Matroid-based packing of spanning arborescences

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Outline

- Motivation
- Definitions
- Problem
- Results
- Proofs
- Conclusion

Motivation

Definition

Packing subgraphs : a set of arc-disjoint subgraphs in a directed graph.

Application of path packings

- Telecommunication
- Transportation
- VLSI

Storing of path packings

Suppose D has a packing of k (s,t)-paths from s to each vertex t in V.

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What is a compact certificate of this fact ?
A packing of k spanning s-arborescences in D.

Packing of spanning s-arborescences

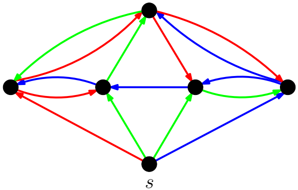
Definition

1. s-arborescence : directed tree, in-degree of every vertex except s is 1,
2. spanning subgraph of D : subgraph that contains all the vertices of D.

Theorem (Edmonds 1973 + Menger 1927)

Let $D = (V + s, A)$, $k \in \mathbb{Z}_+$.

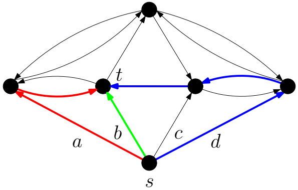
- *D has a packing of k spanning s-arborescences*
- *D has a packing of k paths from s to each vertex in D.*



Motivation

Questions

1. Packing k (s,t)-paths means sending k distinct commodities from s to t by assuming that each arc can transmit at most one commodity.
2. What if commodities have a more involved independence structure ?
3. Suppose that every vertex can receive a sufficient amount of independent commodities to understand the whole structure. Does there exist a compact certificate for such packings of paths ?



Matroids

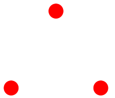
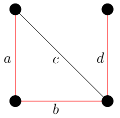
Definition

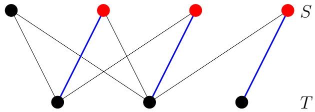
For $\emptyset \neq \mathcal{I} \subseteq 2^E$, $\mathcal{M} = (E, \mathcal{I})$ is a matroid if

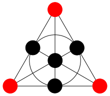
1. If $X \subseteq Y \in \mathcal{I}$, then $X \in \mathcal{I}$,
2. If $X, Y \in \mathcal{I}$ with $|X| < |Y|$ then $\exists\ y \in Y \setminus X$ such that $X \cup y \in \mathcal{I}$.

Examples

1. Free : all subsets of a set,
2. Graphic : edge-sets of forests of a graph,
3. Transversal : end-vertices in S of matchings of bipartite graph $(S, T; E)$
4. Fano : subsets of sets of size 3 not being a line in the Fano plane.





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1. matroid-rooted digraph $(D = (V + s, A), \mathcal{M})$: a matroid $\mathcal{M}$ is given on the set of root arcs (arcs leaving the special vertex s).

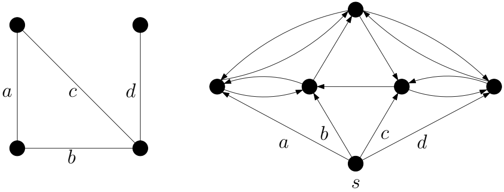

FIGURE: a matroid-rooted digraph

Definition

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2. $\mathcal{M}$-based packing of (s, t)-paths : if the root arcs form a base of $\mathcal{M}$.

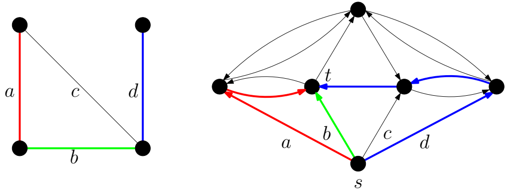

FIGURE: an $\mathcal{M}$-based packing of (s, t)-paths

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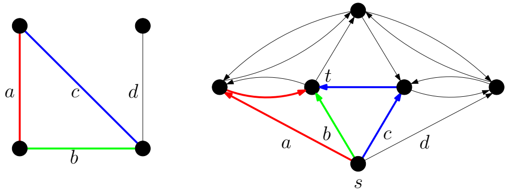

