Sandwich problems on orientations

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Joint work with
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Outline

1. Definitions

2. In-degree constrained orientation

3. Sandwich problems

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   1. Graphs
   2. Functions
   3. Polyhedra
2. In-degree constrained orientation
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    1. Graphs
    2. Functions
    3. Polyhedra
2. In-degree constrained orientation
    1. Characterization
    2. Applications
    3. Algorithm
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   1. Graphs
   2. Functions
   3. Polyhedra
2. In-degree constrained orientation
   1. Characterization
   2. Applications
   3. Algorithm
3. Sandwich problems
   1. Degree constrained
   2. In-degree constrained orientation
      1. Undirected graphs
      2. Mixed graphs

Notations

- Given an undirected graph G and a set X of vertices of G,
  - $d_G(X)$ = number of edges of G entering X,
  - $i_G(X)$ = number of edges of G in X,
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- Given a directed graph D and a set X of vertices of D,
  - $d_D^-(X)$ = number of arcs of D entering X,
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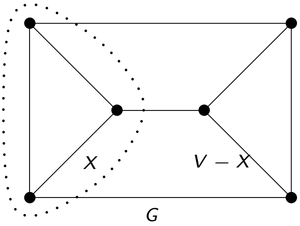

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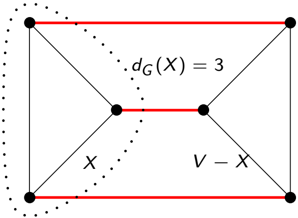

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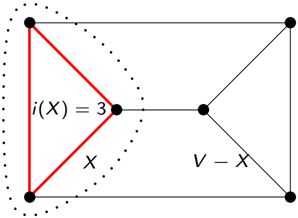

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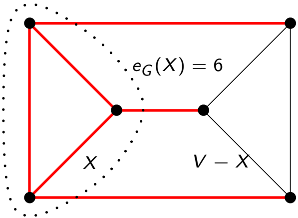

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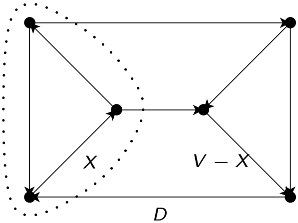

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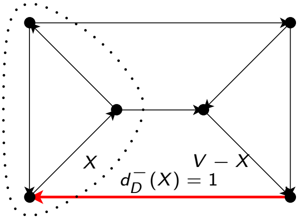

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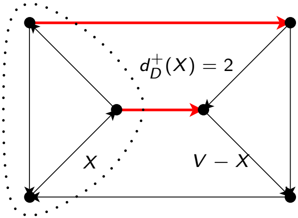

Functions

Definition

- A set function b on V is submodular if for all $X, Y \subset V$,

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y).$$

- The function b is called supermodular if $-b$ is submodular.
- The function b is called modular if b is submodular and supermodular.

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Examples

- Submodular functions :
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  - the function $m(X) = \sum_{x \in X} m(x)$.

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Matroids

Definition

A set system $M = (V, \mathcal{M})$ is called a matroid if $\mathcal{M}$ satisfies :

1. $\emptyset \in \mathcal{M}$,
2. if $F \in \mathcal{M}$ and $F' \subseteq F$, then $F' \in \mathcal{M}$,
3. if $F, F' \in \mathcal{M}$ and $|F| > |F'|$, then $\exists\ f \in F \setminus F' : F' \cup f \in \mathcal{M}$.

The rank of M is the maximum size of a set in $\mathcal{M}$.

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Examples

1. Forests of a graph,
2. Linearly independent vectors of a vector space.

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Algorithmic aspects

1. Matroid is given by an oracle that answers if $F \in \mathcal{M}$.
2. Greedy algorithm finds a set of M of maximum size,
3. more generally, given a matroid M, $F_1 \in \mathcal{M}$ and $|F_1| \leq k \leq$ rank of M, it finds $F \in \mathcal{M}$ that contains F_1 and that has size k.

Generalized Polymatroids

Definition

1. A pair (p, b) of set functions on V is a strong pair if
   - p is supermodular,
   - b is submodular,
   - they are compliant : for all $X, Y \subset V$,

$$p(X) - p(X \setminus Y) \leq b(Y) - b(Y \setminus X).$$

2. If (p, b) is a strong pair then the polyhedron

$$Q(p, b) = \{z \in \mathbb{R}^V : p(X) \leq z(X) \leq b(X)\ \forall X \subseteq V\}$$

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Remarks

1. A pair (m_1, m_2) of modular functions is a strong pair if and only if

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Generalized Polymatroid Intersection Theorem

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Theorem (Frank, Tardos '88)

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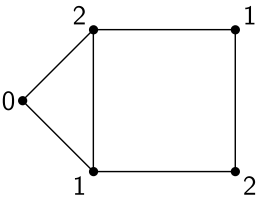
In-degree constrained orientation : Characterization

m-orientation Problem

Instance : Given a graph $G = (V, E)$ and $m : V \to \mathbb{Z}_+$.



