Edge-connectivity augmentation of graphs over symmetric parity families

Zoltán Szigeti

Laboratoire G-SCOP
INP Grenoble, France

27 octobre 2010
Outline

1. Edge-connectivity
2. $T$-cuts
3. Symmetric parity families
Outline

1. Edge-connectivity
   1. Definitions
   2. Cut equivalent trees
   3. Edge-connectivity augmentation

2. $T$-cuts

3. Symmetric parity families
Outline

Edge-connectivity
1. Definitions
2. Cut equivalent trees
3. Edge-connectivity augmentation

$T$-cuts
1. Definitions
2. Minimum $T$-cut
3. Augmentation of minimum $T$-cut

Symmetric parity families
Outline

1. Edge-connectivity
   1. Definitions
   2. Cut equivalent trees
   3. Edge-connectivity augmentation

2. $T$-cuts
   1. Definitions
   2. Minimum $T$-cut
   3. Augmentation of minimum $T$-cut

3. Symmetric parity families
   1. Definition, Examples
   2. Minimum cut over a symmetric parity family
   3. Augmentation of minimum cut over a symmetric parity family
Global edge-connectivity

Given a graph $G = (V, E)$ and an integer $k$, $G$ is called $k$-edge-connected if each cut contains at least $k$ edges.
Definitions

Global edge-connectivity

Given a graph $G = (V, E)$ and an integer $k$, $G$ is called $k$-edge-connected if each cut contains at least $k$ edges.

Local edge-connectivity

Given a graph $G = (V, E)$ and $u, v \in V$, the local edge-connectivity $\lambda_G(u, v)$ is defined as the minimum cardinality of a cut separating $u$ and $v$. 

![Diagram of a graph illustrating global and local edge-connectivity.](null)
Theorem (Gomory-Hu)

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and a weight function $c : E' \rightarrow \mathbb{Z}$ such that for all $u, v \in V$

1. the local edge-connectivity $\lambda_G(u, v)$ is equal to the minimum value $c(e)$ of the edges $e$ of the unique $(u, v)$-path in $H$,

2. if $e$ achieves this minimum, then the fundamental cut of $H - e$ provides a minimum cut of $G$ separating $u$ and $v$. 
Theorem (Gomory-Hu)

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and a weight function $c : E' \rightarrow \mathbb{Z}$ such that for all $u, v \in V$

1. the local edge-connectivity $\lambda_G(u, v)$ is equal to the minimum value $c(e)$ of the edges $e$ of the unique $(u, v)$-path in $H$,

2. if $e$ achieves this minimum, then the fundamental cut of $H - e$ provides a minimum cut of $G$ separating $u$ and $v$. 

Graph $G = (V, E)$

Cut equivalent tree $H = (V, E')$
Theorem (Gomory-Hu)

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and a weight function $c : E' \to \mathbb{Z}$ such that for all $u, v \in V$

1. the local edge-connectivity $\lambda_G(u, v)$ is equal to the minimum value $c(e)$ of the edges $e$ of the unique $(u, v)$-path in $H$,

2. if each achieves this minimum, then the fundamental cut of $H - e$ provides a minimum cut of $G$ separating $u$ and $v$. 

\[ \begin{align*}
\text{Graph } G &= (V, E) \\
\text{Cut equivalent tree } H &= (V, E')
\end{align*} \]
**Theorem (Gomory-Hu)**

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and a weight function $c : E' \to \mathbb{Z}$ such that for all $u, v \in V$

1. the local edge-connectivity $\lambda_G(u, v)$ is equal to the minimum value $c(e)$ of the edges $e$ of the unique $(u, v)$-path in $H$,

2. if $e$ achieves this minimum, then the fundamental cut of $H - e$ provides a minimum cut of $G$ separating $u$ and $v$. 

Graph $G = (V, E)$

Cut equivalent tree $H = (V, E')$
Theorem (Gomory-Hu)

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and a weight function $c : E' \rightarrow \mathbb{Z}$ such that for all $u, v \in V$

1. the local edge-connectivity $\lambda_G(u, v)$ is equal to the minimum value $c(e)$ of the edges $e$ of the unique $(u, v)$-path in $H$,

2. if $e$ achieves this minimum, then the fundamental cut of $H - e$ provides a minimum cut of $G$ separating $u$ and $v$. 

Graph $G = (V, E)$

Cut equivalent tree $H = (V, E')$
Theorem (Gomory-Hu)

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and a weight function $c : E' \rightarrow \mathbb{Z}$ such that for all $u, v \in V$

1. the local edge-connectivity $\lambda^G_{uv}$ is equal to the minimum value $c(e)$ of the edges $e$ of the unique $(u, v)$-path in $H$,

2. if $e$ achieves this minimum, then the fundamental cut of $H - e$ provides a minimum cut of $G$ separating $u$ and $v$. 

Graph $G = (V, E)$

Cut equivalent tree $H = (V, E')$
Theorem (Gomory-Hu)

For every graph \( G = (V, E) \), we can find, in \textit{polynomial} time, a \textit{tree} \( H = (V, E') \) and a \textit{weight function} \( c : E' \rightarrow \mathbb{Z} \) such that for all \( u, v \in V \):

1. the local edge-connectivity \( \lambda_G(u, v) \) is equal to the minimum value \( c(e) \) of the edges \( e \) of the unique \((u, v)\)-path in \( H \),
2. if \( e \) achieves this minimum, then the \textit{fundamental cut} of \( H - e \) provides a \textit{minimum cut} of \( G \) separating \( u \) and \( v \).
Theorem (Gomory-Hu)

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and a weight function $c : E' \to \mathbb{Z}$ such that for all $u, v \in V$

1. the local edge-connectivity $\lambda_G(u, v)$ is equal to the minimum value $c(e)$ of the edges $e$ of the unique $(u, v)$-path in $H$,

2. if $e$ achieves this minimum, then the fundamental cut of $H - e$ provides a minimum cut of $G$ separating $u$ and $v$. 

Graph $G = (V, E)$

Cut equivalent tree $H = (V, E')$
Theorem (Gomory-Hu)

For every graph \( G = (V, E) \), we can find, in polynomial time, a tree \( H = (V, E') \) and a weight function \( c : E' \rightarrow \mathbb{Z} \) such that for all \( u, v \in V \):

1. the local edge-connectivity \( \lambda_G(u, v) \) is equal to the minimum value \( c(e) \) of the edges \( e \) of the unique \((u, v)\)-path in \( H \),
2. if \( e \) achieves this minimum, then the fundamental cut of \( H - e \) provides a minimum cut of \( G \) separating \( u \) and \( v \).
Theorem (Gomory-Hu)

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and a weight function $c : E' \to \mathbb{Z}$ such that for all $u, v \in V$

1. the local edge-connectivity $\lambda_G(u, v)$ is equal to the minimum value $c(e)$ of the edges $e$ of the unique $(u, v)$-path in $H$,

2. if $e$ achieves this minimum, then the fundamental cut of $H - e$ provides a minimum cut of $G$ separating $u$ and $v$. 

Graph $G = (V, E)$

Cut equivalent tree $H = (V, E')$
Theorem (Gomory-Hu)

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and a weight function $c : E' \rightarrow \mathbb{Z}$ such that for all $u, v \in V$

1. the local edge-connectivity $\lambda_G(u, v)$ is equal to the minimum value $c(e)$ of the edges $e$ of the unique $(u, v)$-path in $H$,

2. if $e$ achieves this minimum, then the fundamental cut of $H - e$ provides a minimum cut of $G$ separating $u$ and $v$. 

Graph $G = (V, E)$

Cut equivalent tree $H = (V, E')$
Theorem (Gomory-Hu)

For every graph $G = (V, E)$, we can find, in polynomial time, a tree $H = (V, E')$ and a weight function $c : E' \rightarrow \mathbb{Z}$ such that for all $u, v \in V$

1. the local edge-connectivity $\lambda_G(u, v)$ is equal to the minimum value $c(e)$ of the edges $e$ of the unique $(u, v)$-path in $H$,

2. if $e$ achieves this minimum, then the fundamental cut of $H - e$ provides a minimum cut of $G$ separating $u$ and $v$. 

