Edge-connectivity of permutation hypergraphs

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Outline

1. Permutation graphs
2. Splitting off in graphs
3. Permutation hypergraphs
4. Splitting off in hypergraphs
Definition

Given a graph $G$ on $n$ vertices and a permutation $\pi$ of $[n]$, we define the permutation graph $G_{\pi}$ as follows:

1. we take 2 disjoint copies $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of $G$,
2. for every vertex $v_i \in V_1$, we add an edge between $v_i$ of $G_1$ and $v_{\pi(i)}$ of $G_2$, this edge set is denoted by $E_3$,
3. $G_{\pi} = (V_1 \cup V_2, E_1 \cup E_2 \cup E_3)$.

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Edge-connectivity of $G$: $\lambda(G) = \min \{ d_G(X) : \emptyset \neq X \subset V \}$,
Minimum degree of $G$: $\delta(G) = \min \{ d_G(v) : v \in V \}$. 
Connectivity of permutation graphs

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**Remark**

\[ \lambda(G_\pi) \leq \delta(G_\pi) = \delta(G) + 1. \]
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$\lambda(G_\pi) \leq \delta(G_\pi) = \delta(G) + 1$.

Theorem (Goddard, Raines, Slater)

*For a simple graph $G$ without isolated vertices, there exists a permutation $\pi$ such that $\lambda(G_\pi) = \delta(G) + 1$ if and only if $G \neq 2K_k$ for some odd $k$.***
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- Extension : $\lambda(H) = \delta(G) + 1$,
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Splitting off in graphs

Given: graph $H = (V + s, E)$, partition $\mathcal{P} = \{P_1, P_2\}$ of $V$, integer $k$.

**Definition**

- **Splitting off at $s$:** replacing $\{su, sv\}$ by $uv$.
- **Complete splitting off at $s$:** a sequence of splitting off isolating $s$.
- It is $k$-admissible if $H' - s$ is $k$-edge-connected.
- It is $\mathcal{P}$-allowed if the new edges are between $P_1$ and $P_2$. 

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Theorem (Bang-Jensen, Gabow, Jordán, Szigeti)

Given : graph $H = (V + s, E)$, partition $\mathcal{P} = \{P_1, P_2\}$ of $V$, integer $k \geq 2$. There exists a $k$-admissible $\mathcal{P}$-allowed complete splitting off at $s$ if and only if

- $H$ is $k$-edge-connected in $V$,
- $d(s, P_1) = d(s, P_2)$,
- $H$ contains no $C_4$-obstacle.

$C_4$, $k = 3$
A partition \( \{A_1, A_2, A_3, A_4\} \) of \( V \) is called a \( C_4 \)-obstacle of \( H \) if

- \( k \) is odd,
- each \( A_i \) is of degree \( k \),
- no edge exists between \( A_i \) and \( A_{i+2} \),
- half of the edges incident to \( s \) are incident to \( P_1 \cap (A_1 \cup A_3) \),
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Theorem (Goddard, Raines, Slater)

For a simple graph $G$ without isolated vertices,

- there exists a permutation $\pi$ such that $\lambda(G_\pi) = \delta(G) + 1$,
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**Hypergraphs**

**Definition**

- **hypergraph**: $\mathcal{G} = (V, \mathcal{E})$, $V =$ set of vertices, $\mathcal{E} =$ set of hyperedges, subsets of $V$.
- $\mathcal{G}$ is $k$-edge-connected if each cut contains at least $k$ hyperedges.

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\[ X \cup V - X \geq k \]
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$\pi = (613425)$

$G$

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- $\mathcal{H}$ is $k$-edge-connected in $V$,
- $d_{\mathcal{H}}(s) \geq 2\omega(\mathcal{H} - s)$,
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Connectivity of permutation hypergraphs

Theorem (Jami, Szigeti)

For a hypergraph $\mathcal{G}$ and an integer $k \geq 2$, there exists a permutation $\pi$ such that $\lambda(\mathcal{G}_\pi) = k$ if and only if

1. $d_G(X) \geq k - |X|$ for all $\emptyset \neq X \subseteq V$,

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- Implied by Theorem of Bernáth, Grappe, Szigeti.
- Implies Theorem of Goddard, Raines, Slater: if $G$ is a simple graph $G$ without isolated vertices and $k = \delta(G) + 1$, then
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Thank you for your attention!