Edge-connectivity augmentation of graphs over symmetric parity families

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Outline

1. Edge-connectivity
2. T-cuts
3. Symmetric parity families

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1. Edge-connectivity
    1. Definitions
    2. Cut equivalent trees
    3. Edge-connectivity augmentation
2. T-cuts
3. Symmetric parity families

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   1. Definitions
   2. Cut equivalent trees
   3. Edge-connectivity augmentation
2. T-cuts
   1. Definitions
   2. Minimum T-cut
   3. Augmentation of minimum T-cut
3. Symmetric parity families

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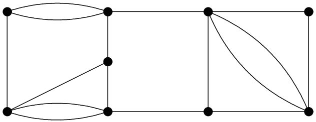
1. Edge-connectivity
    1. Definitions
    2. Cut equivalent trees
    3. Edge-connectivity augmentation
2. T-cuts
    1. Definitions
    2. Minimum T-cut
    3. Augmentation of minimum T-cut
3. Symmetric parity families
    1. Definition, Examples
    2. Minimum cut over a symmetric parity family
    3. Augmentation of minimum cut over a symmetric parity family

Definitions

Global edge-connectivity

Given a graph $G = (V, E)$ and an integer k, G is called k-edge-connected if each cut contains at least k edges.



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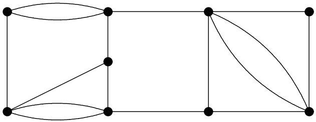
Given a graph $G = (V, E)$ and an integer k, G is called *k-edge-connected* if each cut contains at least k edges.

Local edge-connectivity

Given a graph $G = (V, E)$ and $u, v \in V$, the local edge-connectivity $\lambda_G(u, v)$ is defined as the minimum cardinality of a cut separating u and v.



Cut equivalent tree

Theorem (Gomory-Hu)

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and a weight function $c : E' \to \mathbb{Z}$ such that for all $u, v \in V$

1. *the local edge-connectivity $\lambda_G(u, v)$ is equal to the minimum value $c(e)$ of the edges e of the unique (u, v)-path in H,*
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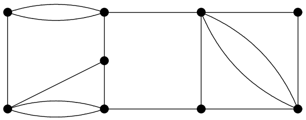
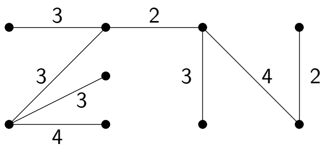


Graph $G = (V, E)$



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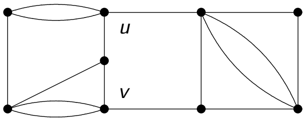

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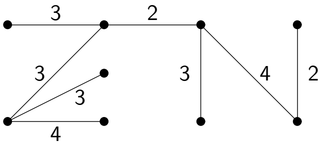

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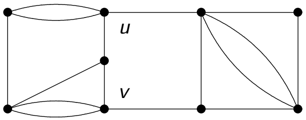

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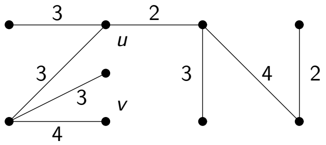

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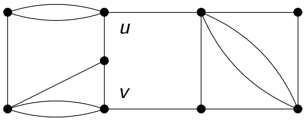

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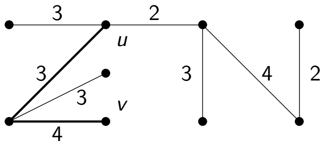

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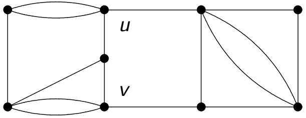

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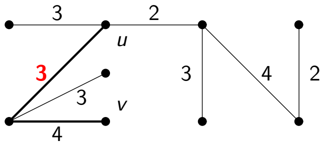

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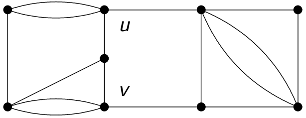

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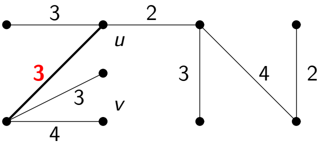

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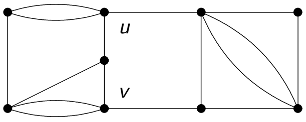