Graph $G = (V, E)$

Cut equivalent tree $H = (V, E')$
Global edge-connectivity augmentation of a graph

Given a graph $G = (V, E)$ and an integer $k \geq 2$, what is the minimum number of new edges whose addition results in a $k$-edge-connected graph?

1. Minimax theorem (Watanabe, Nakamura)
2. Polynomially solvable (Cai, Sun)

Graph $G, k = 4$
Global edge-connectivity augmentation of a graph

Given a graph $G = (V, E)$ and an integer $k \geq 2$, what is the minimum number of new edges whose addition results in a $k$-edge-connected graph?

1. Minimax theorem (Watanabe, Nakamura)
2. Polynomially solvable (Cai, Sun)

Graph $G, k = 4$
Global edge-connectivity augmentation of a graph

Given a graph $G = (V, E)$ and an integer $k \geq 2$, what is the minimum number of new edges whose addition results in a $k$-edge-connected graph?

- Minimax theorem (Watanabe, Nakamura)
- Polynomially solvable (Cai, Sun)

Graph $G, k = 4$
Global edge-connectivity augmentation of a graph

Given a graph $G = (V, E)$ and an integer $k \geq 2$, what is the minimum number of new edges whose addition results in a $k$-edge-connected graph?

1. Minimax theorem (Watanabe, Nakamura)
2. Polynomially solvable (Cai, Sun)

Graph $G$, $k = 4$
Given a graph $G = (V, E)$ and an integer $k \geq 2$, what is the minimum number of new edges whose addition results in a $k$-edge-connected graph?

1. Minimax theorem (Watanabe, Nakamura)
2. Polynomially solvable (Cai, Sun)

Graph $G, k = 4$
Global edge-connectivity augmentation of a graph

Given a graph $G = (V, E)$ and an integer $k \geq 2$, what is the minimum number of new edges whose addition results in a $k$-edge-connected graph?

1. Minimax theorem (Watanabe, Nakamura)
2. Polynomially solvable (Cai, Sun)

$$\text{Opt} \geq \left\lceil \frac{5}{2} \right\rceil = 3$$
Global edge-connectivity augmentation of a graph

Given a graph $G = (V, E)$ and an integer $k \geq 2$, what is the minimum number of new edges whose addition results in a $k$-edge-connected graph?

1. Minimax theorem (Watanabe, Nakamura)
2. Polynomials solvable (Cai, Sun)

Graph $G + F$ is 4-edge-connected and $|F| = 3$
Global edge-connectivity augmentation of a graph

Given a graph $G = (V, E)$ and an integer $k \geq 2$, what is the minimum number of new edges whose addition results in a $k$-edge-connected graph?

1. Minimax theorem (Watanabe, Nakamura)
2. Polynomially solvable (Cai, Sun)

$$\text{Opt} = \lceil \frac{1}{2} \text{maximum deficiency of a subpartition of } V \rceil$$
Global edge-connectivity augmentation of a graph

Given a graph $G = (V, E)$ and an integer $k \geq 2$, what is the minimum number of new edges whose addition results in a $k$-edge-connected graph?

1. Minimax theorem (Watanabe, Nakamura)
2. Polynomially solvable (Cai, Sun)
Frank’s algorithm

1. Minimal extension,
   (i) Add a new vertex \( s \),
   (ii) Add a minimum number of new edges incident to \( s \) to satisfy the edge-connectivity requirements,
   (iii) If the degree of \( s \) is odd, then add an arbitrary edge incident to \( s \).

2. Complete splitting off.

\[
G = (V, E) \quad \text{Minimal Extension} \quad G' \ k\text{-e-c in } V \quad \text{Complete Splitting off} \quad G'' \ k\text{-e-c}
\]
Frank’s algorithm

1. **Minimal extension,**
   - (i) Add a new vertex $s$,
   - (ii) Add a minimum number of new edges incident to $s$ to satisfy the edge-connectivity requirements,
   - (iii) If the degree of $s$ is odd, then add an arbitrary edge incident to $s$.

2. **Complete splitting off.**

---

$G = (V, E)$

$G' \ k$-e-c in $V$

$G'' \ k$-e-c
General method

Frank’s algorithm

1. Minimal extension,
   (i) Add a new vertex $s$,
   (ii) Add a minimum number of new edges incident to $s$ to satisfy the edge-connectivity requirements,
   (iii) If the degree of $s$ is odd, then add an arbitrary edge incident to $s$.

2. Complete splitting off.

\[ G = (V, E) \]  \hspace{1cm} \text{Minimal Extension} \hspace{1cm} G' \text{ k-e-c in } V \hspace{1cm} \text{Complete Splitting off} \hspace{1cm} G'' \text{ k-e-c} \]
General method

Frank’s algorithm

1. Minimal extension,
   (i) Add a new vertex $s$,
   (ii) Add a minimum number of new edges incident to $s$ to satisfy the edge-connectivity requirements,
   (iii) If the degree of $s$ is odd, then add an arbitrary edge incident to $s$.

2. Complete splitting off.

$G = (V, E)$

$G' \text{ k-e-c in } V$

$G'' \text{ k-e-c}$
General method

Frank’s algorithm

1. Minimal extension,
   (i) Add a new vertex $s$,
   (ii) Add a minimum number of new edges incident to $s$ to satisfy the edge-connectivity requirements,
   (iii) If the degree of $s$ is odd, then add an arbitrary edge incident to $s$.

2. Complete splitting off.

$G = (V, E)$

$G' \ k$-e-c in $V$

$G'' \ k$-e-c
A function $p$ on $2^V$ is called **skew-supermodular** if at least one of the following inequalities hold for all $X, Y \subseteq V$:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y),$$

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X).$$

A graph $H$ covers a function $p$ on $2^V$ if each cut $\delta_H(X)$ contains at least $p(X)$ edges.
A function $p$ on $2^V$ is called **skew-supermodular** if at least one of the following inequalities hold for all $X, Y \subseteq V$:

\begin{align*}
p(X) + p(Y) &\leq p(X \cap Y) + p(X \cup Y), \\
p(X) + p(Y) &\leq p(X - Y) + p(Y - X).
\end{align*}

A graph $H$ covers a function $p$ on $2^V$ if each cut $\delta_H(X)$ contains at least $p(X)$ edges.
Minimal extension

Definition

1. A function \( p \) on \( 2^V \) is called **skew-supermodular** if at least one of following inequalities hold for all \( X, Y \subseteq V \):
   \[
   p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y),
   p(X) + p(Y) \leq p(X - Y) + p(Y - X).
   \]

2. A graph \( H \) **covers** a function \( p \) on \( 2^V \) if each cut \( \delta_H(X) \) contains at least \( p(X) \) edges.

Theorem (Frank)

Let \( p : 2^V \to \mathbb{Z} \cup \{-\infty\} \) be a symmetric skew-supermodular function.

1. The minimum number of edges in an extension of the edgeless graph on \( V \) covering \( p \) equals the maximum \( p \)-value of a subpartition of \( V \).

2. An optimal extension can be found in polynomial time in the special cases mentioned in this talk.
Definition

1. A function $p$ on $2^V$ is called **skew-supermodular** if at least one of the following inequalities hold for all $X, Y \subseteq V$:
   
   \[
   p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y),
   \]
   
   \[
   p(X) + p(Y) \leq p(X - Y) + p(Y - X).
   \]

2. A graph $H$ **covers** a function $p$ on $2^V$ if each cut $\delta_H(X)$ contains at least $p(X)$ edges.

Theorem (Frank)

Let $p : 2^V \rightarrow \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew-supermodular function.

1. The minimum number of edges in an **extension** of the edgeless graph on $V$ covering $p$ equals the maximum $p$-value of a subpartition of $V$.

