Reliable Network Design

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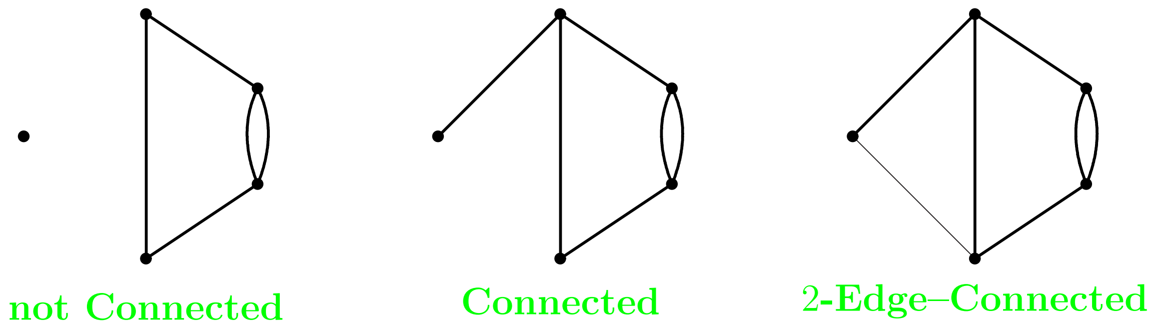
Plan of the talk

Motivations	**: Networks, Frameworks**
Problem	**: Connectivity Augmentation**
Versions	**: Graphs & Bipartite Graphs** **Global & Local Edge-Conn.**
Results	**: Authors**
Method	**: Algorithm of Frank** **(Extension & Splitting)**
Results	**: Extension** **Splitting** **Min-Max Theorems** **Algorithms**

Motivation – Network

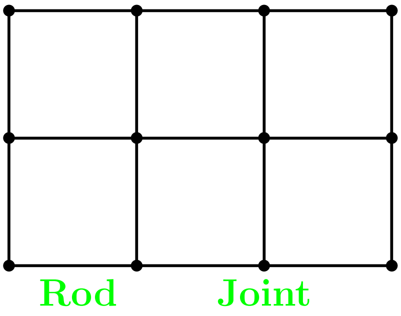


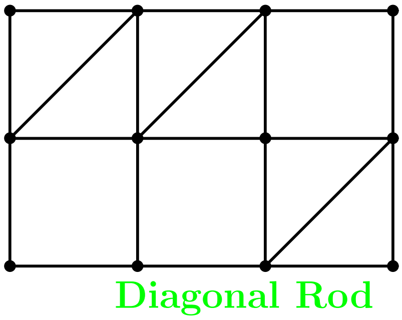
Reliability of Network = Edge-Conn. of Graph

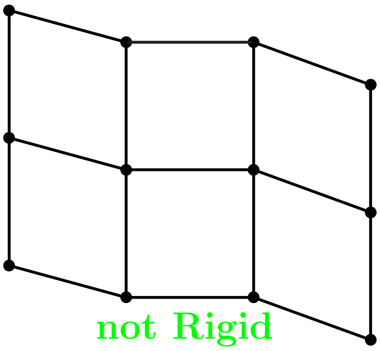
Basic Problem : **Given a graph $G = (V, E)$ and $k \geq 2$, add a minimum number of edges F to G s.t. $G + F$ to be k-edge-connected.**

