

Edge-disjoint paths problem

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Complete packing of paths

An example
An example

Adding the edges $s_i t_i$
An example

The graph $H'$
An example

Complete packing of cycles
$H'$ is planar
An example

$H'$ and his dual $G$
An example

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Complete packing of cycles and cuts
Complete packing of cuts

The graphs are not planar anymore!
Complete packing of cuts

The problem

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Necessary condition

If the graph $G = (V, E + F)$ admits a complete packing of cuts, then $F$ is a join: for every cycle $C$, $|C \cap F| \leq |C \setminus F|$.
Complete packing of cuts

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Given a graph \( G = (V, E + F) \), decide whether there exist \( |F| \) edge-disjoint cuts in \( G \), each containing exactly one edge of \( F \).

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Sufficient condition?

If \( F \) is a join, the graph \( G = (V, E + F) \) admits a complete packing of cuts?
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NOT: $K_4$
Complete packing of cuts

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Theorem (Middendorf, Pfeiffer ’93)

Given a join in a graph, decide whether there exists a complete packing of cuts is an NP-complete problem.
Theorem (Seymour ’77)

If $G$ is a series-parallel graph, then for every join there exists a complete packing of cuts.
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Definition
$G$ is a Seymour graph if for every join there exists a complete packing of cuts.
Seymour graphs

Theorem (Seymour ’77)

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$G$ is a Seymour graph $\iff$ if for every join there exists a complete packing of cuts.
Subclasses

1. Seymour '77 : Graphs without subdivision of $K_4$,
2. Seymour '81 : Graphs without odd cycle,
3. Gerards '92 : Graphs without odd $K_4$ and without odd prism,
4. Szigeti '93 : Graphs without non-Seymour odd $K_4$ and without non-Seymour odd prism.
Around Seymour graphs

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![Graphs and Substructures](image)
### Around Seymour graphs

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![Graphs](image-url)
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**Superclass**

Seymour graph $\implies$ no even subdivision of $K_4$ and of prism.
Preliminaries

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If for a join $F$ of $G$ there exist two $F$-tight cycles whose union is not bipartite, then $G$ is not Seymour.
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Conjecture (Sebő '92)

$G$ is not Seymour if and only if $G$ admits a join $F$ and two $F$-tight cycles whose union is an odd $K_4$ or an odd prism.
Theorem (Ageev, Kostochka, Szigeti ’97)

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Examples

Seymour odd $K_4$  non-Seymour odd prism
Theorem (Ageev, Kostochka, Szigeti ’97)

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**Theorem (Ageev, Benchetrit, Sebő, Szigeti ’11)**

$G$ is non-Seymour if and only if contracting stars and odd cycles it contains an even subdivision of $K_4$. 
Definitions

1. **Matching-covered** = connected and any edge belongs to a perfect matching,

2. **Elementary** = edges belonging to a perfect matching form a connected subgraph,

3. **Barrier** of elementary graph $G = \text{vertex set } X$ such that the number of odd components of $G - X$ is $|X|$. 
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![even subdivision of $K_2^3$]


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1. **Factor-critical** = deleting any vertex results in a graph having a perfect matching,

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3. **Star** = vertex together with its neighbor.

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Matching Theory : Results

Theorems

1. Lovász ’75: A graph is factor-critical if and only contracting odd cycles it can be reduced to a vertex.

2. Lovász-Plummer ’86: Every non-bipartite matching-covered graph contains an even subdivision of $K_4$ or of the prism.
Theorems

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1. Each connected component of an elementary graph minus a maximal barrier is factor-critical, and hence provides a sun.

2. Let $H$ be obtained by gluing $G_1$ and $G_2$ in a vertex set $Y$. If $H/G_2$ is elementary then $H/G_1$ can be obtained from $H$ by contracting suns.
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\[ H/G_2 \]
\[ H/G_1 \]
Theorem (Ageev, Benchetrit, Sebő, Szigeti ’11)

The following conditions are equivalent for any graph $G$:

1. Contracting suns it contains a non-trivial bicritical graph,
2. Contracting suns it contains a non-bipartite matching-covered graph,
3. Contracting suns it contains an even subdivision of $K_4$ or of the prism,
4. Contracting stars and factor-critical graphs it contains an even subdivision of $K_4$ or of the prism,
5. Contracting stars and odd cycles it contains an even subdivision of $K_4$ or of the prism,
6. Contracting stars and odd cycles it contains an even subdivision of $K_4$,
7. Contracting cores it contains an even subdivision of $K_4$ or of the prism or of the biprism.
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**Theorem (Ageev, Benchetrit, Sebő, Szigeti ’11)**

The following conditions are equivalent for any graph $G$:

1. Contracting *suns* it contains a non-trivial bicritical graph,
2. Contracting *suns* it contains a non-bipartite matching-covered graph,
3. Contracting *suns* it contains an even subdivision of $K_4$ or of the prism,
4. Contracting *stars* and factor-critical graphs it contains an even subdivision of $K_4$ or of the prism,
5. Contracting *stars* and *odd cycles* it contains an even subdivision of $K_4$ or of the prism,
6. Contracting *stars* and *odd cycles* it contains an even subdivision of $K_4$,
7. Contracting *cores* it contains an even subdivision of $K_4$ or of the prism or of the biprism.