2. An optimal extension can be found in polynomial time in the special cases mentioned in this talk.
A function $p$ on $2^V$ is called **skew-supermodular** if at least one of the following inequalities hold for all $X, Y \subseteq V$:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y),$$

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X).$$

A graph $H$ covers a function $p$ on $2^V$ if each cut $\delta_H(X)$ contains at least $p(X)$ edges.

**Theorem (Frank)**

Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew-supermodular function.

1. The minimum number of edges in an extension of the edgeless graph on $V$ covering $p$ equals the maximum $p$-value of a subpartition of $V$.
2. An optimal extension can be found in polynomial time in the special cases mentioned in this talk.
Complete splitting off

Definitions

\[ G' \quad \xrightarrow{\text{Splitting off}} \quad G'_{uv} \quad \xrightarrow{\text{Complete splitting off}} \quad G'' \]
Theorem (Mader)

Let $G' = (V + s, E)$ be a graph so that $d(s)$ is even and no cut edge is incident to $s$.

1. Then there exists a complete splitting off at $s$ that preserves the local edge-connectivity between all pairs of vertices in $V$.
2. Such a complete splitting off can be found in polynomial time.
**Theorem (Mader)**

Let $G' = (V + s, E)$ be a graph so that $d(s)$ is even and no cut edge is incident to $s$.

1. Then there exists a **complete splitting off** at $s$ that preserves the local edge-connectivity between all pairs of vertices in $V$.

2. Such a complete splitting off can be found in polynomial time.
Theorem (Mader)

Let $G' = (V + s, E)$ be a graph so that $d(s)$ is even and no cut edge is incident to $s$.

1. Then there exists a complete splitting off at $s$ that preserves the local edge-connectivity between all pairs of vertices in $V$.

2. Such a complete splitting off can be found in polynomial time.
Positive Results

Global edge-connectivity augmentation of a graph

1. Extension works (Frank), $p(X) = k - d_G(X)$ is skew-supermodular,
2. Splitting off works (Mader),
3. proving min-max theorem of Watanabe, Nakamura.
Positive Results

Global edge-connectivity augmentation of a graph

1. Extension works (Frank),
   \[ p(X) = k - d_G(X) \] is skew-supermodular,

2. Splitting off works (Mader),

3. proving min-max theorem of Watanabe, Nakamura.
Global edge-connectivity augmentation of a graph

1. Extension works (Frank),
   \( p(X) = k - d_G(X) \) is skew-supermodular,
2. Splitting off works (Mader),
3. proving min-max theorem of Watanabe, Nakamura.
### Positive Results

#### Global edge-connectivity augmentation of a graph

1. Extension works (Frank),
   
   \[ p(X) = k - d_G(X) \]
   
   is skew-supermodular,

2. Splitting off works (Mader),

3. proving min-max theorem of Watanabe, Nakamura.

#### Local edge-connectivity augmentation of a graph

Given a graph \( G = (V, E) \) and a symmetric function \( r : V \times V \rightarrow \mathbb{Z}_+ \), what is the minimum number of new edges \( F \) such that

\[
\lambda_{G+F}(u, v) \geq r(u, v) \quad \forall (u, v) \in V \times V?
\]

1. Extension works (Frank),

   \[ p(X) = \max \{ r(u, v) : u \in X, v \notin X \} - d_G(X) \]
   
   is skew-supermodular,

2. Splitting off works (Mader),

3. proving min-max theorem of Frank.
## Positive Results

### Global edge-connectivity augmentation of a graph

1. Extension works (Frank),
   
   \[ p(X) = k - d_G(X) \] is skew-supermodular,

2. Splitting off works (Mader),

3. proving min-max theorem of Watanabe, Nakamura.

### Local edge-connectivity augmentation of a graph

Given a graph \( G = (V, E) \) and a symmetric function \( r : V \times V \rightarrow \mathbb{Z}_+ \), what is the minimum number of new edges \( F \) such that

\[
\lambda_{G+F}(u, v) \geq r(u, v) \quad \forall (u, v) \in V \times V
\]

1. Extension works (Frank),
   
   \[ p(X) = \max\{r(u, v) : u \in X, v \notin X\} - d_G(X) \] is skew-supermodular,

2. Splitting off works (Mader),

3. proving min-max theorem of Frank.
Positive Results

Global edge-connectivity augmentation of a graph

1. Extension works (Frank),
   \[ p(X) = k - d_G(X) \] is skew-supermodular,
2. Splitting off works (Mader),
3. proving min-max theorem of Watanabe, Nakamura.

Local edge-connectivity augmentation of a graph

Given a graph \( G = (V, E) \) and a symmetric function \( r : V \times V \to \mathbb{Z}_+ \), what is the minimum number of new edges \( F \) such that
\[
\lambda_{G+F}(u, v) \geq r(u, v) \quad \forall (u, v) \in V \times V
\]

1. Extension works (Frank),
   \[ p(X) = \max\{r(u, v) : u \in X, v \notin X\} - d_G(X) \] is skew-supermodular,
2. Splitting off works (Mader),
3. proving min-max theorem of Frank.
Positive Results

Global edge-connectivity augmentation of a graph

1. Extension works (Frank),
   \[ p(X) = k - d_G(X) \text{ is skew-supermodular}, \]
2. Splitting off works (Mader),
3. proving min-max theorem of Watanabe, Nakamura.

Local edge-connectivity augmentation of a graph

Given a graph \( G = (V, E) \) and a symmetric function \( r : V \times V \to \mathbb{Z}_+ \), what is the minimum number of new edges \( F \) such that

\[ \lambda_{G+F}(u, v) \geq r(u, v) \quad \forall (u, v) \in V \times V. \]

1. Extension works (Frank),
   \[ p(X) = \max\{r(u, v) : u \in X, v \notin X\} - d_G(X) \text{ is skew-supermodular}, \]
2. Splitting off works (Mader),
3. proving min-max theorem of Frank.
Minimum Cover of a Symmetric Skew-Supermodular Function by a Graph

Instance: \( p : 2^V \rightarrow \mathbb{Z} \) symmetric skew-supermodular, \( \gamma \in \mathbb{Z}^+ \).

Question: Does there exist a graph \( H \) on \( V \) with at most \( \gamma \) edges that covers \( p \) that is \( d_H(X) \geq p(X) \quad \forall X \subseteq V \)?
**Negative Result**

**Minimum Cover of a Symmetric Skew-Supermodular Function by a Graph**

**Instance**: $p : 2^V \to \mathbb{Z}$ symmetric skew-supermodular, $\gamma \in \mathbb{Z}^+$.  

**Question**: Does there exist a graph $H$ on $V$ with at most $\gamma$ edges that covers $p$ that is $d_H(X) \geq p(X) \ \forall X \subset V$?

**Theorem (Z. Király, Z. Nutov)**

*The above problem is NP-complete.*
**Definitions**

Given a connected graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even.

1. A subset $X$ of $V$ is called **$T$-odd** if $|X \cap T|$ is odd.
2. A cut $\delta(X)$ is called **$T$-cut** if $X$ is $T$-odd.
3. A subset $F$ of $E$ is called **$T$-join** if $T = \{v \in V : d_F(v) \text{ is odd}\}$.

**Examples**:

(a) $T = \{u, v\}$: a $(u, v)$-path is a $T$-join.
(b) $T = V$: a perfect matching is a $T$-join.
Definitions

Given a connected graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even.

1. A subset $X$ of $V$ is called $T$-odd if $|X \cap T|$ is odd.
2. A cut $\delta(X)$ is called $T$-cut if $X$ is $T$-odd.
3. A subset $F$ of $E$ is called $T$-join if $T = \{v \in V : d_F(v) \text{ is odd}\}$.

Examples:
(a) $T = \{u, v\}$: a $(u, v)$-path is a $T$-join.
(b) $T = V$ : a perfect matching is a $T$-join.
$T$-cut, $T$-join

Definitions

Given a connected graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even.

1. A subset $X$ of $V$ is called $T$-odd if $|X \cap T|$ is odd.
2. A cut $\delta(X)$ is called $T$-cut if $X$ is $T$-odd.
3. A subset $F$ of $E$ is called $T$-join if $T = \{v \in V : d_F(v) \text{ is odd}\}$.