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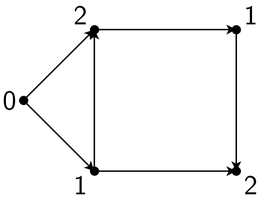

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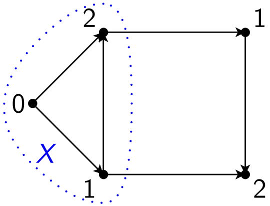
Theorem (Hakimi'65)

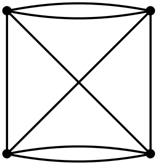
The answer is YES if and only if $m(X) \geq i_G(X)\ \forall X \subseteq V, m(V) = |E|$.



In-degree constrained orientation : Applications

Applications

- Eulerian orientation of an undirected graph (Euler),
- Eulerian orientation of a mixed graph (Ford-Fulkerson),
- Perfect matching in a bipartite graph (Hall, Frobenius),
- f-factor in a bipartite graph (Ore, Tutte).



G

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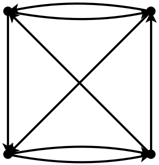

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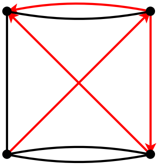

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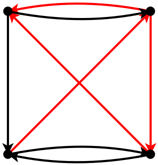

$\vec{G} = (V, \vec{E} \cup A)$

$$m(v) = \frac{d_E(v) + d_A^+(v) + d_A^-(v)}{2} - d_A^-(v) \ \forall v \in V$$

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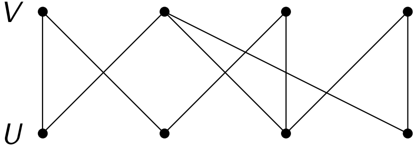

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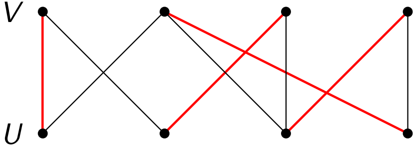

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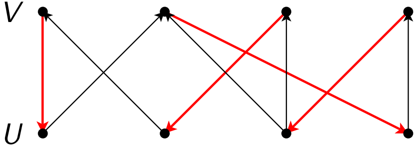

$$G = (U \cup V; E)$$
$$m(u) = 1 \qquad \forall u \in U$$
$$m(v) = d(v) - 1 \ \forall v \in V$$

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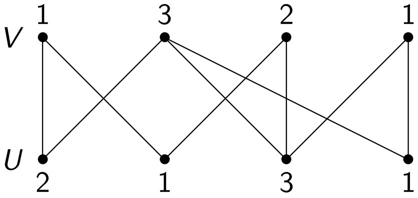

$G = (U \cup V; E), f : U \cup V \to \mathbb{Z}_+$

In-degree constrained orientation : Applications

Applications

- Eulerian orientation of an undirected graph (Euler),
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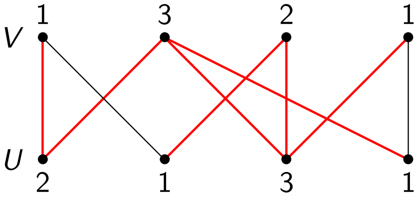

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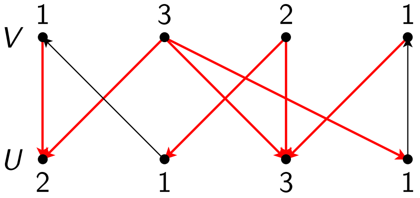

$G = (U \cup V; E)$, f-factor

$$m(u) = f(u) \qquad \forall u \in U$$
$$m(v) = d(v) - f(v) \quad \forall v \in V$$

In-degree constrained orientation : Algorithm

Algorithm 1

The in-degree constrained orientation problem is in P because it is equivalent to the f-factor problem in a bipartite graph.

