On packing of arborescences

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Joint work with :
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Outline

- Motivations
  - Undirected = Orientation + Directed
  - Rigidity
- Results
  - Undirected : Matroid-based packing of rooted-trees
  - Directed : Matroid-based packing of arborescences
  - Orientation : Supermodular function
- Proof

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Motivation 1 : Undirected = Orientation + Directed

Theorem (Tutte, Nash-Williams 1961)

Let G be an undirected graph and k a positive integer.

- *There exists a packing of k spanning trees in G* $\Longleftrightarrow$
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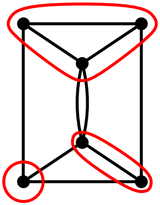

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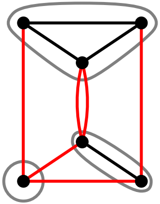

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Theorem (Edmonds 1973)

Let D be an directed graph, s a vertex of D and k a positive integer.

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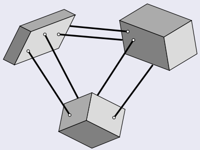
Motivation 2 : Rigidity

Body-Bar Framework



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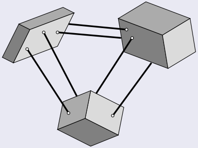

Theorem (Tay 1984)

"Rigidity" of a Body-Bar Framework can be characterized by the existence of a spanning tree decomposition.

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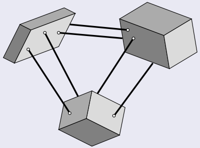
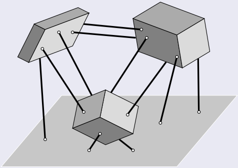
Body-Bar Framework



Body-Bar Framework with Bar-Boundary



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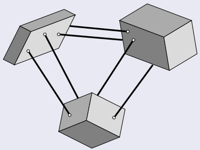

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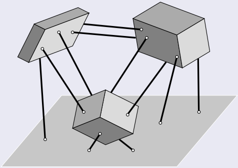


Theorem (Katoh, Tanigawa 2012)

"Rigidity" of a Body-Bar Framework with Bar-Boundary can be characterized by the existence of a matroid-based rooted-tree decomposition.

Definition

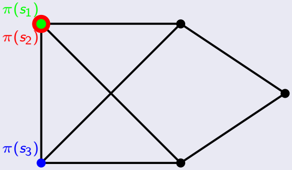
A matroid-based rooted-graph is a quadruple $(G, \mathcal{M}, S, \pi)$:

1. $G = (V, E)$ undirected graph,
2. $\mathcal{M}$ a matroid on a set $S = \{s_1, \ldots, s_t\}$.
3. π a placement of the elements of S at vertices of V.

G

$S = \{s_1, s_2, s_3\}$

$\mathcal{M} = U_{3,2}$



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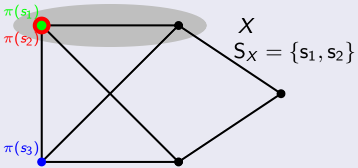

Notation

- S_X = the elements of S placed at X $(= \pi^{-1}(X))$.

$\mathcal{M}$-based packing of rooted-trees

Definition

A packing $\{T_1, \ldots, T_{|S|}\}$ of rooted-trees is called $\mathcal{M}$-based if

1. s_i is the root of T_i for every $s_i \in S$,
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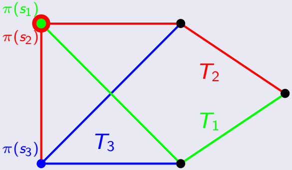

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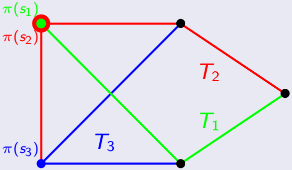

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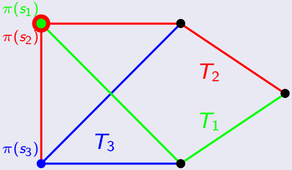

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1. π is $\mathcal{M}$-independent if for every $v \in V$, S_v is independent in $\mathcal{M}$.
2. $(G, \mathcal{M}, \mathrm{S}, \pi)$ is partition-connected if for every partition $\mathcal{P}$ of V,
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Theorem (Katoh, Tanigawa 2012)

Let $(G, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-graph.

