MINIMUM SIZE 2-EDGE-CONNECTED SPANNING SUBGRAPH PROBLEM

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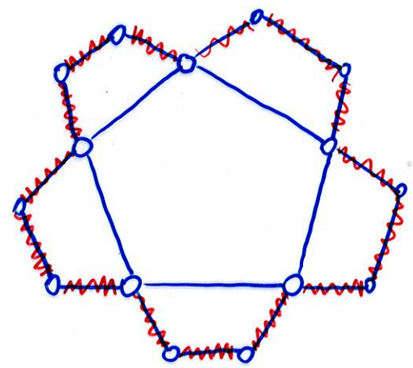
PROBLEM: MINIMUM 2-ECSS

FIND A 2-EDGE-CONNECTED SPANNING SUBGRAPH OF MINIMUM SIZE IN A 2-EDGE-CONNECTED GRAPH.

EXAMPLE:



THM: MIN. 2-ECSS IS NP-HARD.

AIM: APPROXIMATION ALGORITHM

α-APPROXIMATION ALGORITHM WITH α AS SMALL AS POSSIBLE

NEED

- POLYNOMIAL ALGORITHM
- LOWER BOUND ON THE OPTIMUM

2 - APPROXIMATION ALGORITHMS

DEF: DEGREE LOWER BOUND

$$|E_{OPT}| \geq n, \qquad n = |V|.$$

(G IS 2-E-C $\Rightarrow$

$d(v) \geq 2 \quad \forall v \in V \Rightarrow$

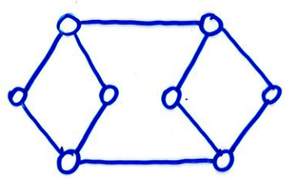
$|E_{OPT}| \geq \frac{2 \cdot n}{2} \cdot n$)

ALL WE NEED: POLYNOMIAL ALGORITHM PROVIDING A SOLUTION WITH $\leq 2 \cdot n$ EDGES.

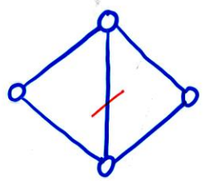
1.

DEF: A 2-EC GRAPH G IS MINIMAL IF EACH EDGE IS CRITICAL THAT IS IF $\forall e \in E(G)$ $G-e$ IS NOT 2-EC.

EXAMPLE:



MIN. 2-EC



NOT MIN. 2-EC

ALGORITHM:

FOR EACH EDGE, CHECK IT IF IT IS CRITICAL. IF NOT, DELETE IT.

THM: (MADER)

EVERY MINIMAL 2-EC GRAPH HAS AT MOST $2n$ EDGES.

2.

ALGORITHM: (NAGAMOCHI - IBARAKI, NISHIZEKI - POLJAK)

$F_1 :=$ SPANNING TREE OF G.

$F_2 :=$ SPANNING FOREST OF $G - F_1$.

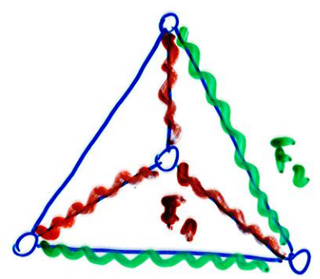
$E' := F_1 \cup F_2$.

THM: (V, E') IS A 2-ECSS OF G.

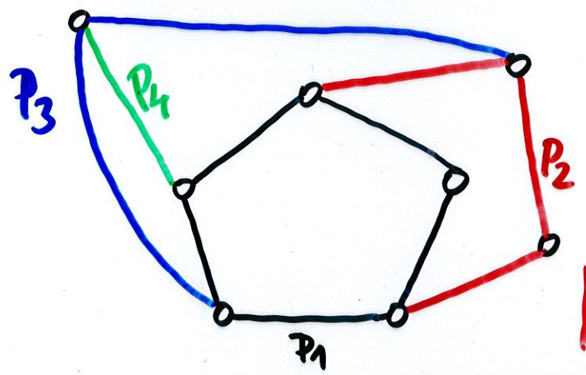
APPROXIMATION GUARANTEE:

$|E'| = |F_1| + |F_2| \leq (n-1) + (n-1) < 2 \cdot n$.