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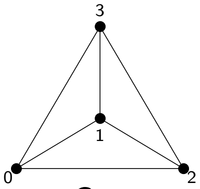

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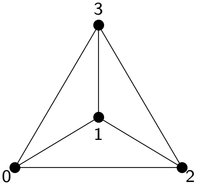

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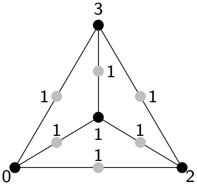

H, f

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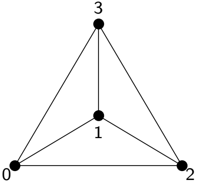

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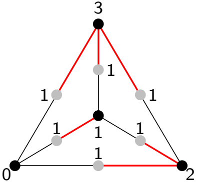

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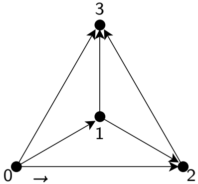
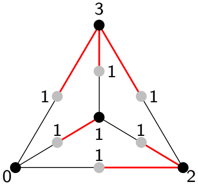


$\vec{G}, m = d^-_{\vec{G}}$



H, f, F

In-degree constrained orientation : Algorithm

Algorithm 2

1. Take an arbitrary orientation $\vec{G}$ of G.
2. If $d^-_{\vec{G}}(v) \leq m(v)\ \forall v$, then it is an m-orientation, Stop.
3. Otherwise, take a big vertex v : $d^-_{\vec{G}}(v) > m(v)$.
4. Let X be the set of vertices u from which there exists a path P_u to v.
5. Take a small vertex $u \in X$: $d^-_{\vec{G}}(u) < m(u)$.
6. Let $\vec{G}'$ be obtained from $\vec{G}$ by reorienting P_u. Go to Step 2.
7. This algorithm finds an m-orientation in polynomial time.

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   (Indeed, $|A| = \sum_{v\in V} d^-_{\vec{G}}(v) \leq \sum_{v\in V} m(v) = m(V) = |E| = |A|$.)
3. Otherwise, take a big vertex v : $d^-_{\vec{G}}(v) > m(v)$.
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5. Take a small vertex $u \in X$: $d^-_{\vec{G}}(u) < m(u)$. (It exists because $\sum_{x \in X} m(x) = m(X) \geq i_G(X) = i_G(X) + d^-_{\vec{G}}(X) = \sum_{x \in X} d^-_{\vec{G}}(x)$.)
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   (It is better : $\sum_{w \in V} |d^-_{\vec{G}'}(w) - m(w)| = \sum_{w \in V} |d^-_{\vec{G}}(w) - m(w)| - 2$.)
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   $(0 \leq \sum_{w \in V} |d^-_{\vec{G}}(w) - m(w)| \leq 2|E|.)$

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Sandwich problems

Graph Sandwich Problem for Property Π

Instance : Given graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subset E_2$.
Question : Does there exist $E_1 \subseteq E \subseteq E_2$ such that the graph $G = (V, E)$ satisfies property Π ?

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Golumbic, Kaplan, Shamir '95

- Split graphs (in P), [V=C+I]
- Cographs (in P), [no induced P_4]
- Eulerian graphs,
- Comparability graphs (NP-complete), [has a transitive orientation]
- Permutation graphs (NP-complete), [intersection graph of the chords of a permutation diagram]
- Interval graphs (NP-complete). [intersection graph of a family of intervals on the real line]

Degree Constrained Sandwich Problems

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G_1, G_2 undirected graphs, $\Pi = \{d_G(v) = m(v)\ \forall v \in V\}$ ($m : V \to \mathbb{Z}_+$).

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Remark

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Exercise

The answer is Yes if and only if $d_{D_2}^-(v) \geq m(v) \geq d_{D_1}^-(v)\ \forall v \in V$.

m-orientation Sandwich Problem 1

UNDIRECTED GRAPHS :

G_1, G_2 undirected graphs, Π $=G$ has an m-orientation ($m : V \to \mathbb{Z}_+$).

m-orientation Sandwich Problem 1

m-orientation Sandwich Problem for Undirected Graphs :

Instance : Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$ and a non-negative integer vector m on V.
Question : Does there exist a sandwich graph $G = (V, E)$ $(E_1 \subseteq E \subseteq E_2)$ that has an orientation $\vec{G}$ whose in-degree vector is m that is $d^-_{\vec{G}}(v) = m(v)\ \forall v \in V$?

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Remark

$E_1 = E_2$: equivalent to Hakimi's Theorem.

Proof

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The answer is YES if and only if $i_{E_1}(X) \leq m(X) \leq e_{E_2}(X)\ \forall X \subseteq V$.

