A NOTE
ON PACKING PATHS
IN PLANAR GRAPHS

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EDGE-DISJOINT PATHS PROBLEM

GIVEN A GRAPH $G$ AND $s_i, t_i \in V(G)$ $i = 1, \ldots, l$.

FIND $l$ EDGE-DISJOINT PATHS CONNECTING THE CORRESPONDING PAIRS $s_i, t_i$.

$G$ SUPPLY GRAPH

$H$ DEMAND GRAPH

FIND $l$ EDGE-DISJOINT CIRCUITS IN $G+H$ SUCH THAT EACH OF THEM CONTAINS EXACTLY ONE DEMAND EDGE.
BY PLANAR DUALIZATION:

FIND \( \ell \) EDGE-DISJOINT CUTS in \( G' \) (DUAL OF \( G+H \)) SUCH THAT EACH OF THEM CONTAINS EXACTLY ONE EDGE OF \( F \) (CORRESPONDING TO \( H \)).
G+H is planar

Even in this case the problem is NP-complete.
(Middendorf, Pfeiffer '93)

Necessary:
Cut criterion

\[ d_G(x) \geq d_H(x) \quad \forall x \in V(G) \]

Not sufficient:
**Theorem (Seymour '81)**

When $G+H$ is planar and Eulerian, the edge-disjoint paths problem has a solution if and only if the cut criterion holds.

**Theorem (Korach, Penn '92)**

Suppose $G+H$ is planar and the cut criterion holds. Then there is at most one demand edge on each bounded face of $G$ so that leaving out these edges from $H$, the problem has a solution.
REMARKS: 1. NO DEMAND EDGE IS LEFT OUT FROM THE INFINITE FACE OF G.

2. IF EACH BOUNDED FACE OF G CONTAINS AT MOST ONE DEMAND EDGE THEN THIS THEOREM SAYS NOTHING.

OUR AIM IS TO STRENGTHEN THE CUT CRITERION

a.) TO BE A SUFFICIENT CONDITION FOR THE PROBLEM.

b.) TO HAVE 1. FOR MORE FACES OF G.
\( d_G(x) \geq d_H(x) \quad \forall x \in V \)

is not sufficient

\( d_G(x) \geq d_H(x) + K \quad \forall x \in V \)

is not sufficient

\[
d_G(x) - d_H(x) = (k+1)2k - (2k^2 - 1 + k + 1) = k
\]
\textbf{THEOREM} (FRANK, SZ. '93)

\textbf{If} $G \oplus H$ \textbf{is planar and}

\[ d_G(x) \geq 2 \cdot d_H(x) \quad \forall x \in V \]

\textbf{then the problem has a solution.}

\textbf{I\textit{t\textbf{.}}}} \quad d_G(x) \geq (2-\varepsilon) \cdot d_H(x) \quad \forall x \in V, \quad \varepsilon > 0

\textbf{is not sufficient}

\[ d_G(x) = 2 \cdot k \quad \Rightarrow \quad d_G(x) = (2 - \frac{2}{k+1}) d_H(x) \]

\[ d_H(x) = k+1 \]
b) Suppose, that the faces of $\sigma$ containing demand edges are partitioned into two groups:

$$C_0, C_1, \ldots, C_k \quad (k \geq 0)$$

$$D_0, \ldots, D_\ell$$

For a cut $\delta_{G+H}(x)$ let $\mu(x)$ be the number of those faces $C_i$ ($i \geq 1$) from which $\delta_{G+H}(x)$ contains at least one demand edge.

**Theorem (Frank, Sz. '93)**

If $G+H$ is planar and

$$d_G(x) \geq d_H(x) + \mu(x) \quad \forall x \in V$$

then it is possible to leave out at most one demand edge from each face $D_0, \ldots, D_\ell$ such that the resulting problem has a solution.
THEOREM: IF $G+H$ IS PLANAR AND
\[d_G(x) \geq d_H(x) + \mu(x) \quad \forall x \in V\]
THEN THE EDGE-DISJOINT PATHS PROBLEM
HAS A SOLUTION.

REMARK: \[d_G(x) \geq d_H(x) + \mu(x) - 2 \quad \forall x \in V\]
IS NOT SUFFICIENT.

OPEN PROBLEM:
\[d_G(x) \geq d_H(x) + \mu(x) - 1 \quad \forall x \in V\]
IS SUFFICIENT OR NOT?