Examples:

(a) $T = \{u, v\}$: a $(u, v)$-path is a $T$-join.
(b) $T = V$: a perfect matching is a $T$-join.
Definitions

Given a connected graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even.

1. A subset $X$ of $V$ is called $T$-odd if $|X \cap T|$ is odd.
2. A cut $\delta(X)$ is called $T$-cut if $X$ is $T$-odd.
3. A subset $F$ of $E$ is called $T$-join if $T = \{v \in V : d_F(v) \text{ is odd}\}$.

Examples:
(a) $T = \{u, v\}$: a $(u, v)$-path is a $T$-join.
(b) $T = V$: a perfect matching is a $T$-join.
**T-cut, T-join**

### Definitions

Given a connected graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even.

1. A subset $X$ of $V$ is called $T$-odd if $|X \cap T|$ is odd.
2. A cut $\delta(X)$ is called $T$-cut if $X$ is $T$-odd.
3. A subset $F$ of $E$ is called $T$-join if $T = \{v \in V : d_F(v) \text{ is odd}\}$.

**Examples:**

(a) $T = \{u, v\}$ : a $(u, v)$-path is a $T$-join.
(b) $T = V$ : a perfect matching is a $T$-join.
Definitions

Given a connected graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even.

1. A subset $X$ of $V$ is called $T$-odd if $|X \cap T|$ is odd.
2. A cut $\delta(X)$ is called $T$-cut if $X$ is $T$-odd.
3. A subset $F$ of $E$ is called $T$-join if $T = \{v \in V : d_F(v) \text{ is odd}\}$.

Examples:

(a) $T = \{u, v\}$: a $(u, v)$-path is a $T$-join.
(b) $T = V$: a perfect matching is a $T$-join.
Definitions

Given a connected graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even.

1. A subset $X$ of $V$ is called $T$-odd if $|X \cap T|$ is odd.

2. A cut $\delta(X)$ is called $T$-cut if $X$ is $T$-odd.

3. A subset $F$ of $E$ is called $T$-join if $T = \{v \in V : d_F(v) \text{ is odd}\}$.

Examples:
(a) $T = \{u, v\}$: a $(u, v)$-path is a $T$-join.
(b) $T = V$: a perfect matching is a $T$-join.
Definitions

Given a connected graph \( G = (V, E) \) and \( T \subseteq V \) with \( |T| \) even.

1. A subset \( X \) of \( V \) is called \( T \)-odd if \( |X \cap T| \) is odd.
2. A cut \( \delta(X) \) is called \( T \)-cut if \( X \) is \( T \)-odd.
3. A subset \( F \) of \( E \) is called \( T \)-join if \( T = \{ v \in V : d_F(v) \text{ is odd} \} \).

Examples:
(a) \( T = \{ u, v \} : \) a \( (u, v) \)-path is a \( T \)-join.
(b) \( T = V : \) a perfect matching is a \( T \)-join.

Properties

1. If \( X, Y \) are \( T \)-odd, then either \( X \cap Y, X \cup Y \) or \( X - Y, Y - X \) are \( T \)-odd.
2. A \( T \)-join and a \( T \)-cut always have an edge in common.
Definitions

Given a connected graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even.

1. A subset $X$ of $V$ is called $T$-odd if $|X \cap T|$ is odd.
2. A cut $\delta(X)$ is called $T$-cut if $X$ is $T$-odd.
3. A subset $F$ of $E$ is called $T$-join if $T = \{v \in V : d_F(v) \text{ is odd}\}$.

Examples:

(a) $T = \{u, v\}$: a $(u, v)$-path is a $T$-join.
(b) $T = V$: a perfect matching is a $T$-join.

Properties

1. If $X, Y$ are $T$-odd, then either $X \cap Y, X \cup Y$ or $X - Y, Y - X$ are $T$-odd.
2. A $T$-join and a $T$-cut always have an edge in common.
Definitions

Given a connected graph \( G = (V, E) \) and \( T \subseteq V \) with \( |T| \) even.

1. A subset \( X \) of \( V \) is called \( T \)-odd if \( |X \cap T| \) is odd.
2. A cut \( \delta(X) \) is called \( T \)-cut if \( X \) is \( T \)-odd.
3. A subset \( F \) of \( E \) is called \( T \)-join if \( T = \{v \in V : d_F(v) \text{ is odd}\} \).

Examples:
   (a) \( T = \{u, v\} \): a \((u, v)\)-path is a \( T \)-join.
   (b) \( T = V \): a perfect matching is a \( T \)-join.

Properties

1. If \( X, Y \) are \( T \)-odd, then either \( X \cap Y, X \cup Y \) or \( X - Y, Y - X \) are \( T \)-odd.
2. A \( T \)-join and a \( T \)-cut always have an edge in common.
Theorem (Edmonds-Johnson)

A minimum $T$-join of $G$ can be found in polynomial time using

1. shortest paths algorithm (Dijkstra) and
2. minimum weight perfect matching algorithm (Edmonds).

Graph $G$ and vertex set $T$
How to find a minimum $T$-join?

**Theorem (Edmonds-Johnson)**

A minimum $T$-join of $G$ can be found in polynomial time using

1. shortest paths algorithm (Dijkstra) and
2. minimum weight perfect matching algorithm (Edmonds).

Graph $G$ and vertex set $T$
How to find a minimum $T$-join?

**Theorem (Edmonds-Johnson)**

A minimum $T$-join of $G$ can be found in *polynomial* time using

1. shortest paths algorithm (Dijkstra) and
2. minimum weight perfect matching algorithm (Edmonds).

Graph $G$ and vertex set $T$
How to find a minimum $T$-join?

**Theorem (Edmonds-Johnson)**

A minimum $T$-join of $G$ can be found in polynomial time using

1. shortest paths algorithm (Dijkstra) and
2. minimum weight perfect matching algorithm (Edmonds).

Graph $G$ and vertex set $T$
Theorem (Edmonds-Johnson)

A minimum $T$-join of $G$ can be found in polynomial time using:

1. shortest paths algorithm (Dijkstra) and
2. minimum weight perfect matching algorithm (Edmonds).

Graph $G$ and vertex set $T$
How to find a minimum $T$-join?

**Theorem (Edmonds-Johnson)**

A minimum $T$-join of $G$ can be found in *polynomial* time using

1. shortest paths algorithm (Dijkstra) and
2. minimum weight perfect matching algorithm (Edmonds).

Graph $G$ and vertex set $T$
How to find a minimum $T$-join?

**Theorem (Edmonds-Johnson)**

A minimum $T$-join of $G$ can be found in *polynomial* time using

1. shortest paths algorithm (Dijkstra) and
2. minimum weight perfect matching algorithm (Edmonds).

Graph $G$ and vertex set $T$
How to find a minimum $T$-join?

**Theorem (Edmonds-Johnson)**

A minimum $T$-join of $G$ can be found in polynomial time using

1. shortest paths algorithm (Dijkstra) and
2. minimum weight perfect matching algorithm (Edmonds).

Graph $G$ and vertex set $T$
How to find a minimum $T$-join?

**Theorem (Edmonds-Johnson)**

A minimum $T$-join of $G$ can be found in *polynomial* time using

1. shortest paths algorithm (Dijkstra) and
2. minimum weight perfect matching algorithm (Edmonds).

Graph $G$ and vertex set $T$
How to find a minimum $T$-join?

Theorem (Edmonds-Johnson)

A minimum $T$-join of $G$ can be found in polynomial time using

1. shortest paths algorithm (Dijkstra) and
2. minimum weight perfect matching algorithm (Edmonds).