EX.:



EAR-DECOMPOSITION



$\Leftrightarrow$ G IS 2-E-C (WHITNEY)

$\#EARS = m - n + 1$

- $\varphi(G)$ = MIN // EVEN EARS IN AN EAR-DECOMP.
- EAR-DECOMP. OPTIMAL IF IT HAS $\varphi(G)$ EVEN EARS
- IT CAN BE COMPUTED IN POLY. TIME (FRANK)

$\varphi(G) = 0 \Leftrightarrow G$ FACTOR-CRITICAL (LOVÁSZ)

$\varphi(G) = \text{MIN}\{|F| : G/F \text{ FACTOR-CRITICAL}\}$ (FRANK)

MINIMAL CRITICAL-MAKING EDGE-SETS FORM THE BASIS OF A MATROID (E.G.)

3.

THM: (WHITNEY)

G IS 2-EC $\Leftrightarrow$ G HAS AN EAR-DECOMPOSITION.

ALGORITHM:

- TAKE AN EAR-DECOMPOSITION OF G.
- DELETE THE 1-EARS.

m' := # OF (REMAINING) EDGES.

$$m' = n + \#\,\text{EARS} - 1$$

APPROXIMATION GUARANTEE:

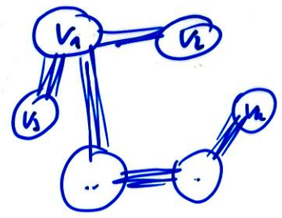
$$m' = n + \#\,\text{EARS} - 1 \leq n + n - 1 - 1 < 2n$$

(SINCE EACH EAR CONTAINS AT LEAST 1 INNER VERTICES)

3/2 - APPROXIMATION ALGORITHMS

(1)

DEF: TREE CARVING PARTITION IS A PARTITION $\{V_1, \dots, V_k\}$ SO THAT $G/V_1/\dots/V_k$ CONTAINS NO CYCLES OF LONGER >2.



DEF: TREE CARVING LOWER BOUND

IF $\{V_1, \dots, V_k\}$ IS A TREE CARVING PARTITION, THEN

$$|E_{OPT}| \geq 2(k-1).$$

ALGORITHM (SKETCH) (KHULLER, VISHKIN)

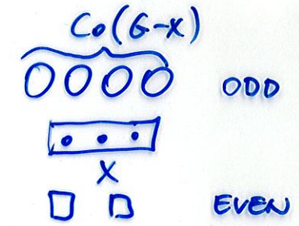
- DFS TREE F
- $F' \subseteq E-F$ (COMPUTED DURING DFS): $(V, \overbrace{F \cup F'}^{E'})$ IS A 2-ECSS
- TREE CARVING PARTITION WITH $|F'|+1$ MEMBERS.

APPROXIMATION GUARANTEE:

$$\frac{|E'|}{|E_{OPT}|} = \frac{|F|+|F'|}{|E_{OPT}|} \leq \frac{(n-1)+|F'|}{\max(n, 2|F'|)} \leq \frac{n-1}{n} + \frac{|F'|}{2|F'|} < \boxed{\frac{3}{2}}.$$

2.

DEF: $def(G) := \max_{X \subset V} \left(c_o(G-X) - |X| \right).$



DEF: DEFICIENCY LOWER BOUND

$$|E_{OPT}| \geq n + def(G) - 1.$$

REMARK: WE MAY SUPPOSE: G IS 2-VERTEX-CONNECTED

ALGORITHM: (CHERIYAN, THURIMELLA)

$F :=$ MINIMUM EDGE COVER $\left(|F| = \frac{n + def(G)}{2}\right)$

$F' :=$ MINIMAL EDGE SET IN $E-F$: $(V, \overbrace{F \cup F'}^{E'})$ IS 2-VERTEX-CONNECTED

THM: (MADER)

F' IS ACYCLIC, i.e. $|F'| \leq n-1$.