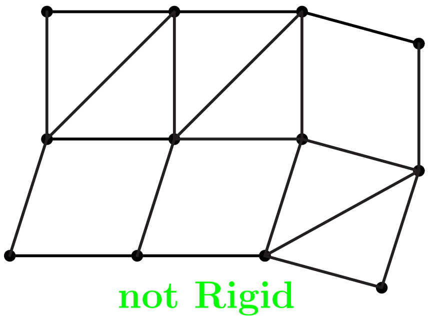
Motivation – Frameworks in Statics

Rigidity of 2-dim. Square Grid Frameworks

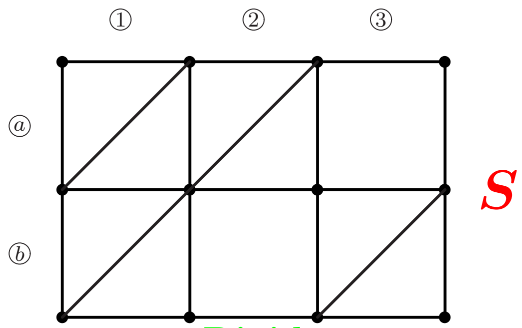




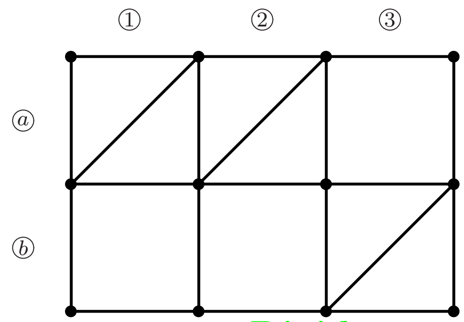




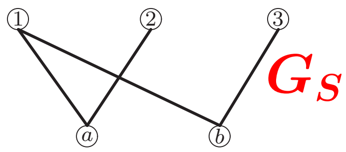
Cont. – Frameworks in Statics



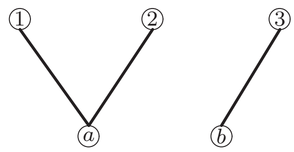
Rigid



not Rigid



Connected



not Connected

Theorem (Bolker-Crapo)

S is Rigid iff G_S iss Connected.

Reliability of the Framework

Problem : Given a framework S and $k \geq 2$, add a minimum number of diagonal rods to S s.t. the deletion of any $k-1$ diagonal rods does not destroy the rigidity of the framework.

Reliability of S = Edge-Connectivity of G_S

Main Problem : Given a bipartite graph $G = (A, B; E)$ and $k \geq 2$, add a minimum number of edges F between A and B s.t. $G + F$ to be k-edge-connected.

Problem

Edge-Connectivity Augmentation of an undirected graph by adding a minimum number of new edges

In this talk we do not consider :

vertex-connectivity
directed graphs
hypergraphs
minimum cost (NP-complete)

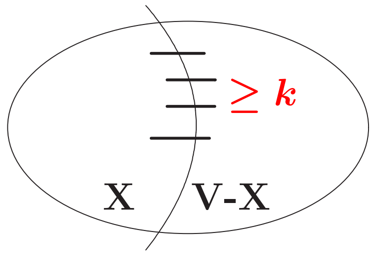
In this talk we consider :

Graphs & **Bipartite Graphs**
Global E-C & **Local E-C**

Definitions

Global Edge-Connectivity

$G = (V, E)$ is
k-edge-connected if
$d(X) \geq k \; \forall X \subset V$.