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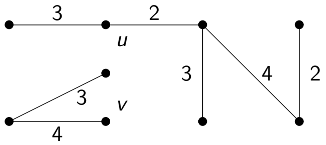

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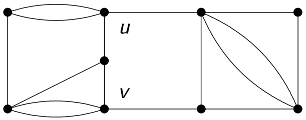

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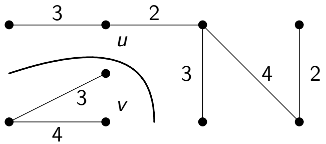

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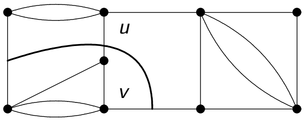

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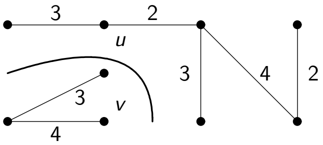

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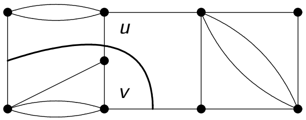

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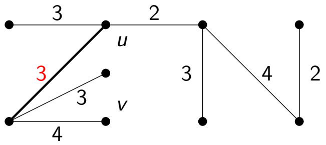

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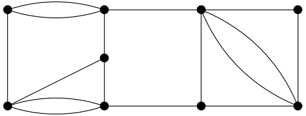

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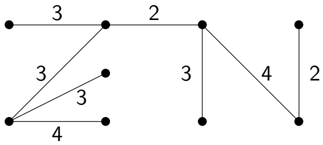


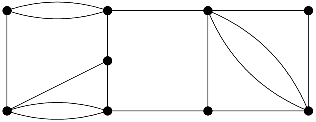
Cut equivalent tree $H = (V, E')$

Edge-Connectivity Augmentation

Global edge-connectivity augmentation of a graph

Given a graph $G = (V, E)$ and an integer $k \geq 2$, what is the minimum number of new edges whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Watanabe, Nakamura)
2. Polynomially solvable (Cai, Sun)



Graph $G, k = 4$

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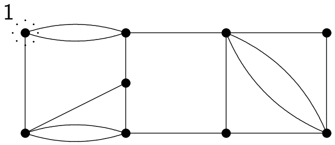

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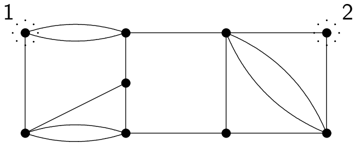

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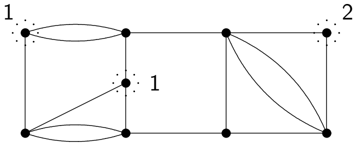

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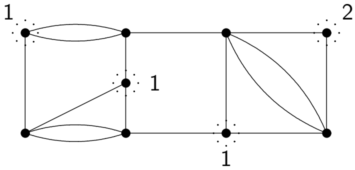

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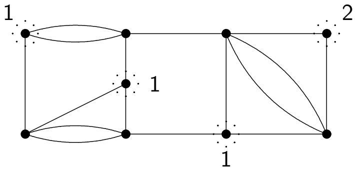

$$\text{Opt} \geq \left\lceil \frac{5}{2} \right\rceil = 3$$

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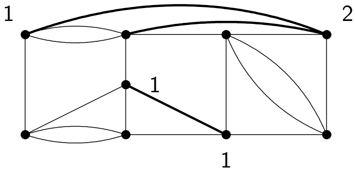

Graph $G + F$ is 4-edge-connected and $|F| = 3$

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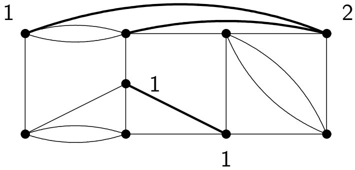

$$\text{Opt} = \left\lceil \frac{1}{2} \text{maximum deficiency of a subpartition of } V \right\rceil$$

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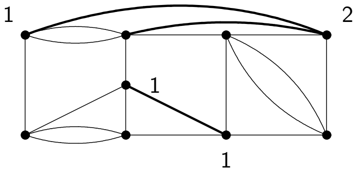
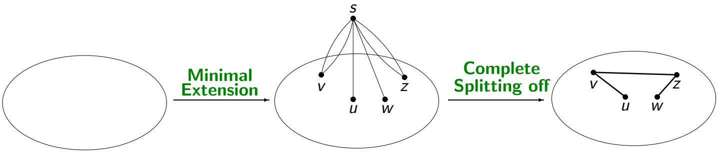


General method

Frank's algorithm

1. Minimal extension,
   (i) Add a new vertex s,
   (ii) Add a minimum number of new edges incident to s to satisfy the edge-connectivity requirements,
   (iii) If the degree of s is odd, then add an arbitrary edge incident to s.
2. Complete splitting off.