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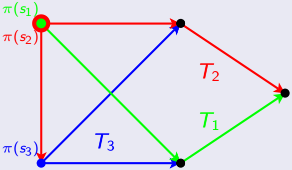

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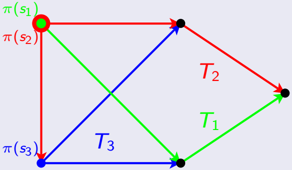

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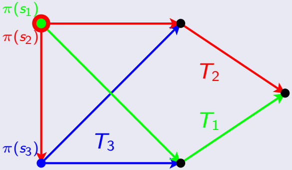

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$(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected if for every $\emptyset \neq X \subseteq V$,

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A packing $\{T_1, \dots, T_{|\mathrm{S}|}\}$ of arborescences is called $\mathcal{M}$-based if

1. s_i is the root of T_i for every $\mathrm{s}_i \in \mathrm{S}$,
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$(D, \mathcal{M}, \mathrm{S}, \pi)$ is rooted-connected if for every $\emptyset \neq X \subseteq V$,

$$\rho_D(X) \geq r_{\mathcal{M}}(\mathrm{S}) - r_{\mathcal{M}}(\mathrm{S}_X).$$

Theorem (O. Durand de Gevigney, V. H. Nguyen, Z. Szigeti 2012)

Let $(D, \mathcal{M}, \mathrm{S}, \pi)$ be a matroid-based rooted-digraph.

- *There is a matroid-based packing of arborescences in $(D, \mathcal{M}, \mathrm{S}, \pi)$*

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Proof of necessity

- Let $\{T_1, \dots, T_{|S|}\}$ be a matroid-based packing of arborescences in $(D, \mathcal{M}, S, \pi)$ and $v \in X \subseteq V$.
- Let $B = \{s_i \in S : v \in V(T_i)\}$, $B_1 = B \cap S_X$ and $B_2 = B \setminus B_1$.
- Since $S_v \subseteq B_1 \subseteq B$ is a base of $\mathcal{M}$, π is $\mathcal{M}$-independent.
- Since, for each root s_i in B_2, there exists an arc of T_i that enters X and the arborescences are arc-disjoint, $\rho_D(X) \geq |B_2| = |B| - |B_1| = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(B_1) \geq r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X)$ that is $(D, \mathcal{M}, S, \pi)$ is rooted-connected.

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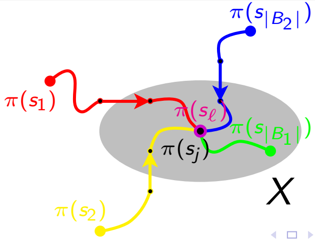

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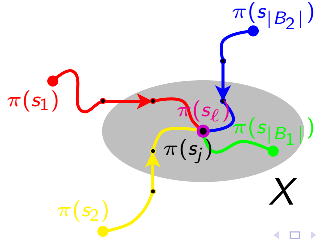

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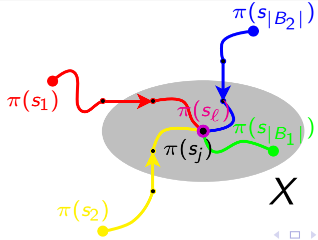

Orientation results

Theorem (Frank 1980)

Let $G = (V, E)$ be an undirected graph and $h : 2^V \to \mathbb{Z}_+$ an intersecting supermodular non-increasing set-function.