1. Necessity : if sandwich graph G that has an m-orientation exists
   1. Each edge that contributes to $i_{E_1}(X)$ must contribute to $m(X)$ and
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   1. Let $\mathcal{M} = \{F \subseteq E_2 : m(X) \geq i_F(X)\ \forall X \subseteq V\}$.
   2. $\mathcal{M}$ is a matroid of rank $\min\{m(V(F)) + |E_2 \setminus F| : F \subseteq E_2\}$.
   3. By $i_{E_1}(X) \leq m(X)\ \forall X \subseteq V$, $E_1 \in \mathcal{M}$.
   4. For all $F \subseteq E_2$, by $m(X) \leq e_{E_2}(X)\ \forall X \subseteq V$, applied for $V \setminus V(F)$, and by 2, rank of $\mathcal{M}$ is $\geq m(V)$.
   5. By 3 and 4, there exists $E \in \mathcal{M}$ that contains E_1, of size $m(V)$.
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   5. By 3 and 4, there exists $E \in \mathcal{M}$ that contains E_1, of size $m(V)$.
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Theorem (de Gevigney, Klein, Nguyen, Szigeti 2010)

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m-orientation Sandwich Problem 2

Mixed Graphs :

G_1, G_2 mixed graphs, $\Pi = G$ has an m-orientation ($m : V \to \mathbb{Z}_+$).

m-orientation Sandwich Problem 2

m-orientation Sandwich Problem for Mixed Graphs :

Instance : Given mixed graphs $G_1 = (V, E_1 \cup A_1)$ and $G_2 = (V, E_2 \cup A_2)$ with $E_1 \subseteq E_2$, $A_1 \subseteq A_2$ and a non-negative integer vector m on V.
Question : Does there exist a sandwich mixed graph $G = (V, E \cup A)$ with $E_1 \subseteq E \subseteq E_2$ and $A_1 \subseteq A \subseteq A_2$ that has an orientation $\vec{G} = (V, \overrightarrow{E} \cup A)$ whose in-degree vector is m that is $d^-_{\vec{G}}(v) = m(v)\ \forall v \in V$?

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Special cases

1. $E_2 = \emptyset$: result on the In-degree Constrained Sandwich Problem.
2. $A_2 = \emptyset$: result on m-orient. Sandwich Problem for Undirected Graphs.

Proof

1. Suppose that $E_1 \subseteq E \subseteq E_2$ has been choosen and oriented with in-degree vector m_1.
2. Then the problem is reduced to the Dir. Degree Const. Sandw. Problem with $m_2(v) = m(v) - m_1(v)\ \forall v \in V$ for $A_1 \subseteq A_2$,
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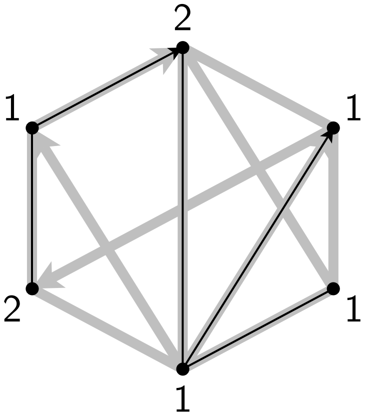
Theorem (de Gevigney, Klein, Nguyen, Szigeti 2010)

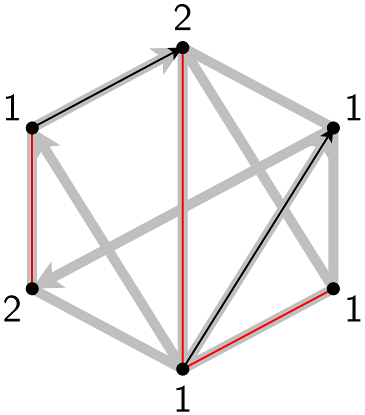
The answer is YES *if and only if*
$i_{E_1}(X) + \sum_{v \in X} d^-_{A_1}(v) \leq m(X) \leq e_{E_2}(X) + \sum_{v \in X} d^-_{A_2}(v)$ $\forall X \subseteq V$.

1. Decide : The answer is YES if and only if both submodular functions $b_1^*(X) = b_1(X) - p_2(X)$ and $b_2^*(X) = b_2(X) - p_1(X)$ have minimum value 0. Submodular function minimization is polynomial.
2. Find : $\vec{G} = (V, \overrightarrow{E} \cup A)$ whose in-degree vector is m.
   1. m_1 : $Q(p_1, b_1)$ is a box, so $R = Q(p_1, b_1) \cap Q(p_2, b_2)$ is a g-polymatroid, hence an integer vector m_1 can be found in R by greedy algorithm.
   2. $\vec{E}$: m_1-ORIENTATION SANDWICH PROBLEM FOR UNDIRECTED GRAPHS for $E_1 \subseteq E_2$,
   3. A : DIR. DEGREE CONST. SANDW. PROBLEM with $m_2 = m - m_1$ for $A_1 \subseteq A_2$.

