**Definition:** A graded ear-decomposition of $G$ is a sequence $G_0, G_1, \ldots, G_k = G$
- $G_0$ is a cycle, (of even length)
- $G_{i+1} = G_i + P_{i_1} + \ldots + P_{i_k}$, disjoint odd paths
- $G_k$ is matching-covered, nice.

**Definition:** 2-graded ear-decomposition if in each step we add $\leq 2$ ears.

**Theorem:** (Lovász-Plummer)
- A graph is matching-covered if and only if it has a 2-graded ear-decomposition.
DEFINITION: $G$ is elementary if the edges which belong to some perfect matching of $G$ form a connected spanning subgraph.

$\phi(G)$ := number of perfect matchings of $G$.

THEOREM (Lovász-Plummer)

$G$ is elementary, $e_1, e_2, \ldots, e_k \in E(G)$.

If $\phi(G + e_1 + e_2 + \ldots + e_k) > \phi(G)$ then

$\exists i, j : \phi(G + e_i + e_j) > \phi(G)$.

SHORT PROOF: 2.52.
Theorem (Lovász-Plummer)

$G$ is matching-covered, $e_1, \ldots, e_k \in E(G)$:

$G + e_1 + \ldots + e_k$ is matching-covered.

Then $\exists i \neq j : G + e_i + e_j$ is matching-covered.

Short proof : 2.82.
**Theorem:** \( G \) is elementary, \( e_1, e_2, e_3 \in E(G) \)

- \( G + e_i \) has a perfect matching \( M \) containing \( e_1, e_2, e_3 \).
- \( G + e_i \) has no perfect matching containing \( e_i \) \( \ (i = 1, 2, 3) \)

Then \( \forall e_i : \exists e_j : G + e_i + e_j \) has a perfect matching containing \( e_i \) and \( e_j \).

**Definition:** \( G \) is elementary, \( X \subseteq V(G) \).

\( X \) is a strong barrier if \( G - X \) has \( |X| \) components and all of them are factor-critical.

**Lemma:** If \( G \) is elementary and \( X \) is a strong barrier of \( G \), then each edge leaving \( X \) belongs to a perfect matching of \( G \).
**Proof:** Suppose $G' = G + e_1, e_2$ has no perfect matching containing $e_1$ and $e_2$.

1. **A strong barrier $X$ in $G'$ containing $e_1$.**

2. $e_2 \in F$.

3. $e_3$ connects $F_i$ and $F_j$.

4. $e_4$ connects $X$, $F_i$, and $F_j$.

5. $G + e_2$ has a perfect matching containing $e_2$. 

\[ e_i, e_j \]