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2. A vertex set Y dominates a vertex set X if $\mathsf{S}_X \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_Y)$. (Note that domination is a transitive relation.)
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Remark

Only good arcs uv can be used in an arborescence rooted at u, since there must exist $\mathsf{s} \in \mathsf{S}_u$ such that $\mathsf{S}_v \cup \mathsf{s}$ is independent in $\mathcal{M}$.

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Every vertex v of a tight set X containing only bad arcs dominates X.

Proof

1. Let Y be the set of vertices from which v is reachable in $D[X]$.
2. v dominates Y : Since domination is transitive, v dominates each vertex of Y and hence Y.
3. Y dominates X : Using that every arc of D that enters Y enters X.

A matroid-based packing of arborescences in $(D, \mathcal{M}, \mathrm{S}, \pi)$

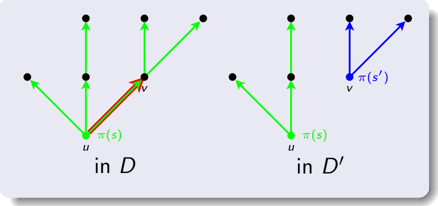
1. Take $|\mathrm{S}_v|$ times each vertex v.
2. S_v is a spanning set of $\mathcal{M}$ for all $v \in V$ by Claim since V is tight,
3. S_v is independent in $\mathcal{M}$ for all $v \in V$ since π is $\mathcal{M}$-independent.

Proof of sufficiency : Case 2 (Good arcs exist.)

Definition

For $uv \in A, \mathsf{s} \in S_u$, let

$$\begin{aligned} D' &= D - uv, \\ \mathsf{S}' &= \mathsf{S} \cup \mathsf{s}', \\ \pi'|\mathsf{S} &= \pi;\ \pi(\mathsf{s}') = v, \\ \mathcal{M}'|\mathsf{S} &= \mathcal{M};\ \mathsf{s}' \text{ parallel to } \mathsf{s}. \end{aligned}$$



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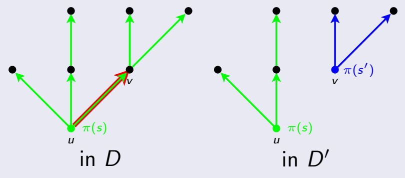

Remarks

1. Packing containing uv in $(D, \mathcal{M}, \mathsf{S}, \pi) \iff$ Packing in $(D', \mathcal{M}', \mathsf{S}', \pi')$
2. π' is $\mathcal{M}'$-independent $\iff \pi$ is $\mathcal{M}$-independent and $\mathsf{s} \notin Span(\mathsf{S}_v)$
3. $(D', \mathcal{M}', \mathsf{S}', \pi')$ is rooted-connected $\iff (D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected and uv does not enter a tight set X that dominates u.

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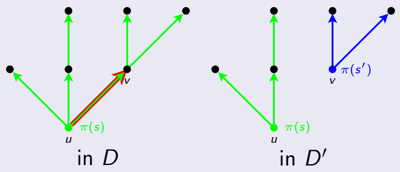

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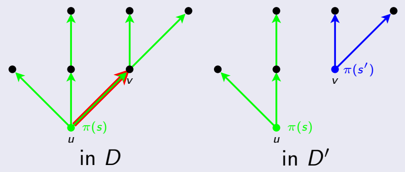

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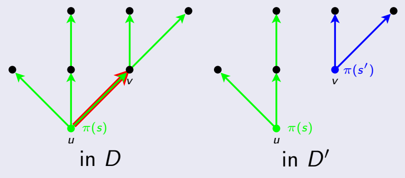

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Proof of sufficiency : Case 2 (Good arcs exist.)

Proof :

1. Wlog. each good arc uv enters a tight set X that dominates u.
2. Choose (uv, X) with X minimal.
3. X dominates u, v does not dominate u so v does not dominate X.
4. By Claim, there exists a good arc $u'v'$ in $D[X]$.
5. $u'v'$ enters a tight set Y that dominates u'.
6. $u'v'$ enters the tight set $X \cap Y$ that dominates u'.
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Thank you for your attention !