FIGURE: Not an $\mathcal{M}$-based packing of (s, t)-paths

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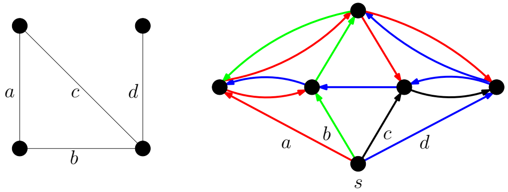

FIGURE: an $\mathcal{M}$-based packing of s-arborescences

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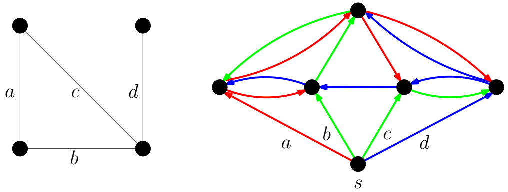

FIGURE: an $\mathcal{M}$-based packing of spanning s-arborescences

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Let $(D = (V + s, A), \mathcal{M})$ be matroid-rooted digraph.

- There exists an $\mathcal{M}$-based packing of (s, t)-paths for all $t \in V$
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Question

Can Edmonds' theorem be extended for $\mathcal{M}$-based packings ?

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Conjecture (Bérczi-Frank 2015)

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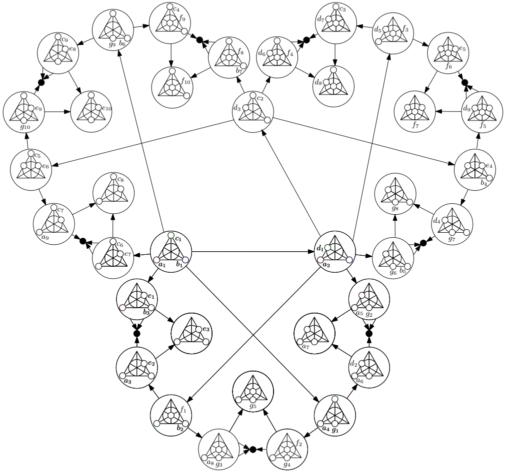
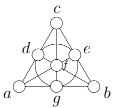
   3. transversal matroids.

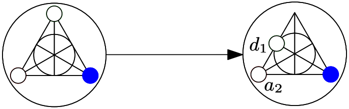
Counterexample

Digraph : acyclic, in-degree 3 for all $v \in V$, 46 vertices and 135 arcs,
Matroid : parallel extension of Fano with 64 elements,
Remark : matroid-based packing of (s, t)-paths exists for all t.

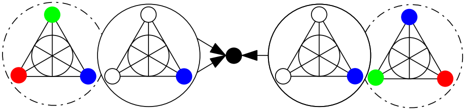




Operation 1



Operation 2



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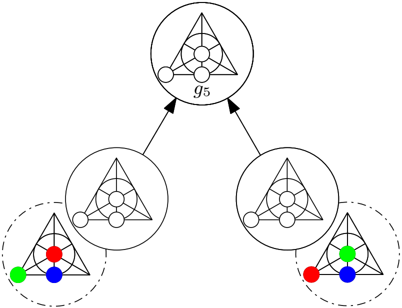
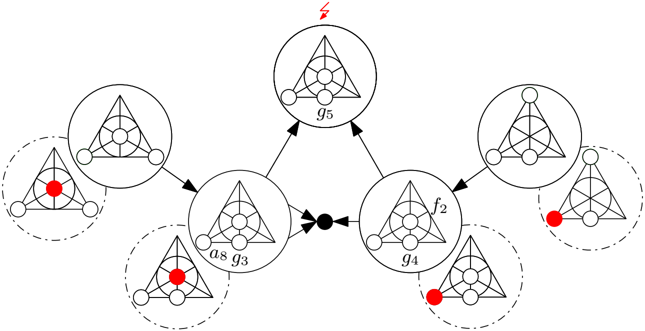
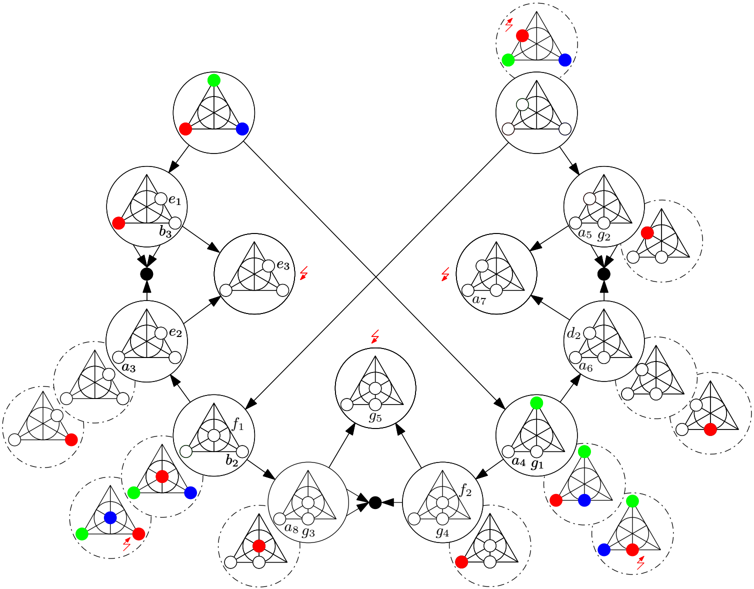