Local Edge-Connectivity

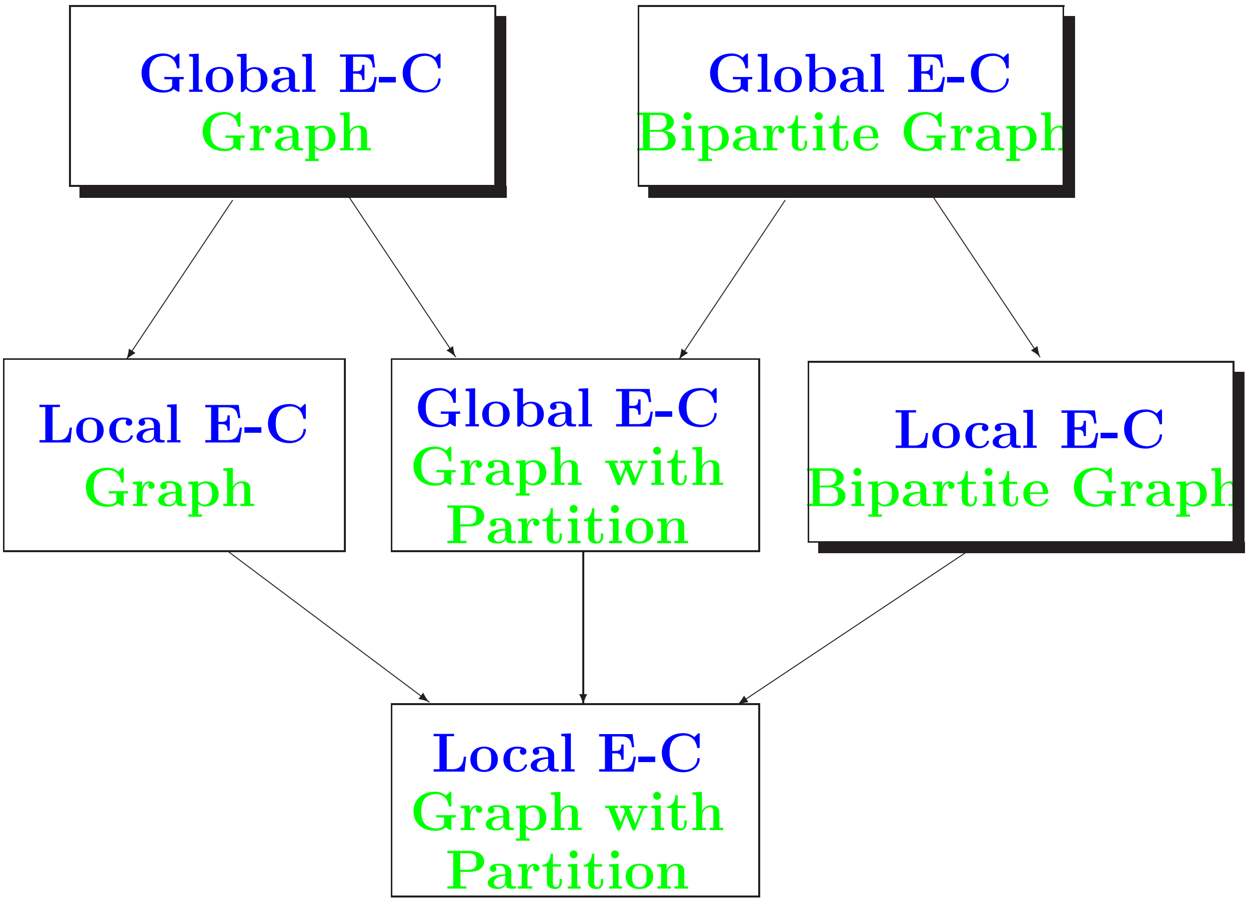
$G = (V, E), u, v \in V$

$\lambda(u, v) := \min\{d(X) : u \in X, v \in V - X\}$

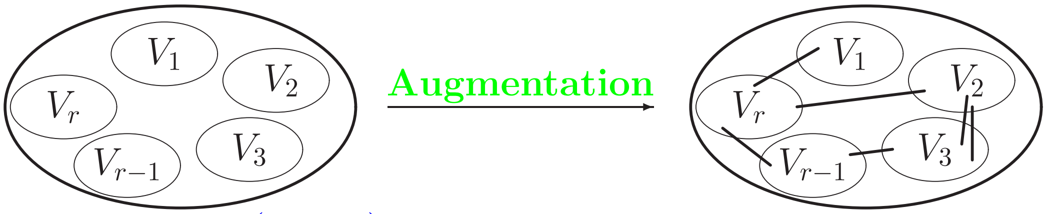
Global Edge-Connectivity II

$G = (V + s, E)$ is k-edge-connected in V if
$d(X) \geq k \; \forall X \subset V$.