APPROXIMATION GUARANTEE:

$$\frac{|E'|}{|E_{OPT}|} = \frac{|F| + |F'|}{|E_{OPT}|} \leq \frac{(n-1) + \frac{n + def(G)}{2}}{\max(n, n + def(G) - 1)} \leq$$

$$\leq \frac{n}{n} + \frac{1}{2} \cdot \frac{n + def(G) - 2}{n + def(G) - 1} < 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$$

3.

DEF: **FOR A 2-EC GRAPH G,**

$\varphi(G)$:= MINIMUM NUMBER OF EVEN EARS IN AN EAR-DECOMPOSITION OF G.

AN EAR-DECOMPOSITION IS OPTIMAL IF IT CONTAINS $\varphi(G)$ EVEN EARS.

THM: **(FRANK)** AN OPTIMAL EAR-DECOMPOSITION CAN BE CONSTRUCTED IN POLYNOMIAL TIME.

REMARK:

MIN. 2-ECSS PROBLEM = FIND AN EAR-DEC. WITH MAXIMUM NUMBER OF 1-EARS.

RELAXATION: FIND AN EAR-DECOMPOSITION WITH MAXIMUM NUMBER OF ODD EARS.

$\Vert$

FIND AN EAR-DECOMPOSITION WITH MINIMUM NUMBER OF EVEN EARS.

DEF: φ-LOWER BOUND

$$|E_{OPT}| \geq n+\varphi-1.$$

$$\left(|E_{OPT}| = n + \#\text{ears} - 1 \geq n + \varphi(G_{OPT}) - 1 \geq n+\varphi-1.\right)$$

ALGORITHM: (CHERIYAN, SEBŐ, Sz.)

- TAKE AN OPTIMAL EAR-DECOMPOSITION OF G.
- DELETE THE 1-EARS.

APPROXIMATION GUARANTEE:

$$\frac{|E'|}{|E_{OPT}|} = \frac{n+\#\text{ears}-1}{|E_{OPT}|} = \frac{n + \overbrace{\#\text{odd ears}}^{\leq n/2} + \overbrace{\#\text{even ears}}^{=\varphi} - 1}{|E_{OPT}|} \leq$$

$$\leq \frac{n/2 + n+\varphi-1}{\max(n, n+\varphi-1)} \leq \frac{n/2}{n} + \frac{n+\varphi-1}{n+\varphi-1} = \frac{1}{2}+1 = \boxed{\frac{3}{2}}.$$

EVEN BETTER APPROXIMATION ALGORITHM:

DEF.: COMPONENT LOWER BOUND

$$|E_{OPT}| \geq n - |S| + c(G-S) \qquad S \neq \emptyset.$$

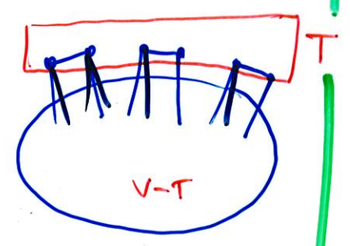
ASSUMPTION: G IS 2-VERTEX-CONNECTED.

ALGORITHM: (CHERIYAN, SEBŐ, SZ.)

- TAKE AN OPEN OPTIMAL EAR-DECOMPOSITION.
- USING EAR SPLICING STEPS, FIND AN OPTIMAL EAR-DECOMPOSITION SO THAT

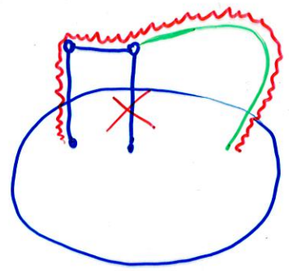
(T := SET OF INNER VERTICES OF THE 3-EARS)

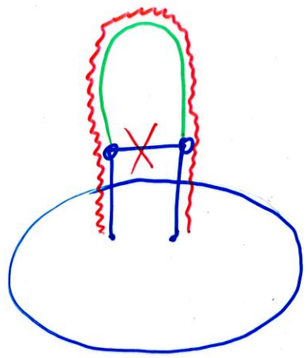
IF AN EAR P STARTS IN $u \in T$ THEN $P = uv$ (1-EAR) AND $v \notin T$.