Graph $G$ and minimum $T$-join

Z. Szigeti (G-SCOP, Grenoble)
How to find a minimum \( T \)-cut?

**Theorem (Padberg-Rao)**

A minimum \( T \)-cut of \( G \) can be found in *polynomial* time

1. using a cut equivalent tree \( H \) of \( G \);
2. taking the set \( J(H) \) edges \( e \) of \( H \) for which the two connected components of \( H - e \) are \( T \)-odd,
3. taking the minimum value \( c(e^*) \) of an edge of \( J(H) \),
4. taking the cut of \( G \) defined by the fundamental cut of \( H - e^* \).

Graph \( G \) and vertex set \( T \)
How to find a minimum $T$-cut?

**Theorem (Padberg-Rao)**

A minimum $T$-cut of $G$ can be found in *polynomial* time

1. using a cut equivalent tree $H$ of $G$;
2. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,
3. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
4. taking the cut of $G$ defined by the fundamental cut of $H - e^*$.

Graph $G$ and vertex set $T$

Cut equivalent tree $H$
How to find a minimum $T$-cut?

**Theorem (Padberg-Rao)**

A minimum $T$-cut of $G$ can be found in polynomial time

1. using a cut equivalent tree $H$ of $G$;
2. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,
3. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
4. taking the cut of $G$ defined by the fundamental cut of $H - e^*$.

Graph $G$ and vertex set $T$

Cut equivalent tree $H$
How to find a minimum $T$-cut?

**Theorem (Padberg-Rao)**

A minimum $T$-cut of $G$ can be found in polynomial time

1. using a cut equivalent tree $H$ of $G$;
2. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,
3. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
4. taking the cut of $G$ defined by the fundamental cut of $H - e^*$.

---

**Graph $G$ and vertex set $T$**

**Cut equivalent tree $H$**
How to find a minimum $T$-cut?

**Theorem (Padberg-Rao)**

A minimum $T$-cut of $G$ can be found in polynomial time

1. using a cut equivalent tree $H$ of $G$
2. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,
3. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
4. taking the cut of $G$ defined by the fundamental cut of $H - e^*$.

Graph $G$ and vertex set $T$

Cut equivalent tree $H$
How to find a minimum $T$-cut?

**Theorem (Padberg-Rao)**

A minimum $T$-cut of $G$ can be found in **polynomial** time

1. using a cut equivalent tree $H$ of $G$;
2. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,
3. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
4. taking the cut of $G$ defined by the fundamental cut of $H - e^*$.

![Graph G and vertex set T](image1)

![Cut equivalent tree H](image2)
How to find a minimum $T$-cut?

**Theorem (Padberg-Rao)**

A minimum $T$-cut of $G$ can be found in polynomial time

1. **using a cut equivalent tree $H$ of $G$;**
2. **taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,**
3. **taking the minimum value $c(e^*)$ of an edge of $J(H),**
4. **taking the cut of $G$ defined by the fundamental cut of $H - e^*.**

Graph $G$ and vertex set $T$

Cut equivalent tree $H$
How to find a minimum $T$-cut?

**Theorem (Padberg-Rao)**

A minimum $T$-cut of $G$ can be found in *polynomial* time

1. using a cut equivalent tree $H$ of $G$;
2. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,
3. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
4. taking the cut of $G$ defined by the fundamental cut of $H - e^*$.

Graph $G$ and vertex set $T$

Cut equivalent tree $H$
How to find a minimum $T$-cut?

**Theorem (Padberg-Rao)**

A minimum $T$-cut of $G$ can be found in *polynomial* time

1. using a cut equivalent tree $H$ of $G$;
2. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,
3. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
4. taking the cut of $G$ defined by the fundamental cut of $H - e^*$.

Graph $G$ and vertex set $T$

Cut equivalent tree $H$ and edge set $J(H)$. 

Z. Szigeti (G-SCOP, Grenoble)
How to find a minimum $T$-cut?

Theorem (Padberg-Rao)

A minimum $T$-cut of $G$ can be found in polynomial time

1. using a cut equivalent tree $H$ of $G$;
2. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,
3. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
4. taking the cut of $G$ defined by the fundamental cut of $H - e^*$.

Graph $G$ and vertex set $T$

Cut equivalent tree $H$
How to find a minimum $T$-cut?

**Theorem (Padberg-Rao)**

A minimum $T$-cut of $G$ can be found in polynomial time

1. using a cut equivalent tree $H$ of $G$;
2. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,
3. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
4. taking the cut of $G$ defined by the fundamental cut of $H - e^*$.

Graph $G$ and vertex set $T$

Cut equivalent tree $H$
How to find a minimum $T$-cut?

**Theorem (Padberg-Rao)**

A minimum $T$-cut of $G$ can be found in polynomial time

1. using a cut equivalent tree $H$ of $G$;
2. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are $T$-odd,
3. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
4. taking the cut of $G$ defined by the fundamental cut of $H - e^*$.

Minimum $T$-cut in $G$

Cut equivalent tree $H$
Proof

Lemma

For any $T$-cut $\delta(X)$ there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$. 
Lemma

For any \( T \)-cut \( \delta(X) \) there exist \( x \in X, y \notin X \) such that \( \lambda_G(x, y) \geq c(e^*) \).

Proof : \( J(H) \) is a \( T \)-join so there exists \( xy \in J(H) \cap \delta_H(X) \) and \( \lambda_G(x, y) = c(xy) \geq c(e^*) \).
Proof

Lemma
For any $T$-cut $\delta(X)$ there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$.

Proof : $J(H)$ is a $T$-join so there exists $xy \in J(H) \cap \delta_H(X)$ and $\lambda_G(x, y) = c(xy) \geq c(e^*)$.

Correctness of Padberg-Rao’s algorithm
Let $\delta(X)$ be a minimum $T$-cut and $\delta(Y)$ the $T$-cut defined by $e^*$. By the lemma, there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$. Then

$$c(e^*) = d(Y) \geq d(X) \geq \lambda_G(x, y) \geq c(e^*).$$
Proof

Lemma
For any $T$-cut $\delta(X)$ there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$.

Proof: $J(H)$ is a $T$-join so there exists $xy \in J(H) \cap \delta_H(X)$ and

$$\lambda_G(x, y) = c(xy) \geq c(e^*)$$

Correctness of Padberg-Rao’s algorithm

Let $\delta(X)$ be a minimum $T$-cut and $\delta(Y)$ the $T$-cut defined by $e^*$.
By the lemma, there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$.
Then

$$c(e^*) = d(Y) \geq d(X) \geq \lambda_G(x, y) \geq c(e^*)$$
Proof

Lemma

For any $T$-cut $\delta(X)$ there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$.

Proof: $J(H)$ is a $T$-join so there exists $xy \in J(H) \cap \delta_H(X)$ and $\lambda_G(x, y) = c(xy) \geq c(e^*)$.

Correctness of Padberg-Rao’s algorithm

Let $\delta(X)$ be a minimum $T$-cut and $\delta(Y)$ the $T$-cut defined by $e^*$. By the lemma, there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$. Then

$$c(e^*) = d(Y) \geq d(X) \geq \lambda_G(x, y) \geq c(e^*).$$
Proof

Lemma

For any $T$-cut $\delta(X)$ there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$.

Proof: $J(H)$ is a $T$-join so there exists $xy \in J(H) \cap \delta_H(X)$ and $\lambda_G(x, y) = c(xy) \geq c(e^*)$.

Correctness of Padberg-Rao’s algorithm

Let $\delta(X)$ be a minimum $T$-cut and $\delta(Y)$ the $T$-cut defined by $e^*$. By the lemma, there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$. Then

$$c(e^*) = d(Y) \geq d(X) \geq \lambda_G(x, y) \geq c(e^*).$$
Proof

Lemma
For any $T$-cut $\delta(X)$ there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$.

Proof: $J(H)$ is a $T$-join so there exists $xy \in J(H) \cap \delta_H(X)$ and $\lambda_G(x, y) = c(xy) \geq c(e^*)$.

Correctness of Padberg-Rao's algorithm
Let $\delta(X)$ be a minimum $T$-cut and $\delta(Y)$ the $T$-cut defined by $e^*$. By the lemma, there exist $x \in X, y \notin X$ such that $\lambda_G(x, y) \geq c(e^*)$. Then
\[
c(e^*) = d(Y) \geq d(X) \geq \lambda_G(x, y) \geq c(e^*).
\]
How to augment a minimum $T$-cut?

