Edge-connectivity of permutation hypergraphs

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Outline

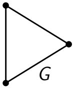
1. Permutation graphs
2. Splitting off in graphs
3. Permutation hypergraphs
4. Splitting off in hypergraphs

Permutation graphs

Definition

Given a graph G on n vertices and a permutation π of $[n]$, we define the permutation graph $\mathbf{G}_\pi$ as follows :

1. we take 2 disjoint copies $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G,
2. for every vertex $v_i \in V_1$, we add an edge between v_i of G_1 and $v_{\pi(i)}$ of G_2, this edge set is denoted by E_3,
3. $G_\pi = (V_1 \cup V_2, E_1 \cup E_2 \cup E_3)$.



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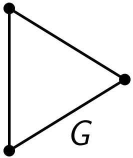

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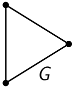

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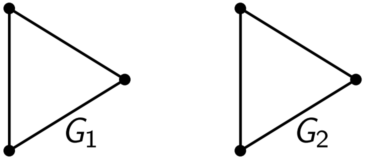

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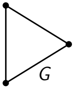

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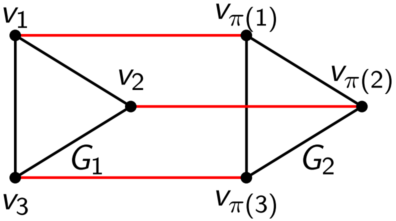

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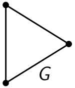

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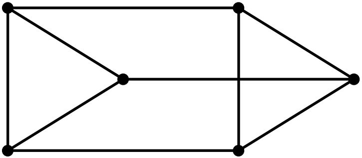

G_π

Connectivity of permutation graphs

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Edge-connectivity of G : $\boldsymbol{\lambda}(\mathbf{G}) = \min\{d_G(X) : \emptyset \neq X \subset V\}$,
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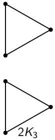
For a simple graph G without isolated vertices, there exists a permutation π such that $\lambda(G_\pi) = \delta(G) + 1$ if and only if $G \neq 2K_k$ for some odd k.



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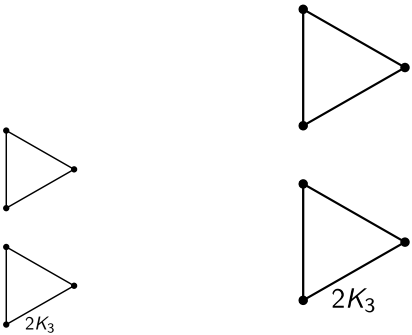

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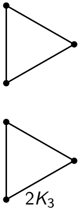

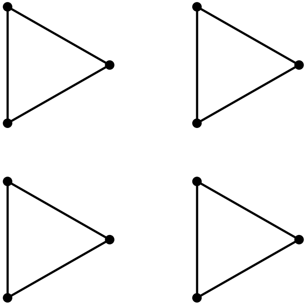

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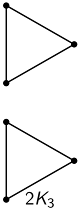

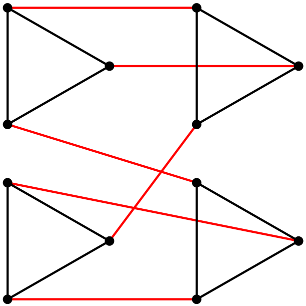

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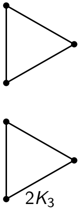

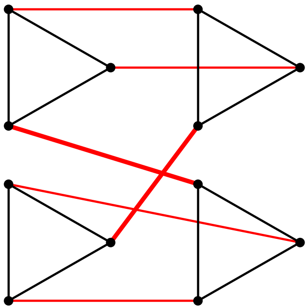

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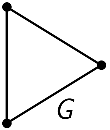

- Extension : $\lambda(H) = \delta(G) + 1$,
- Splitting off : between G_1 and G_2, maintaining edge-connectivity

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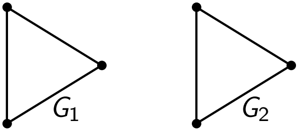

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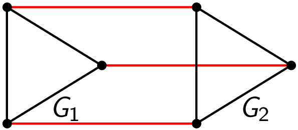

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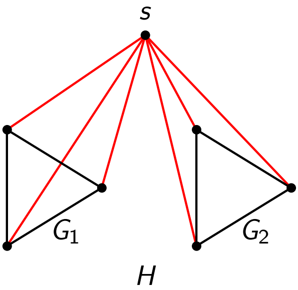

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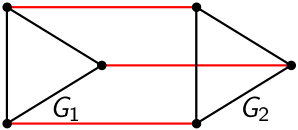

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Splitting off in graphs

Given : graph $H = (V + s, E)$, partition $\mathcal{P} = \{P_1, P_2\}$ of V, integer k.

