Old and new results on packing arborescences

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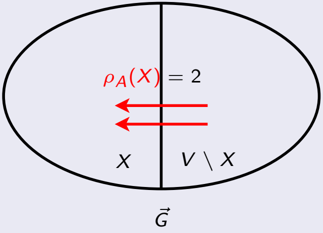
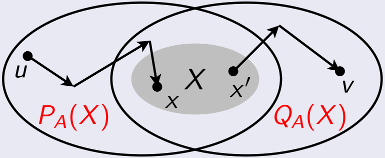
Reachability in digraph

Definition

Let $\vec{G} = (V, A)$ be a digraph and $X \subseteq V$.

1. $\rho_A(X)$ is the number of arcs entering X,
2. $P_A(X)$ is the set of vertices from which X can be reached in $\vec{G}$,
3. $Q_A(X)$ is the set of vertices that can be reached from X in $\vec{G}$.



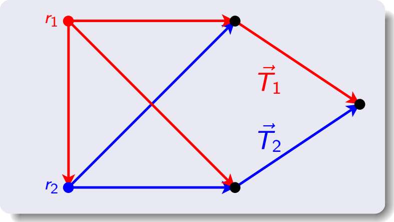


Arborescences

Definition

Let $\vec{G} = (V, A)$ be a digraph and $r \in V$.

1. A subgraph $\vec{T} = (U, B)$ of $\vec{G}$ is an *r*-arborescence if
   1. $r \in U$ with $\rho_B(r) = 0$,
   2. $\rho_B(u) = 1$ for all $u \in U \setminus r$ and
   3. $\rho_B(X) \geq 1$ for all $X \subseteq V \setminus r$, $X \cap U \neq \emptyset$.
2. An r-arborescence $\vec{T}$ is
   1. spanning if $U = V$,
   2. maximal if $U = Q_A(r)$.
3. Packing of arborescences is a set of pairwise arc-disjoint arborescences.

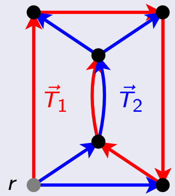


Packing spanning arborescences

Theorem (Edmonds 1973)

Let $\vec{G} = (V, A)$ be a digraph, $r \in V$ and k a positive integer.

1. *There exists a packing of k spanning r-arborescences* $\Longleftrightarrow$
2. $\rho_A(X) \geq k$ *for all* $\emptyset \neq X \subseteq V \setminus r$.

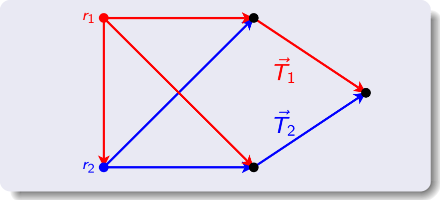


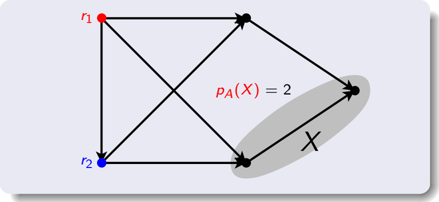
Packing maximal arborescences

Definition

Let $\vec{G} = (V, A)$ be a digraph and $(r_1, \ldots, r_t) \in V^t$.

1. A packing of maximal arborescences is a set $\{\vec{T}_1, \ldots, \vec{T}_t\}$ of pairwise arc-disjoint maximal r_i-arborescences $\vec{T}_i$ in $\vec{G}$; that is for every $v \in V$, $\{r_i : v \in V(\vec{T}_i)\} = \{r_i \in P_A(v)\}$.
2. For $X \subseteq V$, $p_A(X) = |\{r_i \in P_A(X) \setminus X\}|$.





Packing maximal arborescences

Theorem (Kamiyama, Katoh, Takizawa 2009)

Let $\vec{G} = (V, A)$ be a digraph and $(r_1, \dots, r_t) \in V^t$.

1. *There exists a packing of maximal arborescences* $\iff$
2. $\rho_A(X) \geq p_A(X)$ *for all* $X \subseteq V$.

Remark

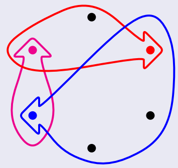
It implies Edmonds' theorem.

1. Let $r_1 = \cdots = r_k = r$.
2. $\rho_A(X) \geq k$ for all $\emptyset \neq X \subseteq V \setminus r$ implies the above condition and that each vertex is reachable from r.
3. Hence there exists a packing of maximal r-arborescences that is a packing of spanning r-arborescences.