Operation 4



Operation 5



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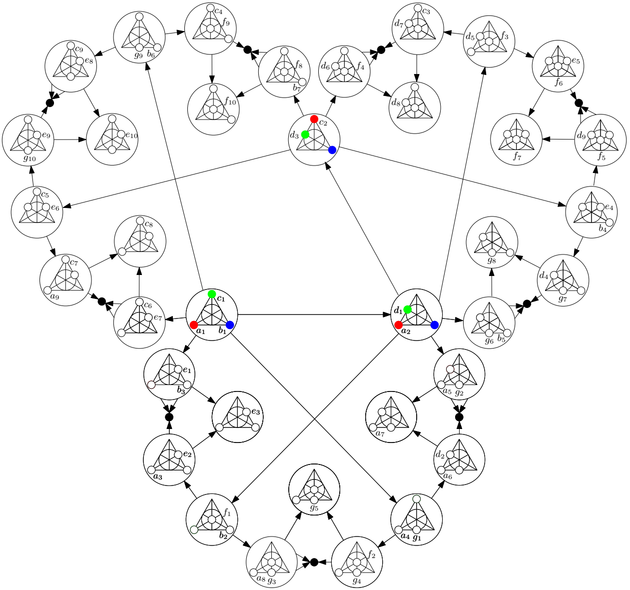

Graphic matroids

Theorem (FKSzT)

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Remark

A packing of k spanning s-arborescences in $D = (V + s, A)$ can be obtained as an $\mathcal{M}$-based packing of s'-arborescences in $D' = (V + s + s', A \cup A')$, where $A' = \{k \times s's\}$ and free matroid $\mathcal{M}$ on A'.

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2. If $= 0$, then $\partial(s, v)$ is a base of $\mathcal{M}$; done.
3. If > 0, then apply shifting operation.
4. $\exists$ a shifting s.t. $(D', \mathcal{M}')$ satisfies 1 and 2.
5. By induction $\exists$ a packing for $(D', \mathcal{M}')$.
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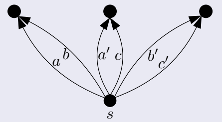

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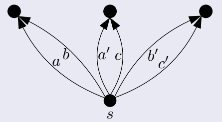

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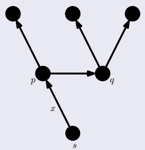

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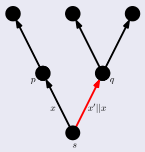

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Proof

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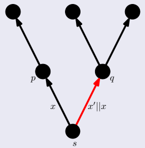

Matroid-based packing of s-arborescences

Theorem (Durand de Gevigney, Nguyen, Szigeti 2013)

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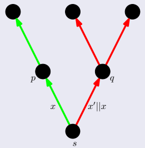

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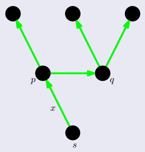

Graphic matroids

Theorem (FKSzT)

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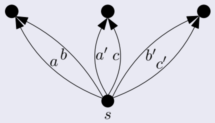