Versions to consider



Generalization



Graph $G = (V, E)$
Partition $\mathcal{P}$ of V

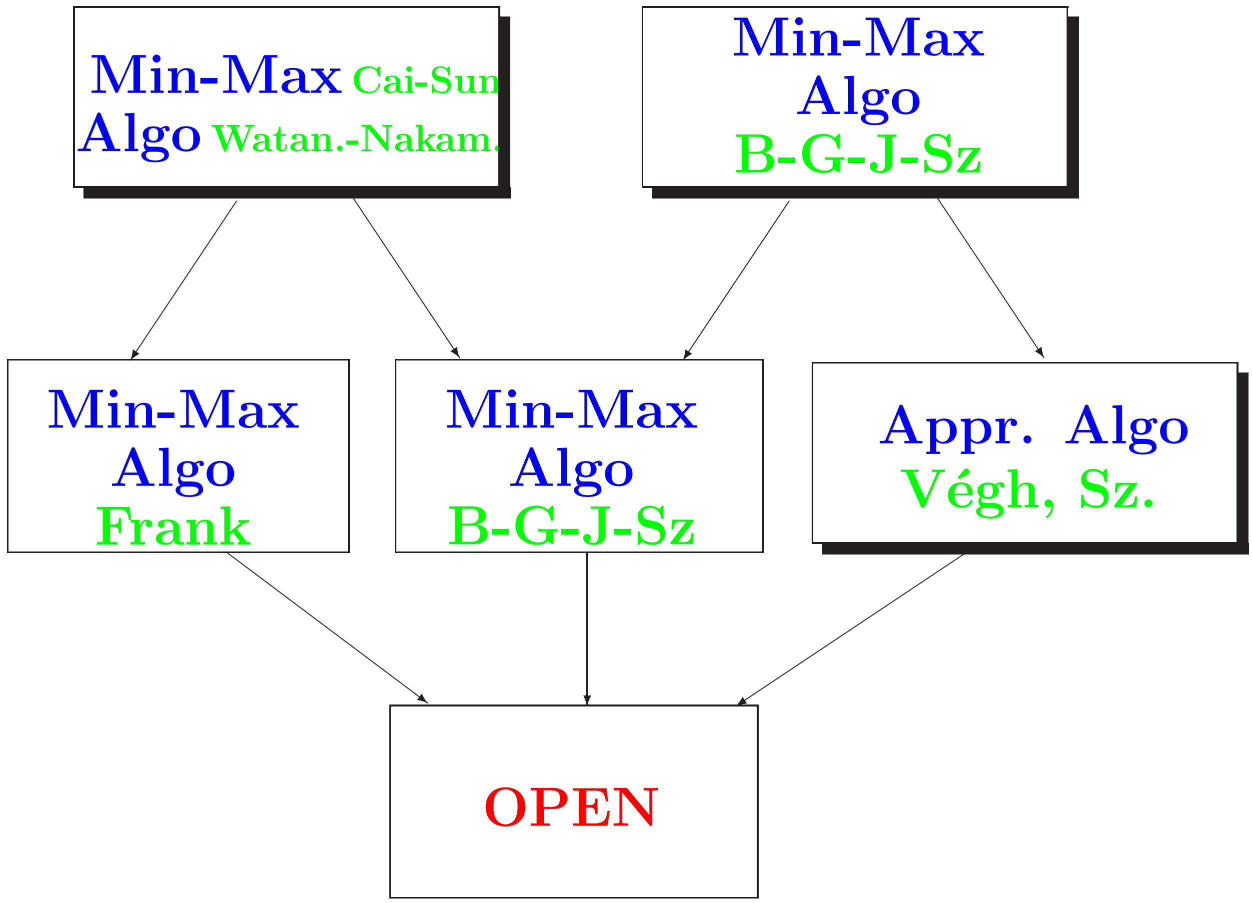
G' k-E-C

Special Cases :

(1) $G = (A, B; E), \mathcal{P} = \{A, B\}$ = **Main Prob.**

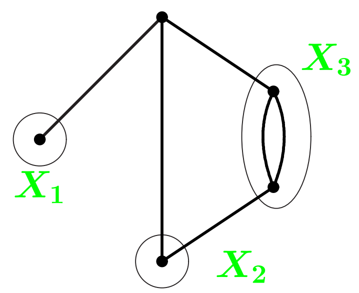
(2) $G = (V, E), \mathcal{P} = \{\{v\} : v \in V\}$ = **Basic Prob.**

Results


Min-Max Algo Cai-Sun Watan.-Nakam.
Min-Max Algo B-G-J-Sz
Min-Max Algo Frank
Min-Max Algo B-G-J-Sz
Appr. Algo Végh, Sz.
OPEN