$(1) \implies (2)$: OK, $(2) \implies (1)$: Contract suns of a maximal barrier
**Equivalent forms**

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$(2) \implies (3)$: Lovász-Plummer ’86, $(3) \implies (2)$: OK
### Equivalent forms

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$(3) \implies (4) :$ OK, $(4) \implies (3) :$ ?
Theorem (Ageev, Benchetrit, Sebő, Szigeti ’11)

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$(4) \implies (5)$: Lovász ’75, $(5) \implies (4)$: OK
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$(5) \implies (6)$: Contract an odd cycle of the even subdivision of the prism to get an even subdivision of $K_4$. $(6) \implies (5)$: OK.
Equivalent forms

Theorem (Ageev, Benchetrit, Sebő, Szigeti ’11)

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To see that (6) $\implies$ (3), we need (7).
Equivalent forms

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3 graphs

$K_4$ prism bi-prism
3 graphs

$K_4$  prism  bi-prism

and their even subdivisions
Graphs

3 graphs

- $K_4$
- prism
- bi-prism

and their even subdivisions
3 graphs

- $K_4$
- prism
- bi-prism

and their even subdivisions
Core-contraction to $K_4$

**$K_4$-obstruction**

An odd $K_4$ subgraph $H$ of $G$ with disjoint sets $U_i \subseteq V(H)$ such that

1. $H[U_i \cup N_H(U_i)]$ is an even subdivision of a 3-star,
2. contracting each $U_i \cup N_G(U_i)$, $H$ transforms into an even subdivision of $K_4$. 
Core-contraction to the prism or to the biprism

Prism- or biprism-obstruction

An odd prism subgraph $H$ of $G$ with disjoint sets $U_i \subseteq V(H)$ such that

1. $H[U_i \cup N_H(U_i)]$ is an even subdivision of a 2- or 3-star,

2. contracting each $U_i \cup N_G(U_i)$, $H$ transforms into an even subdivision of the prism or of the biprism (no edge of $G$ connects the two connected components of the biprism minus its separator).
Remark:

1. The contraction of a core in an obstruction changes the parity of the three paths of the obstruction that contain the core.

2. Their main role is to be able to change the odd $K_4$ (or odd prism) into an even subdivision of $K_4$ (or of the prism).
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![Graphs](image-url)
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![Diagram](image-url)
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Remark:

1. The contraction of a core in an obstruction changes the parity of the three paths of the obstruction that contain the core.

2. Their main role is to be able to change the odd $K_4$ (or odd prism) into an even subdivision of $K_4$ (or of the prism).
Contracting *stars* and *odd cycles* it contains an even subdivision of $K_4$, or contracting *cores* it contains an even subdivision of $K_4$ or of the *prism* or of the *biprism*. 
(6) implies (7)

(6) and (7)

(6) Contracting stars and odd cycles it contains an even subdivision of $K_4$,

(7) It contains an $K_4$- or prism- or biprism-obstruction.
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(6) and (7)

(6) Contracting stars and odd cycles it contains an even subdivision of $K_4$, 
(7) It contains an $K_4$- or prism- or biprism-obstruction.

Lemma

If $G/C$ (C : star or odd cycle) contains an obstruction then so does $G$. 
(7) implies (3)

(7) and (3)

(7) Contracting cores it contains an even subdivision of $K_4$ or of the prism or of the biprism.

(3) Contracting suns it contains an even subdivision of $K_4$ or of the prism.
(7) implies (3)

(7) and (3)

(7) Contracting cores it contains an even subdivision of $K_4$ or of the prism or of the biprism.

(3) Contracting suns it contains an even subdivision of $K_4$ or of the prism.

Lemma

1. A core-contraction can be replaced by some sun-contractions.
2. An even subdivision of the biprism can be sun-contracted to an even subdivision of the $K_4$. 
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1. A core-contraction can be replaced by some sun-contractions.
2. An even subdivision of the biprism can be sun-contracted to an even subdivision of the $K_4$.

Both are implied by the lemma about the contraction of elementary graphs because an even subdivision of $K^3_2$ (and of $K_4$) is matching-covered.
Returning to non-Seymour graphs

Equivalence to non-Seymour graphs

1. Non-Seymour graph implies (1) : by structure theorem of Sebő ’90.

2. (7) implies non-Seymour graph : by lemma of Sebő ’92 : a join of $G$ and two tight cycles whose union is an odd $K_4$ or an odd prism can be easily found in an obstruction.
Equivalence to non-Seymour graphs

1. Non-Seymour graph implies (1) : by structure theorem of Sebő '90.
2. (7) implies non-Seymour graph : by lemma of Sebő '92: a join of $G$ and two tight cycles whose union is an odd $K_4$ or an odd prism can be easily found in an obstruction.
Algorithmic aspects

What we can not do

1. Given a graph $G$, decide whether it is a Seymour graph.
2. Given a graph $G$ and a join $F$ in $G$, decide whether there exists an $F$-complete packing of cuts.
Algorithmic aspects

What we can not do

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What we can do

Given a graph $G$ and a join $F$ in $G$,

1. either provide an $F$-complete packing of cuts
2. or show that $G$ is not Seymour.
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What we can do

Given a matching-covered graph, decide if it is Seymour or not:

1. if it is bipartite then it is Seymour,
2. if it is not bipartite then it is not Seymour.
Algorithmic aspects

What we can not do

1. Given a graph $G$, decide whether it is a Seymour graph.
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Given a graph $G$ and a join $F$ in $G$,
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Given a graph $G$ and a join $F$ in $G$,

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What we can do

Given a matching-covered graph, decide if it is Seymour or not:

1. if it is bipartite then it is Seymour,
2. if it is not bipartite then it is not Seymour.
Open problem

NP characterization?
Open problem

NP characterization?
Find a construction for Seymour graphs!
Thanks!