Dypergraphs

Definition

1. Directed hypergraph (shortly dypergraph) is $\vec{\mathcal{G}} = (V, \mathcal{A})$, where
   - V denotes the set of vertices and
   - $\mathcal{A}$ denotes the set of hyperarcs of $\vec{\mathcal{G}}$.
2. Hyperarc is a pair (Z, z) such that $z \in Z \subseteq V$, where
   - z is the head of the hyperarc (Z, z) and
   - the elements of $Z \setminus z \neq \emptyset$ are the tails of the hyperarc (Z, z).

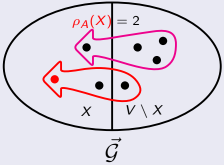


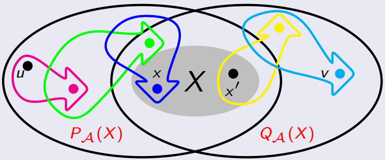
Reachability in dypergraph

Definition

Let $\vec{\mathcal{G}} = (V, \mathcal{A})$ be a dypergraph and $X \subseteq V$.

1. Hyperarc (Z, z) enters X if $z \in X$ and $(Z \setminus z) \cap (V \setminus X) \neq \emptyset$,
2. $\rho_{\mathcal{A}}(X)$ is the number of hyperarcs entering X,
3. Path from u to x in $\vec{\mathcal{G}}$ is $v_1(= u), (Z_1, v_2), v_2, \dots, v_i, (Z_i, v_{i+1}), v_{i+1}, \dots, v_j(= x)$ such that v_i is a tail of (Z_i, v_{i+1}).
4. $P_{\mathcal{A}}(X)$ is the set of vertices from which X can be reached in $\vec{\mathcal{G}}$,
5. $Q_{\mathcal{A}}(X)$ is the set of vertices that can be reached from X in $\vec{\mathcal{G}}$.

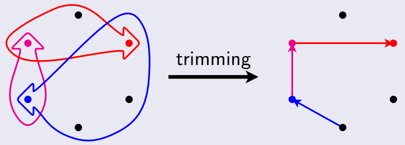




Trimming

Definition

Trimming the dypergraph $\vec{\mathcal{G}}$ means replacing each hyperarc (K, v) of $\vec{\mathcal{G}}$ by an arc uv where u is one of the tails of the hyperarc (K, v).



Definition

h is supermodular : $h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) \quad \forall\, X, Y \subseteq V$.

Theorem (Frank 2011)

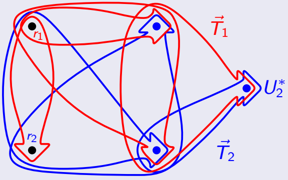
Let $\vec{\mathcal{G}} = (V, \mathcal{A})$ be a dypergraph and h an integer-valued, intersecting supermodular function on V such that $h(\emptyset) = 0 = h(V)$.
If $\rho_{\mathcal{A}}(X) \geq h(X)$ for all $X \subseteq V$, then $\vec{\mathcal{G}}$ can be trimmed to a digraph $\vec{G}$ such that $\rho_A(X) \geq h(X)$ for all $X \subseteq V$.

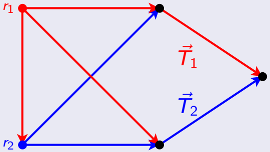
Hyper-arborescences

Definition

Let $\vec{\mathcal{G}} = (V, \mathcal{A})$ be a dypergraph and $r \in V$.

1. A subgraph $\vec{\mathcal{T}} = (U, \mathcal{B})$ of $\vec{\mathcal{G}}$ is an *r*-hyper-arborescence if it can be trimmed to an r-arborescence on $U^* \cup r$, where $U^* = \{u : \rho_{\mathcal{B}}(u) \neq 0\}$; that is
   1. $r \in U \setminus U^*$,
   2. $\rho_{\mathcal{B}}(u) = 1$ for all $u \in U^*$ and
   3. $\rho_{\mathcal{B}}(X) \geq 1$ for all $X \subseteq V \setminus r$, $X \cap U^* \neq \emptyset$.
2. The r-hyper-arborescence $\vec{T}$ is
   1. spanning if $U^* = V \setminus r$,
   2. maximal if $U^* = Q_{\mathcal{A}}(r) \setminus r$.





Packing spanning hyper-arborescences

Theorem (Frank, T. Király, Kriesell 2003)

Let $\vec{\mathcal{G}} = (V, \mathcal{A})$ be a dypergraph, $r \in V$ and k a positive integer.

1. *There exists a packing of k spanning r-hyper-arborescences* $\Longleftrightarrow$
2. $\rho_{\mathcal{A}}(X) \geq k$ *for all* $\emptyset \neq X \subseteq V \setminus r$.