Method
Basic Problem

Bound for min. number γ of new edges



$G, k = 3$

Deficient set : X_1

deficiency of X_1 **:** $k - d(X_1) = 2$

Def. Subpart. : $\{X_1, X_2, X_3\} = \mathcal{X}$

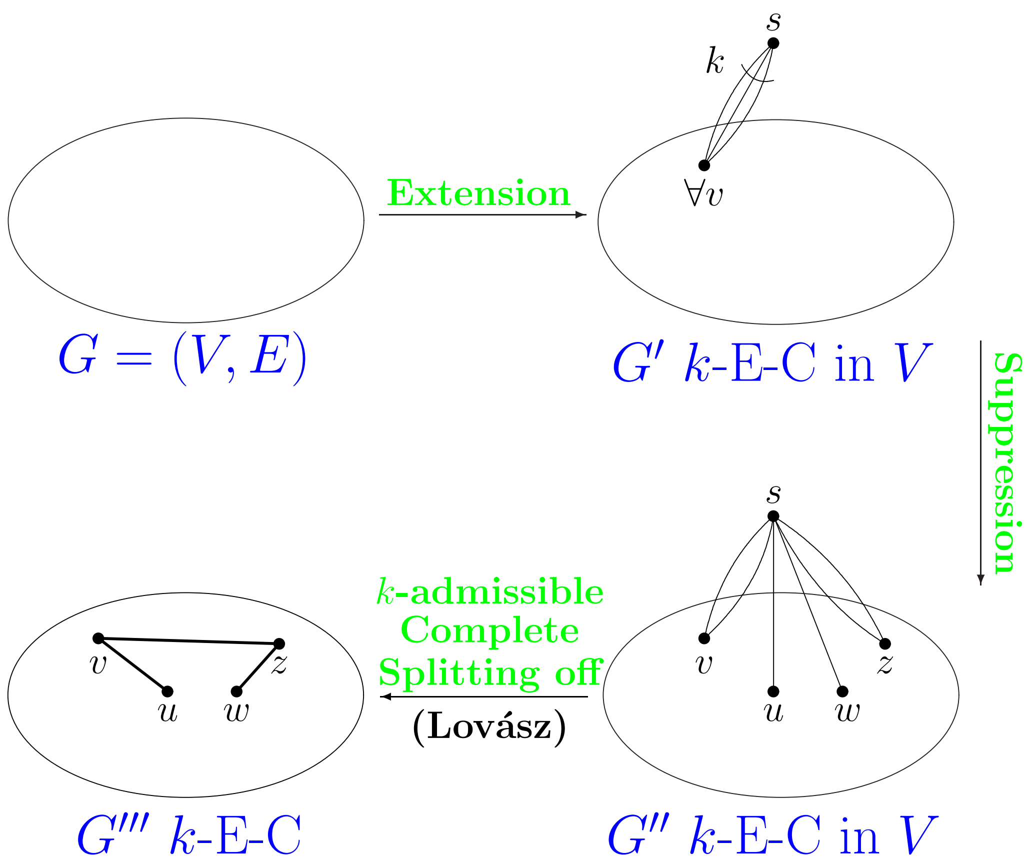
def. of $\mathcal{X}$ **:** $\sum_{X_i \in \mathcal{X}} (k - d(X_i)) = 4$

$\mathbf{def}_{G+e}(\mathcal{X}) \geq \mathbf{def}_G(\mathcal{X}) - 2$

$\gamma \geq \alpha := \lceil \frac{1}{2}$ max. deficiency of subpart. of $V \rceil$

Theorem (Cai, Sun) $\gamma = \alpha$.