$G = (V, E)$ $\qquad$ G' k-e-c in V $\qquad$ G'' k-e-c

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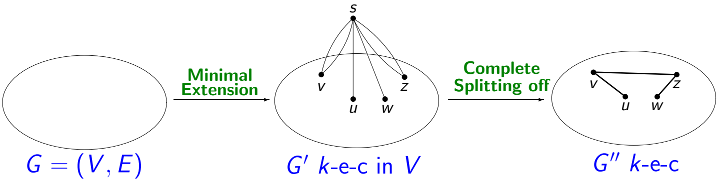

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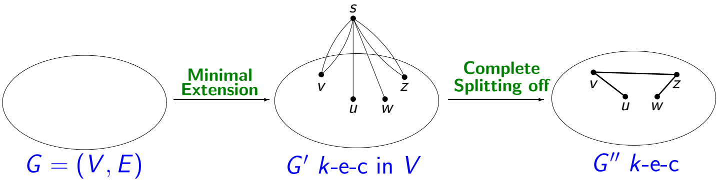

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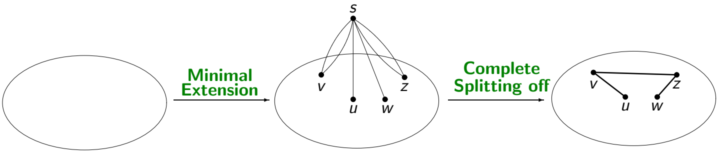

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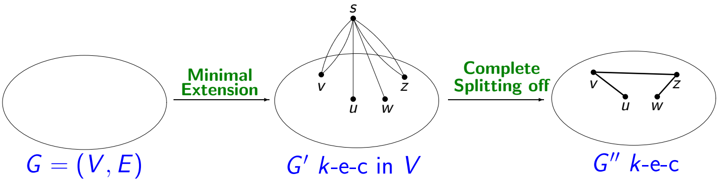

Minimal extension

Definition

1. A function p on 2^V is called skew-supermodular if at least one of following inequalities hold for all $X, Y \subseteq V$:
$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y),$$
$$p(X) + p(Y) \leq p(X - Y) + p(Y - X).$$
2. A graph H covers a function p on 2^V if each cut $\delta_H(X)$ contains at least $p(X)$ edges.

Minimal extension

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1. A function p on 2^V is called skew-supermodular if at least one of following inequalities hold for all $X, Y \subseteq V$:
$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y),$$
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Theorem (Frank)

Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew-supermodular function.

1. *The minimum number of edges in an extension of the edgeless graph on V covering p equals the maximum p-value of a subpartition of V.*
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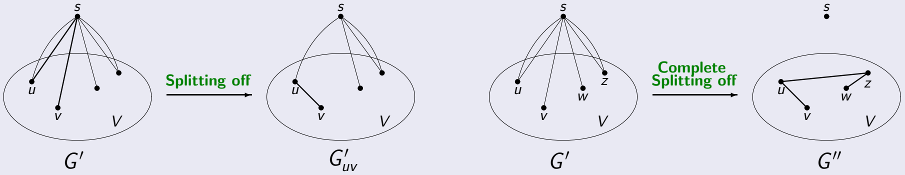

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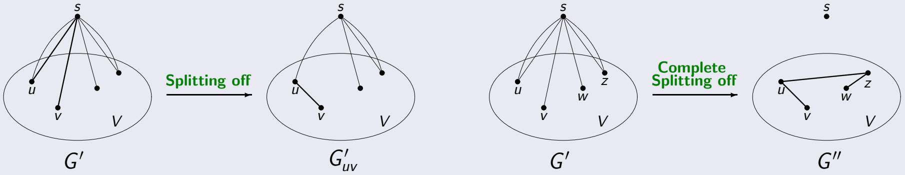

Theorem (Mader)

Let $G' = (V + s, E)$ be a graph so that $d(s)$ is even and no cut edge is incident to s.

