Covering symmetric skew-supermodular functions

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Outline

What is the problem ?

Questions

1. What is a skew-supermodular function ?
2. What is a covering of a function ?
3. Why to cover a function ?

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2. What is a covering of a function ?
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Definition : symmetric skew-supermodular function

Symmetric function

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Skew-supermodular function

$p : 2^V \to \mathbb{R}$ is called **skew-supermodular** if $\forall X, Y \subset V$:

- either $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$,
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Examples : symmetric skew-supermodular function

Well-known examples

1. $p(X) = k$,
2. $R(X) = \max\{r(x,y) : x \in X, y \in V - X\}$, $(r : V \times V \to \mathbb{R})$,
3. $p(X) = k - d_G(X)$, ($d_G(X)$ is the degree function of a graph),
4. $p(X) = R(X) - d_G(X)$,
5. $p(X) = p'(X) - d_G(X)$, ($p'(X)$ is a symmetric skew-supermodular function).

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Definition : covering a function

Covering

A graph $H = (V, F)$ **covers** a function $p : 2^V \to \mathbb{R}$ if

$$d_H(X) \geq p(X) \quad \forall X \subset V.$$

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Minimization problem

Given a symmetric skew-supermodular function p on V, what is the minimum number of edges of a graph $H = (V, F)$ that covers p ?

Motivation

Global edge-connectivity augmentation of a graph

- Given a graph $G = (V, E)$ and an integer k, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph ?
- $\gamma := \min\{|F| : d_{G+F}(X) \geq k \ \forall \emptyset \neq X \subset V\}$
  $= \min\{|F| : d_{(V,F)}(X) \geq k - d_G(X) \ \forall \emptyset \neq X \subset V\}$.
- As the function $p(X) = k - d_G(X)$ is symmetric skew-supermodular, this is a special case of our problem.

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Definition : local edge-connectivity

$\lambda(u, v)$

Given a graph $G = (V, E)$ and $u, v \in V$, we define the local edge-connectivity between u and v as follows :

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r-edge-connected graph

Given a graph $G = (V, E)$ and a function $r : V \times V \to \mathbb{R}$, we say that G is **r**-edge-connected if

$$\lambda(u, v) \geq r(u, v) \quad \forall\, u, v \in V.$$

Motivation 2

Local edge-connectivity augmentation of a graph

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  where $\mathbf{R(X)} = \max\{r(x, y) : x \in X, y \in V - X\}$.
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Negative Result

Minimum Cover of a Symmetric Skew-Supermodular Function by a Graph

Instance : $p : 2^V \to \mathbb{Z}$ symmetric skew-supermodular, $\gamma \in \mathbb{Z}^+$.
Question : Does there exist a graph on V with at most γ edges that covers p?

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The above problem is NP-complete, even if p is $0 - 1$ valued.

Known Results

Global edge-connectivity augmentation of a graph

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$p(X) = R(X) - d_{\mathcal{G}}(X)$: NP-complete (Cosh, Jackson, Z. Király)

New Results

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$p(X) = Q(X) - d_G(X)$, where $Q(X) = k$ if $|X|$ is odd, 0 otherwise ($|V|$ even) : Polynomially solvable (Sz.)

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$p(X) = R'(X) - d_G(X)$:

- NP-complete in general,
- Polynomially solvable if $R'(X) \neq 1 \ \forall X$.

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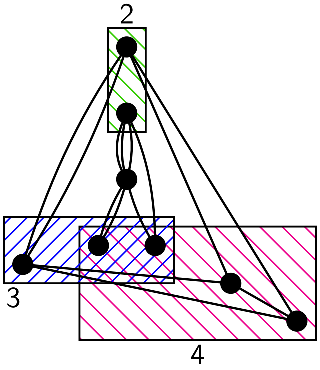
(Ishii, Hagiwara ; simple proof : Grappe, Sz.)

Definition of a node to area connected graph

Definition

Given a graph $G = (V, E)$, a set $\mathcal{A}$ of subsets of V (areas), $r : \mathcal{A} \rightarrow \mathbb{Z}^+$, G is node to area r-edge-conneted if there exist $r(A)$ edge disjoint paths between each area A and each node $v \notin A$.



Reformulation

Cut condition (Menger)

G is node to area r-edge-connected if and only if $\forall X \subset V$,

$$d_G(X) \geq R'(X),$$

where $R'(X) = \max\{r(A), A \in \mathcal{A}, A \subseteq X \text{ or } A \cap X = \emptyset\}$.

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Property of R'

R is symmetric skew-supermodular. In fact, it is **semi-monotone**, that is $\forall X \subset V$, $R'(X) \leq R'(X')$:

- either $\forall X' \subseteq X$,
- or $\forall X' \subseteq V - X$.

Augmentation problem

Node to area global edge-connectivity augmentation

- Given a graph $G = (V, E)$, a set $\mathcal{A}$ of areas with a function $r : \mathcal{A} \to \mathbb{Z}^+$, what is the minimum number γ of new edges whose addition results in a node to area r-edge-connected graph ?
- $\gamma := \min\{|F| : \lambda_{G+F}(v, A) \geq r(v, A)\ \forall A \in \mathcal{A}, v \notin A\}$
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General case

NODE TO AREA GLOBAL EDGE-CONNECTIVITY AUGMENTATION

Instance : Graph G, areas $\mathcal{A}$, $r : \mathcal{A} \rightarrow \mathbb{Z}^+$, $\gamma \in \mathbb{Z}^+$.
Question : Does there exist a node to area r-edge-connected graph on V with at most γ edges ?

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Theorem (Ishii, Hagiwara)

The above problem is NP-complete, even if $r(A) = 1 \ \forall A \in \mathcal{A}$.

Simple case : $\mathbf{R'(X) \neq 1}, \forall X \subset V$

Towards a lower bound

$\Phi = \max\{\sum_{X \in \mathcal{X}} (R'(X) - d(X)) : \mathcal{X}$ sub-partition of $V\}$

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Min-max Theorem (Ishii, Hagiwara)

- $\lceil \frac{\Phi}{2} \rceil$ is a **lower bound** of the minimum number of edges of a good augmentation,
- for some graphs : **one more edge** is needed,
- **charaterization** of these graphs.

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Short proof

R. Grappe, Z. Szigeti, Covering symmetric semi-monotone functions, Discrete Applied Mathematics 156 (2008) 138-144.

Open Problems

Polinomial or NP-complete ?

- Covering a symm. skew-supermodular function, where $p(X) \neq 1 \ \forall X$.
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