On packing of arborescences

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Joint work with:
Olivier Durand de Gevigney and Viet Hang Nguyen (Grenoble)
Motivations
- Undirected = Orientation + Directed
- Rigidity

Results
- Undirected : Matroid-based packing of rooted-trees
- Directed : Matroid-based packing of arborescences
- Orientation : Supermodular function

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Let $G$ be an undirected graph and $k$ a positive integer.

- There exists a packing of $k$ spanning trees in $G$.
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\[ e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1). \]
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Let $D$ be an directed graph, $s$ a vertex of $D$ and $k$ a positive integer.
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**Theorem (Katoh, Tanigawa 2012)**

"Rigidity" of a Body-Bar Framework with Bar-Boundary can be characterized by the existence of a matroid-based rooted-tree decomposition.
Matroid-based rooted-graphs

Definition

A matroid-based rooted-graph is a quadruple $(G, \mathcal{M}, S, \pi)$:

1. $G = (V, E)$ undirected graph,
2. $\mathcal{M}$ a matroid on a set $S = \{s_1, \ldots, s_t\}$.
3. $\pi$ a placement of the elements of $S$ at vertices of $V$.

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Notation:

\(S_X = \{s_1, s_2\}\)
A packing \( \{ T_1, \ldots, T_{|S|} \} \) of rooted-trees is called \( \mathcal{M}\)-based if:

1. \( s_i \) is the root of \( T_i \) for every \( s_i \in S \),
2. \( \{ s_i \in S : v \in V(T_i) \} \) forms a base of \( \mathcal{M} \) for every \( v \in V \).
\( M \)-based packing of rooted-trees

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\[\pi(s_1) \quad \pi(s_2) \quad \pi(s_3)\]

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\[ \pi \text{ is } \mathcal{M} \text{-independent if for every } v \in V, S_v \text{ is independent in } \mathcal{M}. \]

\[ (G, \mathcal{M}, S, \pi) \text{ is partition-connected if for every partition } \mathcal{P} \text{ of } V, \]
\[ e_G(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} (r_M(S) - r_M(S_X)). \]
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**Theorem (Katoh, Tanigawa 2012)**

Let \((G, \mathcal{M}, S, \pi)\) be a matroid-based rooted-graph.

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![Diagram of arborescences](image)
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Proof of necessity

Let \( \{T_1, \ldots, T_{|S|}\} \) be a matroid-based packing of arborescences in 
\((D, \mathcal{M}, S, \pi)\) and \( v \in X \subseteq V \).

**Let** \( B = \{s_i \in S : v \in V(T_i)\} \), \( B_1 = B \cap S \) and \( B_2 = B \setminus B_1 \).

**Since** \( S_v \subseteq B_1 \subseteq B \) is a base of \( \mathcal{M} \), \( \pi \) is \( \mathcal{M} \)-independent.

**Since**, for each root \( s_i \) in \( B_2 \), there exists an arc of \( T_i \) that enters \( X \) and the arborescences are arc-disjoint,

\[ \rho_D(X) \geq |B_2| = |B| - |B_1| = r_M(S) - r_M(B_1) \geq r_M(S) - r_M(S \setminus X) \]

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Theorem (Frank 1980)

Let $G = (V, E)$ be an undirected graph and $h : 2^V \rightarrow \mathbb{Z}_+$ an intersecting supermodular non-increasing set-function.

- There is an orientation $D$ of $G$ s.t. $\rho_D(X) \geq h(X)$ $\forall \emptyset \neq X \subset V$
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Z. Szigeti (G-SCOP, Grenoble)  
On packing of arborescences  
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Definitions for the Proof

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1. A vertex set $X$ is **tight** if $\rho_D(X) = r_M(S) - r_M(S_X)$.

2. A vertex set $Y$ **dominates** a vertex set $X$ if $S_X \subseteq \text{Span}_M(S_Y)$.
   (Note that domination is a transitive relation.)

3. An arc $uv$ is **good** if $v$ does not dominate $u$, otherwise it is **bad**.
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Remark

Only good arcs $uv$ can be used in an arborescence rooted at $u$, since there must exist $s \in S_u$ such that $S_v \cup s$ is independent in $M$. 
Proof of sufficiency: Case 1 (No good arc exists.)

Claim

Every vertex \( v \) of a tight set \( X \) containing only bad arcs dominates \( X \).
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Every vertex $v$ of a tight set $X$ containing only bad arcs dominates $X$.

Proof

1. Let $Y$ be the set of vertices from which $v$ is reachable in $D[X]$.
2. $v$ dominates $Y$: Since domination is transitive, $v$ dominates each vertex of $Y$ and hence $Y$.
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$$r_M(S) - r_M(S_Y) \leq \rho(Y) \leq \rho(X) = r_M(S) - r_M(S_X) \leq r_M(S) - r_M(S_Y).$$
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1. Take $|S_v|$ times each vertex $v$.
2. $S_v$ is a spanning set of $M$ for all $v \in V$ by Claim since $V$ is tight,
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Proof of sufficiency: Case 2 (Good arcs exist.)

**Definition**

For $uv \in A$, $s \in S_u$, let

- $D' = D - uv$,
- $S' = S \cup s'$,
- $\pi'|S = \pi; \pi(s') = v$,
- $\mathcal{M}'|S = \mathcal{M}$; $s'$ parallel to $s$. 

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Remarks

1. Packing containing $uv$ in $(D, \mathcal{M}, S, \pi)$ $\iff$ Packing in $(D', \mathcal{M}', S', \pi')$

2. $\pi'$ is $\mathcal{M}'$-independent $\iff$ $\pi$ is $\mathcal{M}$-independent and $s \notin \text{Span}(S_v)$

3. $(D', \mathcal{M}', S', \pi')$ is rooted-connected $\iff$ $(D, \mathcal{M}, S, \pi)$ is rooted-connected and $uv$ does not enter a tight set $X$ that dominates $u$. 

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Z. Szigeti (G-SCOP, Grenoble)
On packing of arborescences
November 2012
Proof of sufficiency: Case 2 (Good arcs exist.)

**Definition**

For $uv \in A, s \in S_u$, let

- $D' = D - uv$,
- $S' = S \cup s'$,
- $\pi'|S = \pi; \pi(s') = v$,
- $\mathcal{M}'|S = \mathcal{M}; s'$ parallel to $s$.

**Remarks**

1. Packing containing $uv$ in $(D, \mathcal{M}, S, \pi) \iff$ Packing in $(D', \mathcal{M}', S', \pi')$
2. $\pi'$ is $\mathcal{M}'$-independent $\iff$ $\pi$ is $\mathcal{M}$-independent and $s \notin \text{Span}(S_v)$
3. $(D', \mathcal{M}', S', \pi')$ is rooted-connected $\iff$ $(D, \mathcal{M}, S, \pi)$ is rooted-connected and $uv$ does not enter a tight set $X$ that dominates $u$. Z. Szigeti (G-SCOP, Grenoble)
Proof of sufficiency : Case 2 (Good arcs exist.)

Proof:

1. Wlog. each good arc $uv$ enters a tight set $X$ that dominates $u$.
2. Choose $(uv, X)$ with $X$ minimal.
3. $X$ dominates $u$, $v$ does not dominate $u$ so $v$ does not dominate $X$.
4. By Claim, there exists a good arc $u'v'$ in $D[X]$.
5. $u'v'$ enters a tight set $Y$ that dominates $u'$.
6. $u'v'$ enters the tight set $X \cap Y$ that dominates $u'$.
7. Contradiction.
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Thank you for your attention!