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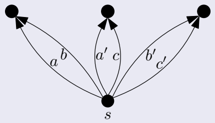

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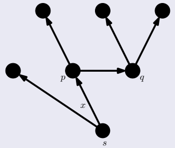

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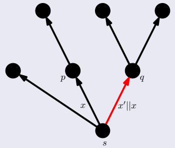

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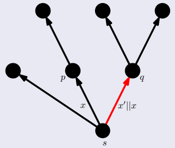

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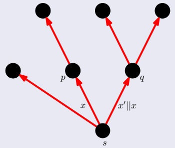

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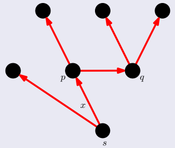

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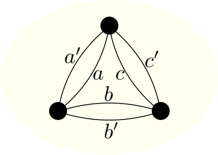
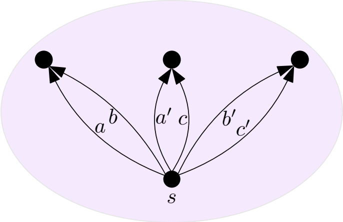
Example concerning parallel elements

Remark

It cannot be required that parallel elements be contained in the same spanning s-arborescence.

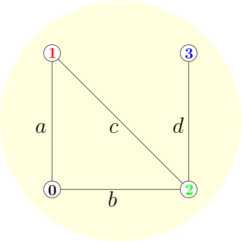


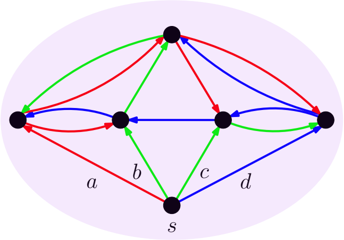


Idea

Let $(D, \mathcal{M})$ be a matroid-rooted digraph where $\mathcal{M}$ is a graphic matroid of rank k and $G = (\{0, 1, \ldots, k\}, E)$ a graph representing $\mathcal{M}$.

($\star$) A root arc $\vec{e}$ of D (that corresponds to an edge $e = ij$ of G) may belong only to the i^{th} or to the j^{th} spanning arborescence.

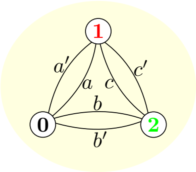


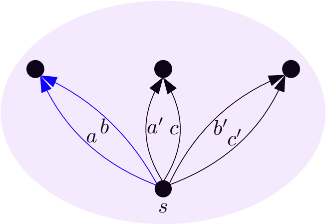


Base case for induction

Remark

1. $\partial(s,v)$ is a base of $\mathcal{M}$ for all $v \in V$.
2. In G, corresponding edge set E_v forms a spanning tree T_v.
3. Orient T_v to get a spanning 0-arborescence $\vec{T}_v$.
4. $\vec{T}_v$ provides a bijection from $\partial(s,v)$ to $\{1,\ldots,r(\mathcal{M})\}$ satisfying $(\star)$.
5. These bijections provide the $\mathcal{M}$-based packing of s-arborescen. in D.





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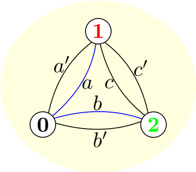

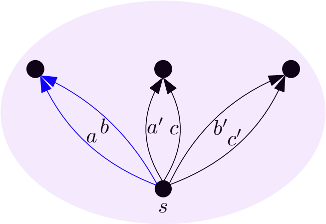

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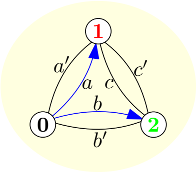

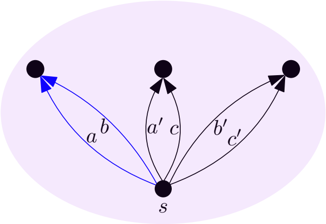

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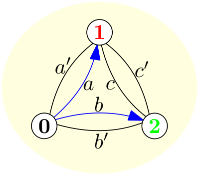

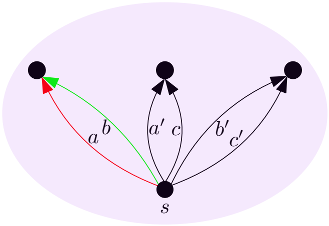

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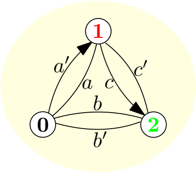