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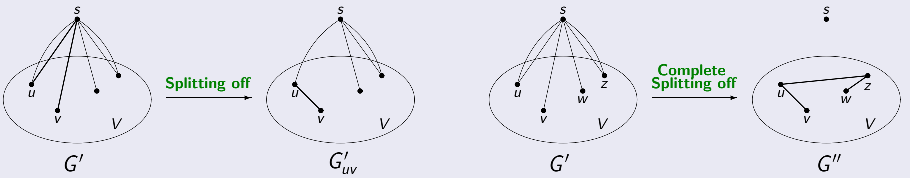

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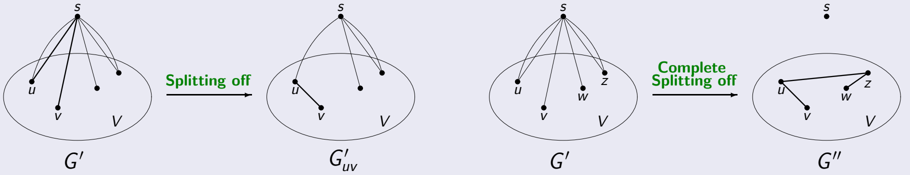

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Positive Results

Global edge-connnectivity augmentation of a graph

1. Extension works (Frank),
   $p(X) = k - d_G(X)$ is skew-supermodular,
2. Splitting off works (Mader),
3. proving min-max theorem of Watanabe, Nakamura.

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Given a graph $G = (V, E)$ and a symmetric function $r : V \times V \to \mathbb{Z}_+$, what is the minimum number of new edges F such that

$$\lambda_{G+F}(u, v) \geq r(u, v) \ \forall (u, v) \in V \times V?$$

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Instance : $p : 2^V \to \mathbb{Z}$ symmetric skew-supermodular, $\gamma \in \mathbb{Z}^+$.
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Theorem (Z. Király, Z. Nutov)

The above problem is NP-complete.

T-cut, T-join

Definitions

Given a connected graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even.

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   (a) $T = \{u, v\}$: a (u, v)-path is a T-join.

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Theorem (Edmonds-Johnson)

A minimum T-join of G can be found in polynomial time using

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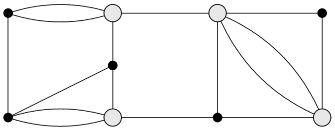

Graph G and vertex set T

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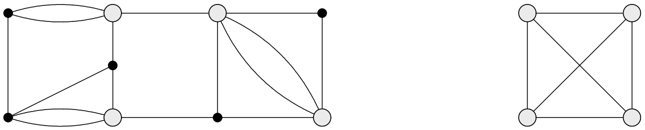

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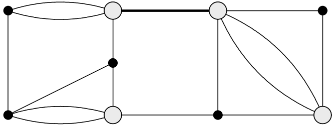

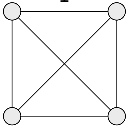

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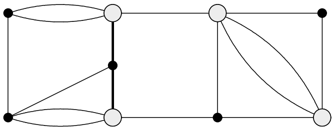

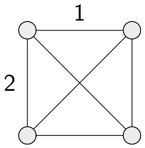

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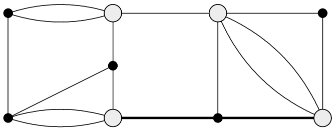

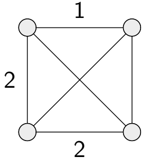

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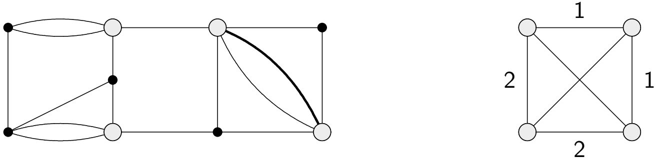

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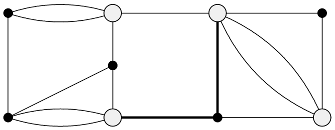

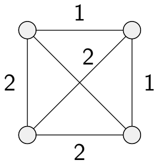

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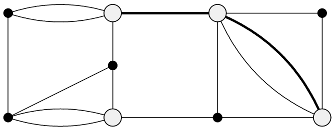