Definition

- Splitting off at s : replacing $\{su, sv\}$ by uv.
- Complete splitting off at s : a sequence of splitting off isolating s.
- it is k-admissible if $H' - s$ is k-edge-connected.
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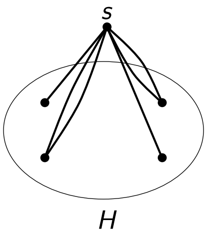

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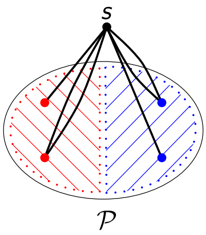

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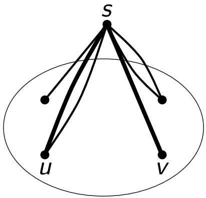

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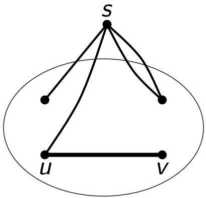

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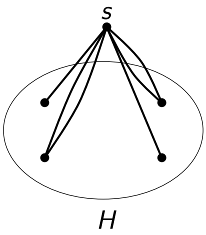

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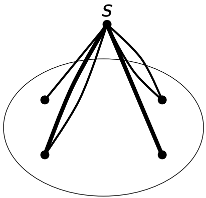

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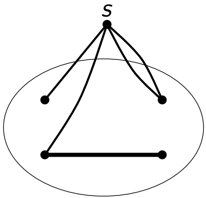

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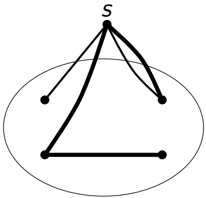

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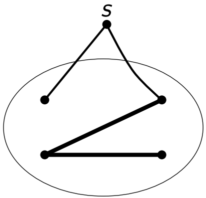

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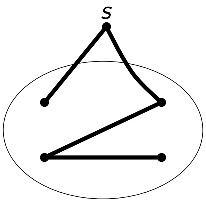

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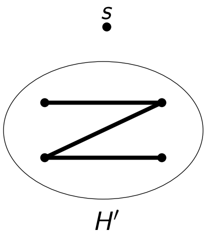

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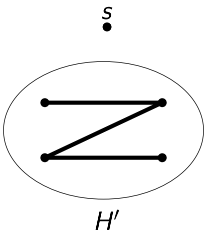

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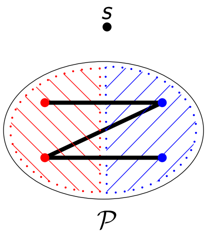
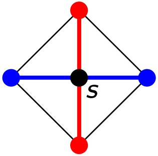


Result on splitting off in graphs

Theorem (Bang-Jensen, Gabow, Jordán, Szigeti)

Given : graph $H = (V + s, E)$, partition $\mathcal{P} = \{P_1, P_2\}$ of V, integer $k \geq 2$. There exists a k-admissible $\mathcal{P}$-allowed complete splitting off at s if and only if

- *H is k-edge-connected in V,*
- *$d(s, P_1) = d(s, P_2)$,*
- *H contains no C_4-obstacle.*



C_4, $k = 3$

C_4-obstacle

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A partition $\{A_1, A_2, A_3, A_4\}$ of V is called a C_4-obstacle of H if

- k is odd,
- each A_i is of degree k,
- no edge exists between A_i and A_{i+2},
- half of the edges incident to s are incident to $P_1 \cap (A_1 \cup A_3)$,
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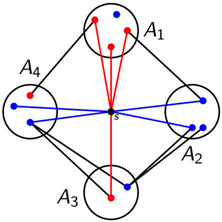