Remark

1. It is proved easily by trimming and Edmonds' theorem.
2. It implies Edmonds' theorem if $\vec{\mathcal{G}}$ is a digraph.

Packing maximal hyper-arborescences

Theorem (Bérczi, Frank 2008)

Let $\vec{\mathcal{G}} = (V, \mathcal{A})$ *be a dypergraph and* $(r_1, \dots, r_t) \in V^t$.

1. *There exists a packing of maximal hyper-arborescences* $\iff$
2. $\rho_{\mathcal{A}}(X) \geq p_{\mathcal{A}}(X)$ *for all* $X \subseteq V$.

Remark

1. It is proved not easily by trimming and Kamiyama, Katoh, Takizawa's theorem since $p_{\mathcal{A}}(X)$ is not intersecting supermodular.
2. It implies
   1. Frank, T. Király, Kriesell's theorem if $r_1 = \cdots = r_k = r$ and $\rho_{\mathcal{A}}(X) \geq k$ for all $\emptyset \neq X \subseteq V \setminus r$,
   2. Kamiyama, Katoh, Takizawa's theorem if $\vec{\mathcal{G}}$ is a digraph.

Matroids

Definition

For $\mathcal{I} \subseteq 2^{\mathsf{S}}$, $\mathcal{M} = (\mathsf{S}, \mathcal{I})$ is a matroid if

1. $\mathcal{I} \neq \emptyset$,
2. If $X \subseteq Y \in \mathcal{I}$ then $X \in \mathcal{I}$,
3. If $X, Y \in \mathcal{I}$ with $|X| < |Y|$ then $\exists\ y \in Y \setminus X$ such that $X \cup y \in \mathcal{I}$.

Examples

1. Sets of linearly independent vectors in a vector space,
2. Edge-sets of forests of a graph,
3. $U_{n,k} = \{X \subseteq \mathsf{S} : |X| \leq k\}$ where $|\mathsf{S}| = n$, free matroid $= U_{n,n}$.

Matroids

Notion

1. independent sets $= \mathcal{I}$,
   1. any subset of an independent set is independent,
2. base $=$ maximal independent set,
   1. all basis are of the same size,
3. rank function : $r(X) = \max\{|Y| : Y \in \mathcal{I}, Y \subseteq X\}$.
   1. non-decreasing,
   2. submodular (that is $-r$ is supermodular),
   3. $X \in \mathcal{I}$ if and only if $r(X) = |X|$.

Matroid-based rooted-digraphs

Definition

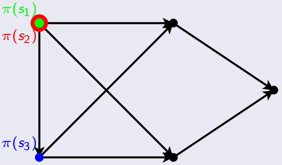
A matroid-based rooted-digraph is a quadruple $(\vec{G}, \mathcal{M}, S, \pi)$:

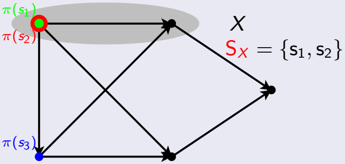
1. $\vec{G} = (V, A)$ is a digraph,
2. $\mathcal{M}$ is a matroid on a set $S = \{s_1, \ldots, s_t\}$.
3. π is a placement of the elements of S at vertices of V such that $S_v \in \mathcal{I}$ for every $v \in V$, where $S_X = \pi^{-1}(X)$, the elements of S placed at X.

$\vec{G}$

$S = \{s_1, s_2, s_3\}$

$\mathcal{M} = U_{3,2}$



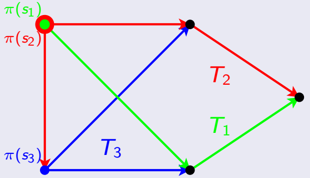


Matroid-based packing of rooted-arborescences

Definition

A rooted-arborescence is a pair $(\vec{T}, \mathrm{s})$ where

1. $\vec{T}$ is an r-arborescence for some vertex r,
2. $\mathrm{s} \in \mathrm{S}$, placed at r.



Definition

A packing $\{(\vec{T}_1, \mathrm{s}_1), \ldots, (\vec{T}_{|\mathrm{S}|}, \mathrm{s}_{|\mathrm{S}|})\}$ of rooted-arborescences is matroid-based if $\{\mathrm{s}_i \in \mathrm{S} : v \in V(\vec{T}_i)\}$ forms a base of S for every $v \in V$.

Remark

For the free matroid $\mathcal{M}$ with all k roots at a vertex r,

1. matroid-based packing of rooted-arborescences $\iff$
2. packing of k spanning r-arborescences.