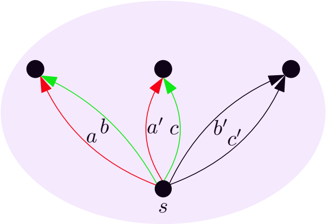

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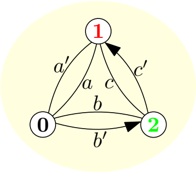
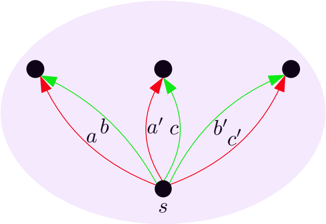




Induction step

To be able to apply shifting

We need $pq \in A(D)$ and $x \in \partial(s, p)$ such that $(D', \mathcal{M}')$ satisfies

(i) $r'(\partial'(s, X)) + |\partial'(V - X, X)| \geq r'(\mathcal{M}')$ for all $\emptyset \neq X \subseteq V$,

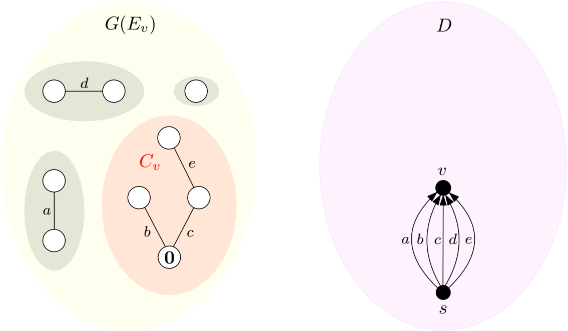
(ii) $\partial'(s, v) \in \mathcal{I}'$ for all $v \in V$,

(iii) x and x' belong to the same spanning arborescence of any matroid-based packing of spanning arborescences of $(D', \mathcal{M}')$ satisfying $(\star)$.

We show how to find one (pq, x) satisfying (ii) and (iii),
one can show that one of them satisfies (i) as in DdG-N-Sz.

Remark : By ($\star$),

- root arcs may belong to 2 arborescences,
- some of them may belong only to 1 arborescence.



Unique arborescence

Remark : By ($\star$),

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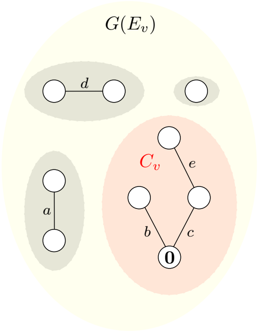

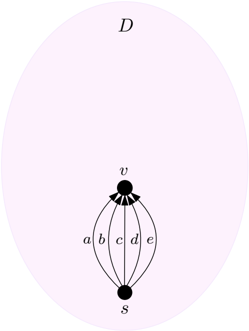

Unique arborescence

Remark : By ($\star$),

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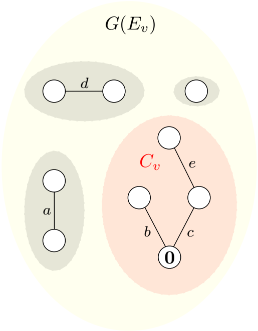

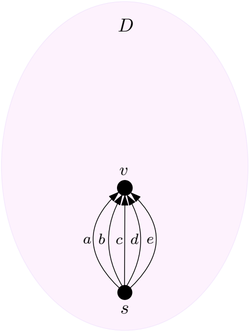

Unique arborescence

Remark : By ($\star$),

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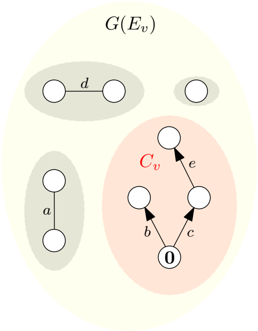
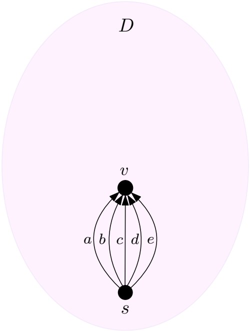
- some of them may belong only to 1 arborescence. Which one ? Those, contained in the connected component C_v of $G(E_v)$ that contains 0.