**Theorem (Szigeti)**

Given a connected graph $G = (V, E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $T$-cut is of size at least $k$ is equal to $\lceil \frac{1}{2} \text{ maximum } p'-\text{value of a subpartition of } V \rceil$. An optimal augmentation can be found in **polynomial** time using

1. Frank’s minimal extension and
2. Mader’s complete splitting off.

**Proof**

1. works because $p'(X) = k - d_G(X)$ if $X$ is $T$-odd and $-\infty$ otherwise is symmetric skew-supermodular

   (i) $k - d_G(X)$ satisfies both inequalities,
   (ii) $X, Y$ are $T$-odd $\implies$ either $X \cap Y$, $X \cup Y$ or $X - Y$, $Y - X$ are $T$-odd.

2. works because for all $T$-odd sets, $d_{G'}(X) \geq k$ and, by the above lemma, $k \leq \lambda_{G'}(x, y) = \lambda_{G''}(x, y) \leq d_{G''}(X)$. 

Z. Szigeti (G-SCOP, Grenoble)
How to augment a minimum $T$-cut?

**Theorem (Szigeti)**

Given a connected graph $G = (V, E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $T$-cut is of size at least $k$ is equal to $\lceil \frac{1}{2} \text{ maximum } p'-\text{value of a subpartition of } V \rceil$. An optimal augmentation can be found in polynomial time using

1. Frank’s minimal extension and
2. Mader’s complete splitting off.

**Proof**

1. works because $p'(X) = k - d_G(X)$ if $X$ is $T$-odd and $-\infty$ otherwise is symmetric skew-supermodular
   
   (i) $k - d_G(X)$ satisfies both inequalities,
   (ii) $X, Y$ are $T$-odd $\implies$ either $X \cap Y, X \cup Y$ or $X - Y, Y - X$ are $T$-odd.

2. works because for all $T$-odd sets, $d_{G'}(X) \geq k$ and, by the above lemma, $k \leq \lambda_{G'}(x, y) = \lambda_{G''}(x, y) \leq d_{G''}(X)$. 

Z. Szigeti (G-SCOP, Grenoble)
How to augment a minimum $T$-cut?

Theorem (Szigeti)

Given a connected graph $G = (V, E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $T$-cut is of size at least $k$ is equal to $\lceil \frac{1}{2} \text{ maximum } p'\text{-value of a subpartition of } V \rceil$. An optimal augmentation can be found in polynomial time using

1. Frank’s minimal extension and
2. Mader’s complete splitting off.

Proof

1. works because $p'(X) = k - d_G(X)$ if $X$ is $T$-odd and $-\infty$ otherwise is symmetric skew-supermodular

   (i) $k - d_G(X)$ satisfies both inequalities,
   (ii) $X, Y$ are $T$-odd $\implies$ either $X \cap Y, X \cup Y$ or $X - Y, Y - X$ are $T$-odd.

2. works because for all $T$-odd sets, $d_G'(X) \geq k$ and, by the above lemma, $k \leq \lambda_G'(x, y) = \lambda_G''(x, y) \leq d_G''(X)$.
How to augment a minimum $T$-cut?

**Theorem (Szigeti)**

Given a connected graph $G = (V, E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $T$-cut is of size at least $k$ is equal to $\lceil \frac{1}{2} \text{ maximum } p'\text{-value of a subpartition of } V \rceil$. An optimal augmentation can be found in polynomial time using

1. Frank’s minimal extension and
2. Mader’s complete splitting off.

**Proof**

1. works because $p'(X) = k - d_G(X)$ if $X$ is $T$-odd and $-\infty$ otherwise. Is symmetric skew-supermodular
   
   (i) $k - d_G(X)$ satisfies both inequalities,
   (ii) $X, Y$ are $T$-odd $\implies$ either $X \cap Y, X \cup Y$ or $X - Y, Y - X$ are $T$-odd.

2. works because for all $T$-odd sets, $d_{G'}(X) \geq k$ and, by the above lemma, $k \leq \lambda_{G'}(x, y) = \lambda_{G''}(x, y) \leq d_{G''}(X)$. 

How to augment a minimum $T$-cut?

**Theorem (Szigeti)**

Given a connected graph $G = (V, E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $T$-cut is of size at least $k$ is equal to $\lceil \frac{1}{2} \text{ maximum } p'\text{-value of a subpartition of } V \rceil$. An optimal augmentation can be found in *polynomial* time using

1. Frank’s minimal extension and
2. Mader’s complete splitting off.

**Proof**

1. works because $p'(X) = k - d_G(X)$ if $X$ is $T$-odd and $-\infty$ otherwise is symmetric skew-supermodular
   
   (i) $k - d_G(X)$ satisfies both inequalities,
   
   (ii) $X, Y$ are $T$-odd $\implies$ either $X \cap Y, X \cup Y$ or $X - Y, Y - X$ are $T$-odd.

2. works because for all $T$-odd sets, $d_G'(X) \geq k$ and, by the above lemma, $k \leq \lambda_{G'}(x, y) = \lambda_{G''}(x, y) \leq d_{G''}(X)$.
How to augment a minimum $T$-cut?

**Theorem (Szigeti)**

Given a connected graph $G = (V, E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $T$-cut is of size at least $k$ is equal to $\lceil \frac{1}{2} \text{ maximum } p' \text{-value of a subpartition of } V \rceil$.

An optimal augmentation can be found in *polynomial* time using

1. Frank’s minimal extension and
2. Mader’s complete splitting off.

**Proof**

1. works because $p'(X) = k - d_G(X)$ if $X$ is $T$-odd and $-\infty$ otherwise is symmetric skew-supermodular
   
   (i) $k - d_G(X)$ satisfies both inequalities,
   
   (ii) $X, Y$ are $T$-odd $\implies$ either $X \cap Y$, $X \cup Y$ or $X - Y$, $Y - X$ are $T$-odd.

2. works because for all $T$-odd sets, $d_{G'}(X) \geq k$ and, by the above lemma, $k \leq \lambda_{G'}(x, y) = \lambda_{G''}(x, y) \leq d_{G''}(X)$. 

Z. Szigeti (G-SCOP, Grenoble) 

Edge-connectivity augmentation

27 octobre 2010 15 / 20
How to augment a minimum $T$-cut?

**Theorem (Szigeti)**

Given a connected graph $G = (V, E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $T$-cut is of size at least $k$ is equal to $\lceil \frac{1}{2} \text{ maximum } p'\text{-value of a subpartition of } V \rceil$.

An optimal augmentation can be found in polynomial time using

1. Frank’s minimal extension and
2. Mader’s complete splitting off.

**Proof**

1. works because $p'(X) = k - d_G(X)$ if $X$ is $T$-odd and $-\infty$ otherwise is symmetric skew-supermodular
   
   (i) $k - d_G(X)$ satisfies both inequalities,
   (ii) $X, Y$ are $T$-odd $\iff$ either $X \cap Y, X \cup Y$ or $X - Y, Y - X$ are $T$-odd.

2. works because for all $T$-odd sets, $d_{G'}(X) \geq k$ and, by the above lemma, $k \leq \lambda_{G'}(x, y) = \lambda_{G''}(x, y) \leq d_{G''}(X)$. 
How to augment a minimum $T$-cut?

**Theorem (Szigeti)**

Given a connected graph $G = (V, E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $T$-cut is of size at least $k$ is equal to $\lceil \frac{1}{2} \text{ maximum } p'\text{-value of a subpartition of } V \rceil$.

An optimal augmentation can be found in polynomial time using

1. Frank’s minimal extension and
2. Mader’s complete splitting off.

Graph $G$, vertex set $T$ and $k = 4$
How to augment a minimum $T$-cut?