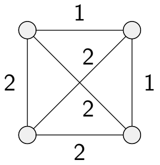

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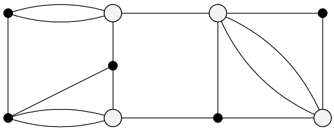

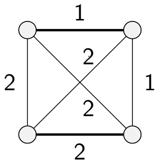

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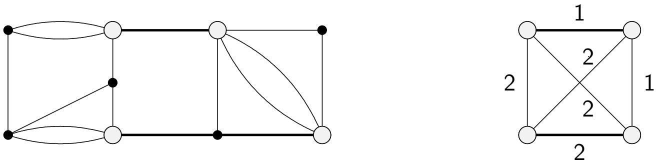

Graph G and minimum T-join

How to find a minimum T-cut ?

Theorem (Padberg-Rao)

A minimum T-cut of G can be found in polynomial time

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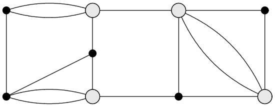

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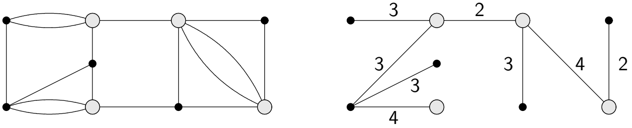

Graph G and vertex set T Cut equivalent tree H

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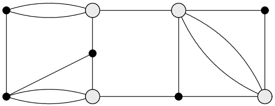

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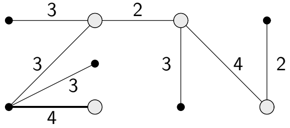

Cut equivalent tree H

How to find a minimum T-cut ?

Theorem (Padberg-Rao)

A minimum T-cut of G can be found in polynomial time

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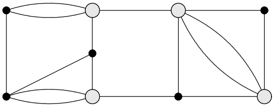

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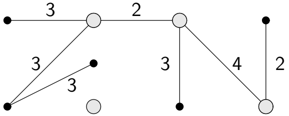

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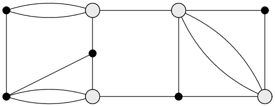

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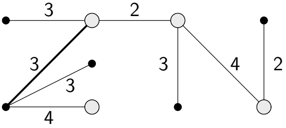

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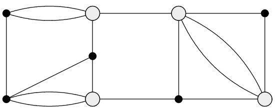

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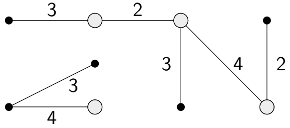

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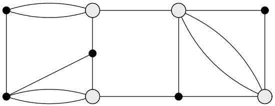

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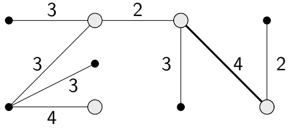

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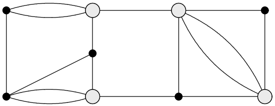

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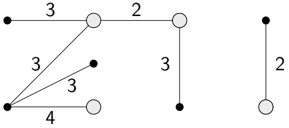

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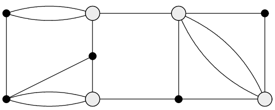

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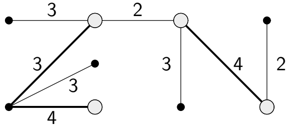

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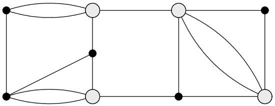

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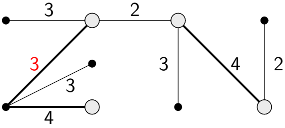

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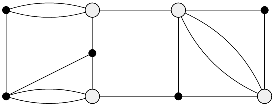

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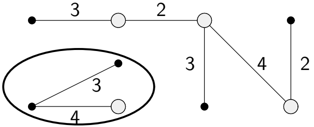

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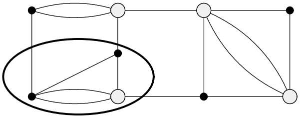
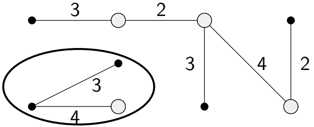


Minimum T-cut in G



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Correctness of Padberg-Rao's algorithm

Let $\delta(X)$ be a minimum T-cut and $\delta(Y)$ the T-cut defined by e^*.
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Theorem (Szigeti)