Method
Algorithm of Frank



Min. Extension

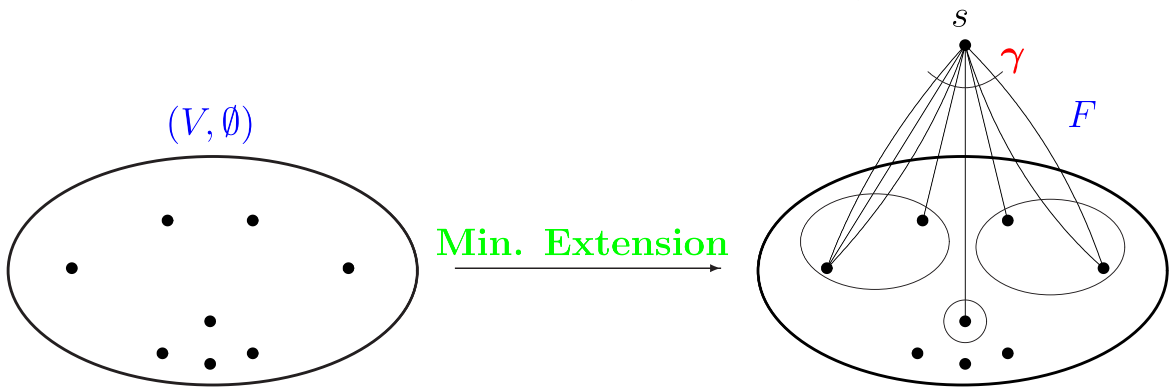
Theorem of Frank

Given $p : V \to Z$ sym. skew-supermod, $(V, \emptyset)$ can be extended to a graph F by adding a new vertex s and γ edges incident to s s.t.

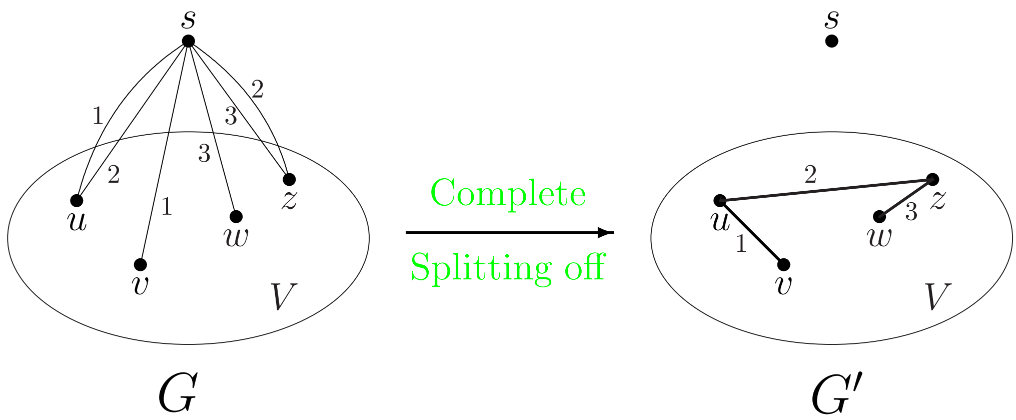
$$d_F(X) \geq p(X) \qquad \forall\ \emptyset \neq X \subset V$$

if and only if

$$\Sigma_1^l p(X_i) \leq \gamma \ \ \forall \text{ subpart. } \{X_1, ..., X_l\} \text{ of } V.$$



Complete Splitting off



Complete Splitting off is

k**-admis. if** G' **is** k**-E-C in** V.
λ**-admis. if** $\lambda_{G'}(x,y) = \lambda_G(x,y)\ \forall x,y \in V$**.**

Theorems

Complete Splitting off

Given

Graph $G = (V + s, E)$ s.t. $d_G(s)$ is even,

(Lovász) **If G is k-E-C in V $(k \geq 2)$,**

$\exists$ k-admis. Complete Splitting off.

(Mader) **If G is 2-E-C,**

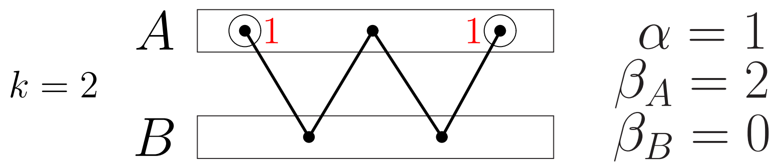
$\exists$ λ-admis. Complete Splitting off.

$\Longrightarrow$ **Global and Local E-C Augmentation can be solved.**

Main Problem
Bounds

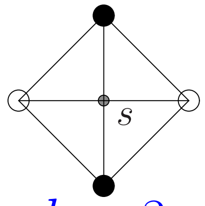
$\boldsymbol{\alpha} := \lceil \frac{1}{2}$ **max. deficiency of subpart. of** $A \cup B \rceil$

$$\boldsymbol{\beta_A := \sum_{\substack{v \in A \\ d(v) < k}} (k - d(v))} \qquad \boldsymbol{\beta_B := \sum_{\substack{v \in B \\ d(v) < k}} (k - d(v))}$$