**Theorem (Szigeti)**

Given a connected graph $G = (V, E)$, $T \subseteq V$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $T$-cut is of size at least $k$ is equal to $\lceil \frac{1}{2} \text{ maximum } p'-\text{value of a subpartition of } V \rceil$.

An optimal augmentation can be found in polynomial time using

1. Frank’s minimal extension and
2. Mader’s complete splitting off.

Minimum $T$-cut in $G + F$ is 4
Definition: symmetric parity family

A family \( \mathcal{F} \) of subsets of \( V \) is called symmetric parity family if

1. \( \emptyset, V \notin \mathcal{F} \),
2. if \( A \in \mathcal{F} \), then \( V - A \in \mathcal{F} \),
3. if \( A, B \notin \mathcal{F} \) and \( A \cap B = \emptyset \), then \( A \cup B \notin \mathcal{F} \).
Definition: symmetric parity family

A family $\mathcal{F}$ of subsets of $V$ is called symmetric parity family if

1. $\emptyset, V \notin \mathcal{F}$,
2. if $A \in \mathcal{F}$, then $V - A \in \mathcal{F}$,
3. if $A, B \notin \mathcal{F}$ and $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{F}$. 
Definition: symmetric parity family

A family $\mathcal{F}$ of subsets of $V$ is called symmetric parity family if

1. $\emptyset, V \notin \mathcal{F}$,
2. if $A \in \mathcal{F}$, then $V - A \in \mathcal{F}$,
3. if $A, B \notin \mathcal{F}$ and $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{F}$. 
Definition: symmetric parity family

A family $\mathcal{F}$ of subsets of $V$ is called symmetric parity family if

1. $\emptyset, V \notin \mathcal{F}$,
2. if $A \in \mathcal{F}$, then $V - A \in \mathcal{F}$,
3. if $A, B \notin \mathcal{F}$ and $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{F}$. 
Definition: symmetric parity family

A family $\mathcal{F}$ of subsets of $V$ is called symmetric parity family if

1. $\emptyset, V \notin \mathcal{F}$,
2. if $A \in \mathcal{F}$, then $V - A \in \mathcal{F}$,
3. if $A, B \notin \mathcal{F}$ and $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{F}$.

Examples

1. $\mathcal{F} := 2^V - \{\emptyset, V\}$
2. $\mathcal{F} := \{X \subset V : X \text{ is } T\text{-odd}\}$ where $T \subseteq V$ with $|T|$ even.
Definition : symmetric parity family

A family $\mathcal{F}$ of subsets of $V$ is called symmetric parity family if

1. $\emptyset, V \notin \mathcal{F}$,
2. if $A \in \mathcal{F}$, then $V - A \in \mathcal{F}$,
3. if $A, B \notin \mathcal{F}$ and $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{F}$.

Examples

1. $\mathcal{F} := 2^V - \{\emptyset, V\}$
2. $\mathcal{F} := \{X \subset V : X \text{ is } T\text{-odd}\}$ where $T \subseteq V$ with $|T|$ even.
Definition: symmetric parity family

A family $\mathcal{F}$ of subsets of $V$ is called symmetric parity family if

1. $\emptyset, V \notin \mathcal{F}$,
2. if $A \in \mathcal{F}$, then $V - A \in \mathcal{F}$,
3. if $A, B \notin \mathcal{F}$ and $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{F}$.

Examples

1. $\mathcal{F} := 2^V - \{\emptyset, V\}$
2. $\mathcal{F} := \{X \subset V : X \text{ is } T-\text{odd}\}$ where $T \subseteq V$ with $|T|$ even.
Definition: symmetric parity family

A family $F$ of subsets of $V$ is called symmetric parity family if

1. $\emptyset, V \notin F$,
2. if $A \in F$, then $V - A \in F$,
3. if $A, B \notin F$ and $A \cap B = \emptyset$, then $A \cup B \notin F$.

Examples

1. $F := 2^V - \{\emptyset, V\}$
2. $F := \{X \subset V : X$ is $T$-odd$\}$ where $T \subseteq V$ with $|T|$ even.

Property

1. If $X, Y$ are in $F$, then either $X \cap Y, X \cup Y$ or $X - Y, Y - X$ are in $F$. 
Definition: symmetric parity family

A family $\mathcal{F}$ of subsets of $V$ is called symmetric parity family if

1. $\emptyset, V \notin \mathcal{F}$,
2. if $A \in \mathcal{F}$, then $V - A \in \mathcal{F}$,
3. if $A, B \notin \mathcal{F}$ and $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{F}$.

Examples

1. $\mathcal{F} := 2^V - \{\emptyset, V\}$
2. $\mathcal{F} := \{X \subset V : X \text{ is } T\text{-odd}\}$ where $T \subseteq V$ with $|T|$ even.

Property

1. If $X, Y$ are in $\mathcal{F}$, then either $X \cap Y, X \cup Y$ or $X - Y, Y - X$ are in $\mathcal{F}$. 
How to find a minimum $\mathcal{F}$-cut?

**Theorem (Goemans-Ramakrishnan)**

Given a connected graph $G$ and a symmetric parity family $\mathcal{F}$, a minimum cut of $G$ over $\mathcal{F}$, (a minimum $\mathcal{F}$-cut) can be found in polynomial time

1. using a cut equivalent tree $H$ of $G$,
2. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are in $\mathcal{F}$,
3. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
4. taking the cut of $G$ defined by the fundamental cut of $H - e^*$.
How to find a minimum $\mathcal{F}$-cut?

**Theorem (Goemans-Ramakrishnan)**

Given a connected graph $G$ and a symmetric parity family $\mathcal{F}$, a minimum cut of $G$ over $\mathcal{F}$, (a minimum $\mathcal{F}$-cut) can be found in polynomial time using a cut equivalent tree $H$ of $G$,

1. taking the set $J(H)$ edges $e$ of $H$ for which the two connected components of $H - e$ are in $\mathcal{F}$,
2. taking the minimum value $c(e^*)$ of an edge of $J(H)$,
3. taking the cut of $G$ defined by the fundamental cut of $H - e^*$. 

Z. Szigeti (G-SCOP, Grenoble)
Lemma

For any $A \in \mathcal{F}$ there exists an edge $xy \in \delta_{J(H)}(A)$, and hence

$$\lambda_G(x, y) = c(xy) \geq c(e^*)$$.
Proof

Lemma

For any $A \in \mathcal{F}$ there exists an edge $xy \in \delta_{J(H)}(A)$, and hence

$\lambda_{G}(x, y) = c(xy) \geq c(e^{*})$.

Proof:

1. Let $H[A_1], \ldots, H[A_k]$ be the connected components of $H[A]$.
2. Since $A \in \mathcal{F}$ and $\bigcup A_i = A$, $\exists i : A_i \in \mathcal{F}$ by (iii).
3. Let $H[B_1], \ldots, H[B_l]$ be the connected components of $H - A_i$.
4. Since $V - A_i \in \mathcal{F}$ by (ii) and $\bigcup B_j = V - A_i$, $\exists j : B_j \in \mathcal{F}$ by (iii).
5. $H$ is a tree, $H[A_i]$ is connected, $H[B_j]$ is a connected component of $H - A_i$, so there exists exactly one edge $e \in H$ between $A_i$ and $B_j$.
6. Then $e \in J(H)$ and $e$ enters $A$. 
Proof

Lemma

For any $A \in \mathcal{F}$ there exists an edge $xy \in \delta_{J(H)}(A)$, and hence $\lambda_G(x, y) = c(xy) \geq c(e^*)$.

Proof:

1. Let $H[A_1], \ldots, H[A_k]$ be the connected components of $H[A]$.
2. Since $A \in \mathcal{F}$ and $\bigcup A_i = A$, $\exists i : A_i \in \mathcal{F}$ by (iii).
3. Let $H[B_1], \ldots, H[B_l]$ be the connected components of $H - A_i$.
4. Since $V - A_i \in \mathcal{F}$ by (ii) and $\bigcup B_j = V - A_i$, $\exists j : B_j \in \mathcal{F}$ by (iii).
5. $H$ is a tree, $H[A_i]$ is connected, $H[B_j]$ is a connected component of $H - A_i$, so there exists exactly one edge $e \in H$ between $A_i$ and $B_j$.
6. Then $e \in J(H)$ and $e$ enters $A$. 
Proof

Lemma

For any $A \in \mathcal{F}$ there exists an edge $xy \in \delta_{J(H)}(A)$, and hence $\lambda_G(x, y) = c(xy) \geq c(e^*)$.