Matroid-based packing of rooted-arborescences

Theorem (Durand de Gevigney, Nguyen, Szigeti 2013)

Let $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-digraph.

1. *There is a matroid-based packing of rooted-arborescences* $\iff$
2. $\rho_A(X) \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X)$ *for all* $\emptyset \neq X \subseteq V$.

Remark

It implies Edmonds' theorem if $\mathcal{M}$ is the free matroid with all k roots at the vertex r.

Maximal-rank packing of rooted-arborescences

Definition

A packing $\{(\vec{T}_1, s_1), \ldots, (\vec{T}_{|S|}, s_{|S|})\}$ of rooted-arborescences is of maximal rank if $\{s_i \in S : v \in V(\vec{T}_i)\}$ forms a base of $S_{P_A(v)}$ for every $v \in V$.

Theorem (Cs. Király 2013)

Let $(\vec{G}, \mathcal{M}, S, \pi)$ be a matroid-based rooted-digraph.

1. *There exists a maximal-rank packing of rooted-arborescences* $\iff$
2. $\rho_A(X) \geq r_{\mathcal{M}}(S_{P_A(X)}) - r_{\mathcal{M}}(S_X)$ *for all* $X \subseteq V$.

Remark

1. It implies
   1. DdG-N-Sz' theorem if $\rho_A(X) \geq r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X)$ for all $\emptyset \neq X \subseteq V$,
   2. Kamiyama, Katoh, Takizawa's theorem if $\mathcal{M}$ is the free matroid.

Matroid-based rooted-dypergraphs

Definition

A matroid-based rooted-dypergraph is a quadruple $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$:

1. $\vec{\mathcal{G}} = (V, \mathcal{A})$ is a dypergraph,
2. $\mathcal{M}$ is a matroid on a set $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_t\}$.
3. π is a placement of the elements of S at vertices of V such that $\mathsf{S}_v \in \mathcal{I}$ for every $v \in V$.

Matroid-based packing of rooted-hyper-arborescences

Definition

1. A rooted-hyper-arborescence is a triple $(\vec{\mathcal{T}}, r, \mathsf{s})$ where $\vec{\mathcal{T}}$ is an r-hyper-arborescence and s is an element of S placed at r.
2. A packing $\{(\vec{\mathcal{T}}_1, r_1, \mathsf{s}_1), \dots, (\vec{\mathcal{T}}_{|\mathsf{S}|}, r_{|\mathsf{S}|}, \mathsf{s}_{|\mathsf{S}|})\}$ of rooted-hyper-arborescences is matroid-based if $\{\mathsf{s}_i \in \mathsf{S} : v \in Q_{\mathcal{A}(\vec{\mathcal{T}}_i)}(r_i)\}$ forms a base of S for every $v \in V$.

Theorem (Léonard, Szigeti 2013)

Let $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-dypergraph.

1. *There is a matroid-based packing of rooted-hyper-arborescences* $\Longleftrightarrow$
2. $\rho_{\mathcal{A}}(X) \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X)$ *for all* $\emptyset \neq X \subseteq V$.

Remark

1. It is proved easily by trimming and DdG-N-Sz' theorem.

Maximal-rank packing of rooted-hyper-arborescences

Definition

Packing $\{(\vec{\mathcal{T}}_1, r_1, s_1), \ldots, (\vec{\mathcal{T}}_{|S|}, r_{|S|}, s_{|S|})\}$ of rooted-hyper-arborescences is of maximal rank if $\{s_i \in S : v \in Q_{\mathcal{A}(\vec{\mathcal{T}}_i)}(r_i)\}$ forms a base of $S_{P_{\mathcal{A}}(v)}$ $\forall v \in V$.

Theorem (Szigeti 2015-)

Let $(\vec{\mathcal{G}}, \mathcal{M}, S, \pi)$ be a matroid-based rooted-dypergraph.

1. There is a maximal-rank packing of rooted-hyper-arborescences $\Longleftrightarrow$
2. $\rho_{\mathcal{A}}(X) \geq r_{\mathcal{M}}(S_{P_{\mathcal{A}}(X)}) - r_{\mathcal{M}}(S_X)$ for all $X \subseteq V$.

Remark

1. It is proved not easily by trimming and Cs. Király's theorem since $r_{\mathcal{M}}(S_{P_{\mathcal{A}}(X)}) - r_{\mathcal{M}}(S_X)$ is not intersecting supermodular.
2. It implies all the previous results.