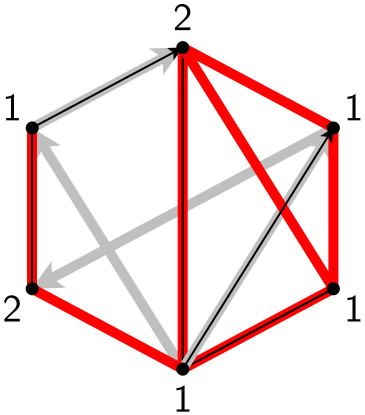
Example



Example



Example



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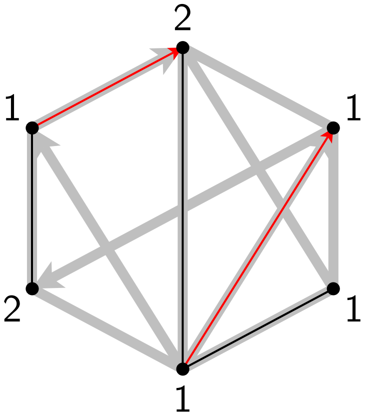
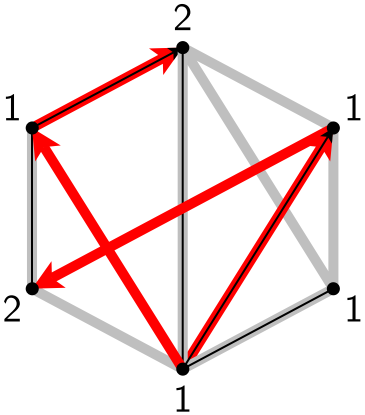
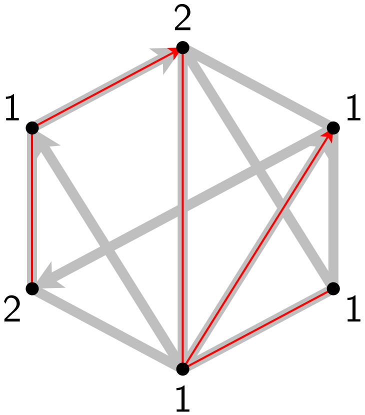


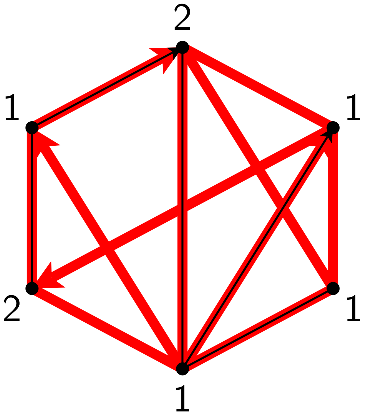
Example



Example



Example



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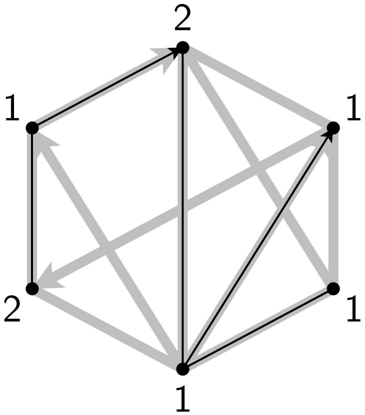

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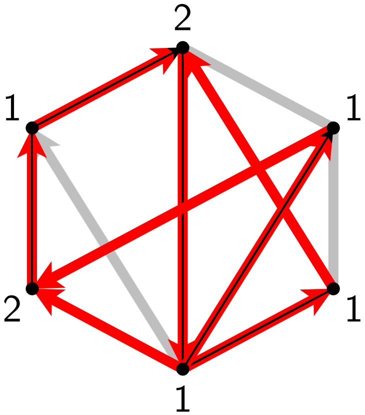

Strongly connected m-orientation Sandwich Problem

Strongly connected m-orientation Sandwich Problem :

G_1, G_2 undirected graphs, Π $=G$ has an m-orientation that is strongly connected ($m : V \to \mathbb{Z}_+$).

Strongly connected m-orientation Sandwich Problem

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Remark

It is NP-complete.

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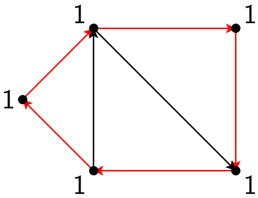
Remark

It is NP-complete. The special case $E_1 = \emptyset, m(v) = 1\ \forall v \in V$ is equivalent to decide if G_2 has a Hamiltonian cycle.



Thank you for your attention !