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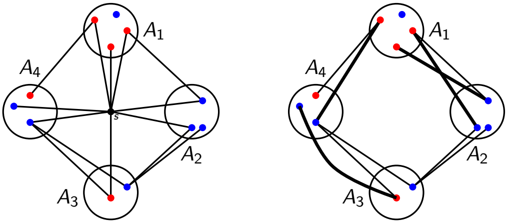

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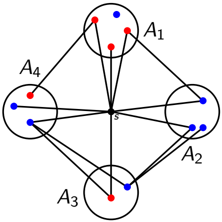

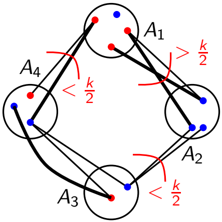

Sufficiency

Theorem (Goddard, Raines, Slater)

For a simple graph G without isolated vertices,

- *there exists a permutation π such that $\lambda(G_\pi) = \delta(G) + 1$,*

  if and only if

- *there exists a k-admissible $\mathcal{P}$-allowed complete splitting off at s in H,*

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- *H contains no C_4-obstacle,*

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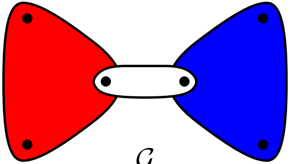
Definition

- hypergraph : $\mathcal{G} = (V, \mathcal{E})$, V = set of vertices, $\mathcal{E}$ = set of hyperedges, subsets of V.
- $\mathcal{G}$ is k-edge-connected if each cut contains at least k hyperedges.



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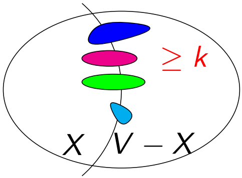
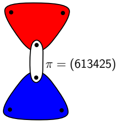


Permutation hypergraphs

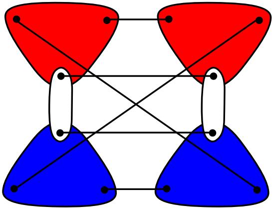
Definition

Given a hypergraph $\mathcal{G}$ on n vertices and a permutation π of $[n]$, we define the **permutation hypergraph $\mathcal{G}_\pi$** as follows :

1. we take 2 disjoint copies $\mathcal{G}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (V_2, \mathcal{E}_2)$ of $\mathcal{G}$,
2. for every vertex $v_i \in V_1$, we add an edge between v_i of $\mathcal{G}_1$ and $v_{\pi(i)}$ of $\mathcal{G}_2$, this edge set is denoted by E_3,
3. $\mathcal{G}_\pi = (V_1 \cup V_2, \mathcal{E}_1 \cup \mathcal{E}_2 \cup E_3)$.



$\mathcal{G}$



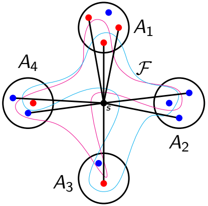
$\mathcal{G}_\pi$

Result on splitting off in hypergraphs

Theorem (Bernáth, Grappe, Szigeti)

Given : hypergraph $\mathcal{H} = (V + s, \mathcal{E})$, where s is incident only to graph edges, partition $\mathcal{P} = \{P_1, P_2\}$ of V, integer k. There exists a k-admissible $\mathcal{P}$-allowed complete splitting off at s if and only if

- *$\mathcal{H}$ is k-edge-connected in V,*
- *$d_{\mathcal{H}}(s) \geq 2\omega(\mathcal{H} - s)$,*
- *$d_{\mathcal{H}}(s, P_1) = d_{\mathcal{H}}(s, P_2)$,*
- *$\mathcal{H}$ contains no $\mathcal{C}_4$-obstacle.*



Connectivity of permutation hypergraphs

Theorem (Jami, Szigeti)

For a hypergraph $\mathcal{G}$ and an integer $k \geq 2$, there exists a permutation π such that $\lambda(\mathcal{G}_\pi) = k$ if and only if

1. $d_{\mathcal{G}}(X) \geq k - |X|$ *for all* $\emptyset \neq X \subseteq V$,
2. $\mathcal{G}$ *is not composed of two connected components both of* k *vertices,* k *being odd.*

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Remark

- Implied by Theorem of Bernáth, Grappe, Szigeti.
- Implies Theorem of Goddard, Raines, Slater :
  if $\mathcal{G}$ is a simple graph G without isolated vertices and $k = \delta(G) + 1$, then
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Thank you for your attention !