How to find (pq, x) satisfying (ii) and (iii) ?

Proof

1. It is enough to find $pq \in A(D)$ such that $C_p \setminus C_q \neq \emptyset$.
2. $v^* \in V(D)$ that minimizes $|C_v|$, $C_{v^*} \subsetneq V(G)$ since not base case.
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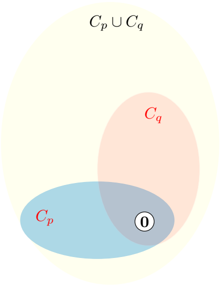

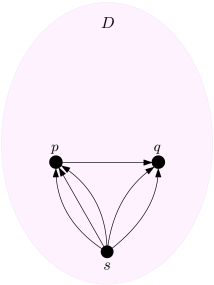

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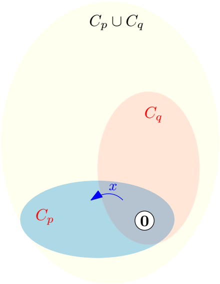

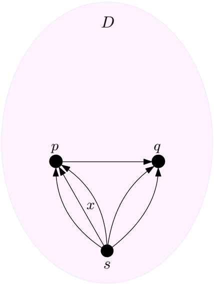

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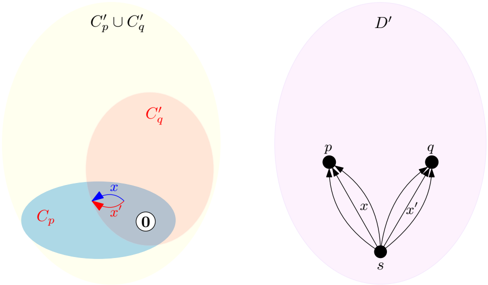

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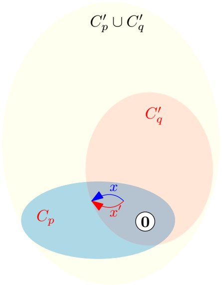

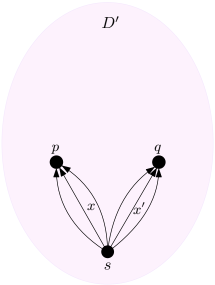

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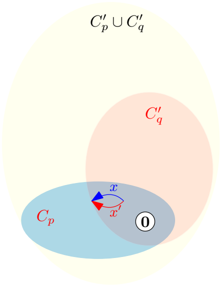

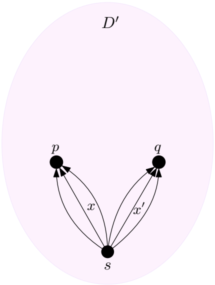

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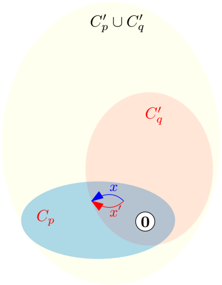

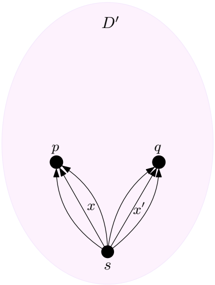

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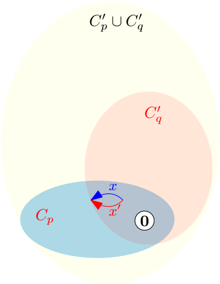

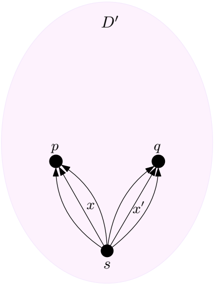

Conclusion

Summary

1. Compact certificate of the existence of a matroid-based packing of paths :
   1. a matroid-based packing of arborescences and
   2. not a matroid-based packing of spanning arborescences ;
   3. a matroid-based packing of spanning arborescences if the matroid is rank 2 or graphic or transversal.
2. The decision problem whether a matroid-rooted graph has a matroid-based packing of spanning arborescences is NP-complet.

Thank you for your attention !