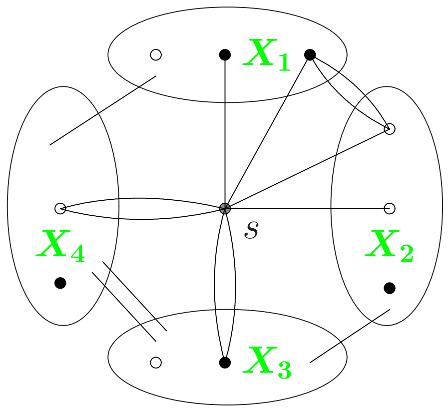
$$\boldsymbol{\gamma \geq \Phi := \max\{\alpha, \beta_A, \beta_B\}}$$

Main Problem



$k = 3$

no k-admis. Complete Allowed Splitting off



C_4-**obstacle**

(i) k **odd**

(ii) $X_1 \dot\cup \ldots \dot\cup X_4 = V$

(iii) $d(X_i) = k \;\; 1 \leq i \leq 4$

(iv) $d(X_1, X_3) = 0 = d(X_2, X_4)$

(v) $N(s) \cap A \subseteq X_1 \cup X_3$

$N(s) \cap B \subseteq X_2 \cup X_4$

(vi) $d(s, A) = d(s, B)$

Main Problem

Theorems (B-G-J-Sz)

Complete Splitting off

Given Graph $G = (A \cup B \cup s, E')$ **s.t.**

$G - s$ **is Bipartite,** $k \geq 2$**.**

$\exists$ k**-admis. Complete Allowed Splitting off**

$$\Longleftrightarrow$$

(a) $d(s, A) = d(s, B)$,

(b) G **is** k**-E-C in** V,

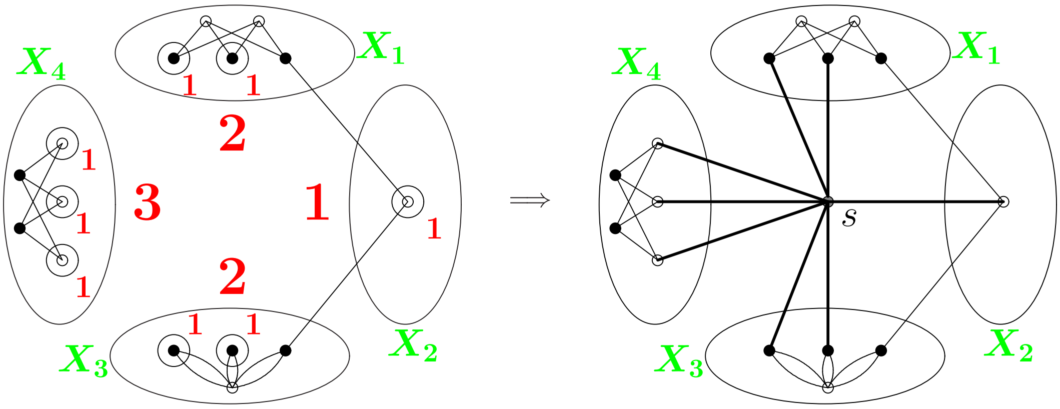
(c) G **contains no** C_4**-obstacle.**

Augmentation

For Bipartite Graph $G = (A, B; E)$, $k \geq 2$,

$$\gamma = \begin{cases} \Phi & \text{if } G \text{ has no } C_4\text{-configuration,} \\ \Phi + 1 & \text{otherwise.} \end{cases}$$

C_4-configuration $\qquad$ C_4-obstacle



$\exists C_4$-**configuration** $\Rightarrow \boldsymbol{\gamma} \geq \boldsymbol{\Phi} + \mathbf{1}$

Main Problem – Local E-C Approximation Algorithm (Végh-Sz)

Given a bipartite graph $G = (A, B; E)$ and $r(u,v) \geq 2 \quad \forall u,v \in A \cup B$, we can find in polynomial time a new edge-set F between A and B s.t.

$$\lambda_{G+F}(u,v) \geq r(u,v) \ \forall u,v \in A \cup B,$$

$$|F| \leq \frac{3}{2} OPT.$$

Is this Problem NP-complete ?