Proof of necessity

Proof

1. Let $\{(\vec{\mathcal{T}}_1, r_1, \mathrm{s}_1), \dots, (\vec{\mathcal{T}}_{|\mathrm{S}|}, r_{|\mathrm{S}|}, \mathrm{s}_{|\mathrm{S}|})\}$ be a maximal-rank packing of rooted-hyper-arborescences in $(\vec{\mathcal{G}}, \mathcal{M}, \mathrm{S}, \pi)$.
2. Let $\mathrm{B}_v = \{\mathrm{s}_i \in \mathrm{S} : v \in Q_{\mathcal{A}(\vec{\mathcal{T}}_i)}(r_i)\}$ (base of $\mathrm{S}_{P_{\mathcal{A}}(v)}$) and $X \subseteq V$.
3. For each root $\mathrm{s}_i \in \bigcup_{v \in X} \mathrm{B}_v \setminus \mathrm{S}_X$, there exists a vertex $v \in X$ such that $\mathrm{s}_i \in \mathrm{B}_v$ and then since $\vec{\mathcal{T}}_i$ is an r_i-hyper-arborescence, $r_i \notin X$ and $v \in Q_{\mathcal{A}(\vec{\mathcal{T}}_i)}(r_i) \cap X$, there exists a hyperarc of $\vec{\mathcal{T}}_i$ that enters X.
4. Since the hyper-arborescences are arc-disjoint,

$$
\begin{aligned}
\rho_{\mathcal{A}}(X) &\geq |\textstyle\bigcup_{v \in X} \mathrm{B}_v \setminus \mathrm{S}_X| \\
&\geq r_{\mathcal{M}}(\textstyle\bigcup_{v \in X} \mathrm{B}_v \setminus \mathrm{S}_X) \\
&\geq r_{\mathcal{M}}(\textstyle\bigcup_{v \in X} \mathrm{B}_v) - r_{\mathcal{M}}(\mathrm{S}_X) \\
&\geq r_{\mathcal{M}}(\textstyle\bigcup_{v \in X} \mathrm{S}_{P_{\mathcal{A}}(v)}) - r_{\mathcal{M}}(\mathrm{S}_X) \\
&= r_{\mathcal{M}}(\mathrm{S}_{P_{\mathcal{A}}(X)}) - r_{\mathcal{M}}(\mathrm{S}_X).
\end{aligned}
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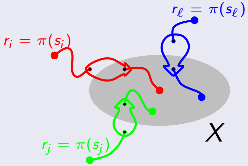
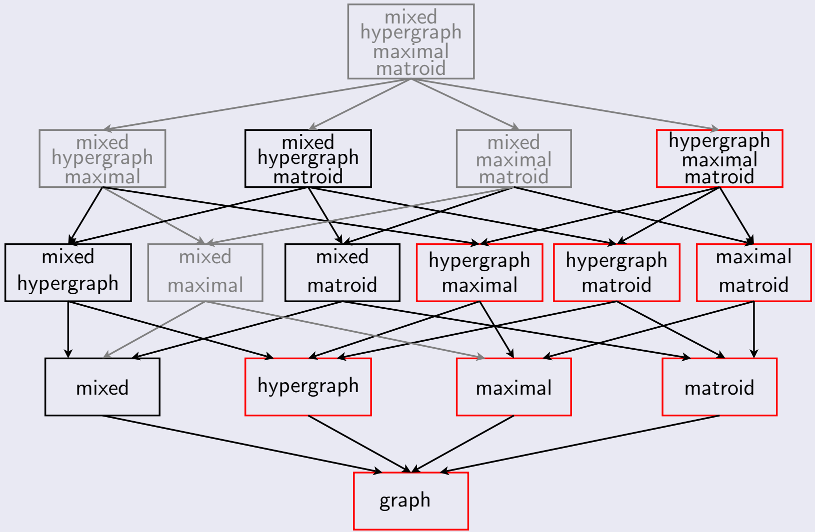


Conclusion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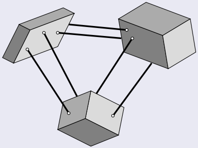
Thank you for your attention !

Motivation : Rigidity

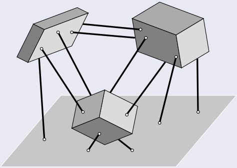
Body-Bar Framework



Theorem (Tay 1984)

"Rigidity" of a Body-Bar Framework can be characterized by the existence of a spanning tree decomposition.

Body-Bar Framework with Bar-Boundary



Theorem (Katoh, Tanigawa 2013)

"Rigidity" of a Body-Bar Framework with Bar-Boundary can be characterized by the existence of a matroid-based rooted-tree decomposition.