Proof:

1. Let $H[A_1], \ldots, H[A_k]$ be the connected components of $H[A]$.
2. Since $A \in \mathcal{F}$ and $\bigcup A_i = A$, $\exists i : A_i \in \mathcal{F}$ by (iii).
3. Let $H[B_1], \ldots, H[B_i]$ be the connected components of $H - A_i$.
4. Since $V - A_i \in \mathcal{F}$ by (ii) and $\bigcup B_j = V - A_i$, $\exists j : B_j \in \mathcal{F}$ by (iii).
5. $H$ is a tree, $H[A_i]$ is connected, $H[B_j]$ is a connected component of $H - A_i$, so there exists exactly one edge $e \in H$ between $A_i$ and $B_j$.
6. Then $e \in J(H)$ and $e$ enters $A$. 
Lemma

For any $A \in \mathcal{F}$ there exists an edge $xy \in \delta_{J(H)}(A)$, and hence $\lambda_G(x, y) = c(xy) \geq c(e^*)$.

Proof:

1. Let $H[A_1], \ldots, H[A_k]$ be the connected components of $H[A]$.
2. Since $A \in \mathcal{F}$ and $\bigcup A_i = A$, $\exists i : A_i \in \mathcal{F}$ by (iii).
3. Let $H[B_1], \ldots, H[B_l]$ be the connected components of $H - A_i$.
4. Since $V - A_i \in \mathcal{F}$ by (ii) and $\bigcup B_j = V - A_i$, $\exists j : B_j \in \mathcal{F}$ by (iii).
5. $H$ is a tree, $H[A_i]$ is connected, $H[B_j]$ is a connected component of $H - A_i$, so there exists exactly one edge $e \in H$ between $A_i$ and $B_j$.
6. Then $e \in J(H)$ and $e$ enters $A$. 
Lemma

For any \( A \in \mathcal{F} \) there exists an edge \( xy \in \delta_{J(H)}(A) \), and hence
\[
\lambda_G(x, y) = c(xy) \geq c(e^*).
\]

Proof:

1. Let \( H[A_1], \ldots, H[A_k] \) be the connected components of \( H[A] \).
2. Since \( A \in \mathcal{F} \) and \( \bigcup A_i = A \), \( \exists i : A_i \in \mathcal{F} \) by (iii).
3. Let \( H[B_1], \ldots, H[B_l] \) be the connected components of \( H - A_i \).
4. Since \( V - A_i \in \mathcal{F} \) by (ii) and \( \bigcup B_j = V - A_i \), \( \exists j : B_j \in \mathcal{F} \) by (iii).
5. \( H \) is a tree, \( H[A_i] \) is connected, \( H[B_j] \) is a connected component of \( H - A_i \), so there exists exactly one edge \( e \in H \) between \( A_i \) and \( B_j \).
6. Then \( e \in J(H) \) and \( e \) enters \( A \).
Lemma
For any $A \in \mathcal{F}$ there exists an edge $xy \in \delta_{J(H)}(A)$, and hence $\lambda_G(x, y) = c(xy) \geq c(e^*)$.

Proof:
1. Let $H[A_1], \ldots, H[A_k]$ be the connected components of $H[A]$.
2. Since $A \in \mathcal{F}$ and $\bigcup A_i = A$, $\exists i : A_i \in \mathcal{F}$ by (iii).
3. Let $H[B_1], \ldots, H[B_l]$ be the connected components of $H - A_i$.
4. Since $V - A_i \in \mathcal{F}$ by (ii) and $\bigcup B_j = V - A_i$, $\exists j : B_j \in \mathcal{F}$ by (iii).
5. $H$ is a tree, $H[A_i]$ is connected, $H[B_j]$ is a connected component of $H - A_i$, so there exists exactly one edge $e \in H$ between $A_i$ and $B_j$.
6. Then $e \in J(H)$ and $e$ enters $A$. 

Z. Szigeti (G-SCOP, Grenoble)
Proof

Lemma

For any \( A \in \mathcal{F} \) there exists an edge \( xy \in \delta_{J(H)}(A) \), and hence \( \lambda_G(x, y) = c(xy) \geq c(e^*) \).

Proof:

1. Let \( H[A_1], \ldots, H[A_k] \) be the connected components of \( H[A] \).
2. Since \( A \in \mathcal{F} \) and \( \bigcup A_i = A \), \( \exists i : A_i \in \mathcal{F} \) by (iii).
3. Let \( H[B_1], \ldots, H[B_l] \) be the connected components of \( H - A_i \).
4. Since \( V - A_i \in \mathcal{F} \) by (ii) and \( \bigcup B_j = V - A_i \), \( \exists j : B_j \in \mathcal{F} \) by (iii).
5. \( H \) is a tree, \( H[A_i] \) is connected, \( H[B_j] \) is a connected component of \( H - A_i \), so there exists exactly one edge \( e \in H \) between \( A_i \) and \( B_j \).
6. Then \( e \in J(H) \) and \( e \) enters \( A \).
Proof

Lemma

For any $A \in \mathcal{F}$ there exists an edge $xy \in \delta_{J(H)}(A)$, and hence $\lambda_G(x, y) = c(xy) \geq c(e^*)$.

Correctness of Goemans-Ramakrishnan’s algorithm

The same proof works as for Padberg-Rao’s algorithm.
Theorem (Szigeti)

Given a connected graph $G$, a symmetric parity family $\mathcal{F}$ and $k \in \mathbb{Z}$, the minimum number of edges whose addition results in a graph so that each $\mathcal{F}$-cut is of size at least $k$ equals $\left\lceil \frac{1}{2} \max p^*\text{-value of a subpartition of } V \right\rceil$. An optimal augmentation can be found in polynomial time using

1. Frank’s minimal extension and
2. Mader’s complete splitting off.

Proof

1. works because $p^*(X) = k - d_G(X)$ if $X \in \mathcal{F}$ and $-\infty$ otherwise is symmetric skew-supermodular

   (i) $k - d_G(X)$ satisfies both inequalities,
   (ii) If $X, Y \in \mathcal{F}$, then either $X \cap Y, X \cup Y \in \mathcal{F}$ or $X - Y, Y - X \in \mathcal{F}$.

2. works because for all $X \in \mathcal{F}$, $d_{G'}(X) \geq k$ and, by the above lemma, $k \leq \lambda_{G'}(x, y) = \lambda_{G''}(x, y) \leq d_{G''}(X)$.
Conclusion

1. Special cases:
   1. Global edge-connectivity augmentation (Watanabe, Nakamura)
   2. Minimum $T$-cut augmentation

2. A new \textit{polynomial} special case of the NP-complete problem
   \textbf{Minimum Cover of a Symmetric Skew-Supermodular Function by a Graph}
Conclusion

1. Special cases:
   1. Global edge-connectivity augmentation (Watanabe, Nakamura)
   2. Minimum $T$-cut augmentation

2. A new polynomial special case of the NP-complete problem
   Minimum Cover of a Symmetric Skew-Supermodular Function by a Graph
Conclusion

Special cases:
1. Global edge-connectivity augmentation (Watanabe, Nakamura)
2. Minimum $T$-cut augmentation

A new polynomial special case of the NP-complete problem
**Minimum Cover of a Symmetric Skew-Supermodular Function by a Graph**
Conclusion

Special cases:
1. Global edge-connectivity augmentation (Watanabe, Nakamura)
2. Minimum $T$-cut augmentation

A new polynomial special case of the NP-complete problem

**Minimum Cover of a Symmetric Skew-Supermodular Function by a Graph**