Given a connected graph $G=(V,E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each T-cut is of size at least k is equal to $\lceil \frac{1}{2}$ maximum p'-value of a subpartition of $V\rceil$.
An optimal augmentation can be found in polynomial time using

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Proof

1. works because $p'(X) = k - d_G(X)$ if X is T-odd and $-\infty$ otherwise is symmetric skew-supermodular

   (i) $k - d_G(X)$ satisfies both inequalities,
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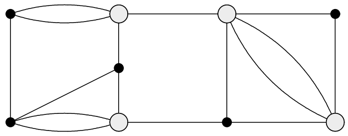

Graph G, vertex set T and $k = 4$

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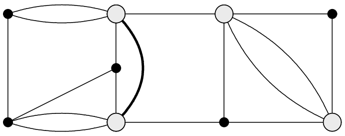

Minimum T-cut in $G+F$ is 4

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A family $\mathcal{F}$ of subsets of V is called symmetric parity family if

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Examples

1. $\mathcal{F} := 2^V - \{\emptyset, V\}$
2. $\mathcal{F} := \{X \subset V : X \text{ is } T\text{-odd}\}$ where $T \subseteq V$ with $|T|$ even.

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How to find a minimum $\mathcal{F}$-cut ?

Theorem (Goemans-Ramakrishnan)

Given a connected graph G and a symmetric parity family $\mathcal{F}$, a minimum cut of G over $\mathcal{F}$, (a minimum $\mathcal{F}$-cut) can be found in polynomial time

1. *using a cut equivalent tree H of G,*
2. *taking the set $J(H)$ edges e of H for which the two connected components of $H - e$ are in $\mathcal{F}$,*
3. *taking the minimum value $c(e^*)$ of an edge of $J(H)$,*
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Lemma

For any $A \in \mathcal{F}$ there exists an edge $xy \in \delta_{J(H)}(A)$, and hence $\lambda_G(x,y) = c(xy) \geq c(e^*)$.

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1. Let $H[A_1], \ldots, H[A_k]$ be the connected components of $H[A]$.
2. Since $A \in \mathcal{F}$ and $\bigcup A_i = A$, $\exists\ i : A_i \in \mathcal{F}$ by (iii).
3. Let $H[B_1], \ldots, H[B_l]$ be the connected components of $H - A_i$.
4. Since $V - A_i \in \mathcal{F}$ by (ii) and $\bigcup B_j = V - A_i$, $\exists\ j : B_j \in \mathcal{F}$ by (iii).
5. H is a tree, $H[A_i]$ is connected, $H[B_j]$ is a connected component of $H - A_i$, so there exists exactly one edge $e \in H$ between A_i and B_j.
6. Then $e \in J(H)$ and e enters A.

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Correctness of Goemans-Ramakrishnan's algorithm

The same proof works as for Padberg-Rao's algorithm.

How to augment a minimum $\mathcal{F}$-cut ?

Theorem (Szigeti)

Given a connected graph G, a symmetric parity family $\mathcal{F}$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $\mathcal{F}$-cut is of size at least k equals $\lceil \frac{1}{2}$ maximum p^*-value of a subpartition of $V \rceil$. An optimal augmentation can be found in polynomial time using

1. *Frank's minimal extension and*
2. *Mader's complete splitting off.*

Proof

1. works because $p^*(X) = k - d_G(X)$ if $X \in \mathcal{F}$ and $-\infty$ otherwise is symmetric skew-supermodular
   (i) $k - d_G(X)$ satisfies both inequalities,
   (ii) If $X, Y \in \mathcal{F}$, then either $X \cap Y, X \cup Y \in \mathcal{F}$ or $X - Y, Y - X \in \mathcal{F}$.
2. works because for all $X \in \mathcal{F}$, $d_{G'}(X) \geq k$ and, by the above lemma, $k \leq \lambda_{G'}(x, y) = \lambda_{G''}(x, y) \leq d_{G''}(X)$.

Conclusion

1. Special cases :
   1. Global edge-connectivity augmentation (Watanabe, Nakamura)
   2. Minimum T-cut augmentation
2. A new polynomial special case of the NP-complete problem MINIMUM COVER OF A SYMMETRIC SKEW-SUPERMODULAR FUNCTION BY A GRAPH

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