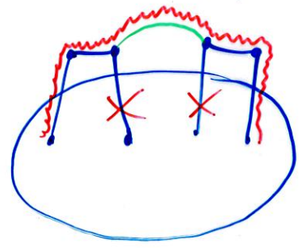


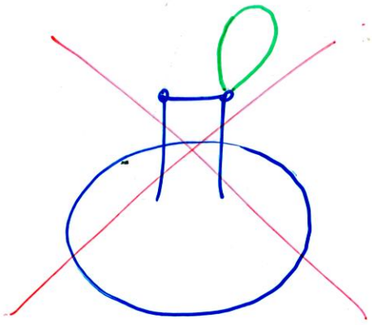
- DELETE 1-EARS.

EAR SPLICING:









FACT:

$$|E_{OPT}| \geq 3 \cdot \ell, \qquad \ell := \#\,3\text{-EARS}$$

$$\left(|E_{OPT}| \geq n - |V-T| + c(G-(V-T)) = n - (n-2\ell) + \ell = 3\ell \right)$$

APPROXIMATION GUARANTEE:

$$\frac{|E'|}{|E_{OPT}|} = \frac{n + \#\text{ears} - 1}{|E_{OPT}|} = \frac{n + \overbrace{(\#\text{even ears})}^{=\varphi} + \overbrace{(\#\,3\text{-ears})}^{=\ell} + \overbrace{(\#\geq 5\text{-ODD ears})}^{\leq \frac{n-2\ell-1}{4}} - 1}{|E_{OPT}|} \leq$$

$$\leq \frac{\frac{5}{4}(n+\varphi-1) + \frac{\ell}{2}}{\max(n+\varphi-1,\ 3\ell)} \leq \frac{5}{4}\underbrace{\frac{(n+\varphi-1)}{n+\varphi-1}} + \frac{\frac{\ell}{2}}{3\ell} = \frac{5}{4} + \frac{1}{6} = \boxed{\frac{17}{12}}.$$

CONCLUSION:

DEF: A SET OF CUTS OF G IS A 2-PACKING OF CUTS IF EACH EDGE OF G BELONGS TO AT MOST 2 OF THEM.

DEF: 2-PACKING LOWER BOUND

IF $\mathcal{P}$ IS A 2-PACKING OF CUTS, THEN

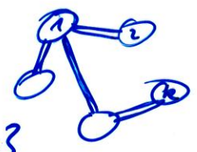
$$|E_{OPT}| \geq |\mathcal{P}|.$$

ALL THE EARLIER BOUNDS ARE SPECIAL CASES OF THIS BOUND.

DEGREE BOUND: $\mathcal{P} := \{\delta(v) : v \in V\}$

$|\mathcal{P}| = n$

TREE CARVING BOUND:



$\mathcal{P} := \{2 \times$ FUNDAMENTAL CUTS OF THE TREE$\}$

$|\mathcal{P}| = 2k-2$

DEFICIENCY BOUND

SPECIAL CASE OF COMPONENT BOUND

COMPONENT BOUND:

$$\mathcal{P} := \{\delta(v) : v \in V - S\} \cup \{\delta(K) : K \text{ COMPONENT OF } G-S\}$$

$$|\mathcal{P}| = n - |S| + c(G-S)$$

φ- BOUND:

$$\max_T \Big(\max_{\substack{\text{2-PACKING} \\ \text{OF T-CUTS}}} |\mathcal{P}_T| \Big) \overset{\text{LOVÁSZ}}{=} 2 \cdot \max_T \Big(\min_{\text{T-JOIN}} |F_T| \Big) =$$

$$= 2 \cdot \frac{n+\varphi-1}{2} = n+\varphi-1$$

$\uparrow$ FRANK