Reachability-based matroid-restricted packing of arborescences

Zoltán Szigeti

Combinatorial Optimization Group, G-SCOP
Univ. Grenoble Alpes, Grenoble INP, CNRS, France

May 2017

Joint work with :

Csaba Király (EGRES, Budapest),
Shin-ichi Tanigawa (University of Tokyo).

Outline

- Packing of arborescences
  - spanning
  - matroid-restricted
  - matroid-based
  - reachability
  - reachability-based
- New result
  - matroid-based
    matroid-restricted
  - reachability-based
    matroid-restricted
- Algorithmic aspects,
  weighted case for
  - matroid-based
    matroid-restricted
  - reachability-based
    matroid-restricted

Outline

- Packing of arborescences
  - spanning
  - matroid-restricted
  - matroid-based
  - reachability
  - reachability-based
- New result
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted

matroid intersection

- Algorithmic aspects, weighted case for
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted

Outline

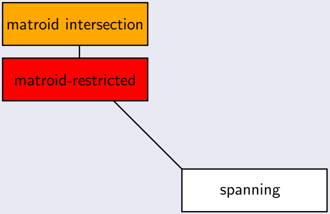
- Packing of arborescences
  - spanning
  - matroid-restricted
  - matroid-based
  - reachability
  - reachability-based
- New result
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted
- Algorithmic aspects, weighted case for
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted

matroid intersection

spanning

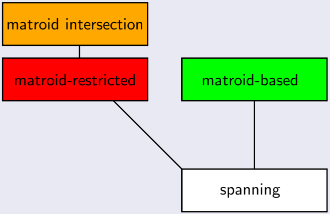
Outline

- Packing of arborescences
  - spanning
  - matroid-restricted
  - matroid-based
  - reachability
  - reachability-based
- New result
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted
- Algorithmic aspects, weighted case for
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted



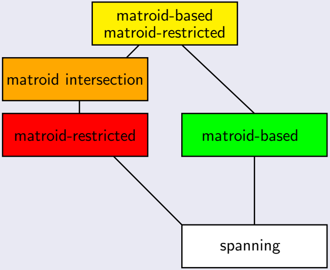
Outline

- Packing of arborescences
  - spanning
  - matroid-restricted
  - matroid-based
  - reachability
  - reachability-based
- New result
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted
- Algorithmic aspects, weighted case for
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted



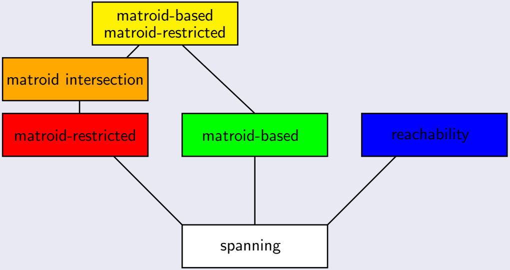
Outline

- Packing of arborescences
  - spanning
  - matroid-restricted
  - matroid-based
  - reachability
  - reachability-based
- New result
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted
- Algorithmic aspects, weighted case for
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted



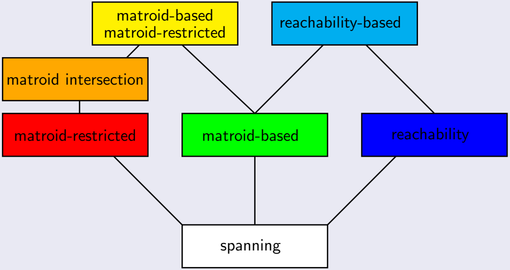
Outline

- Packing of arborescences
  - spanning
  - matroid-restricted
  - matroid-based
  - reachability
  - reachability-based
- New result
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted
- Algorithmic aspects, weighted case for
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted



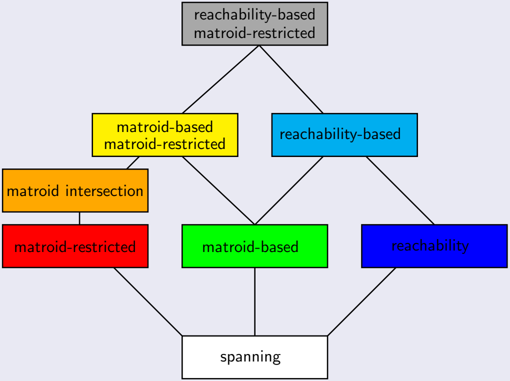
Outline

- Packing of arborescences
  - spanning
  - matroid-restricted
  - matroid-based
  - reachability
  - reachability-based
- New result
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted
- Algorithmic aspects, weighted case for
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted



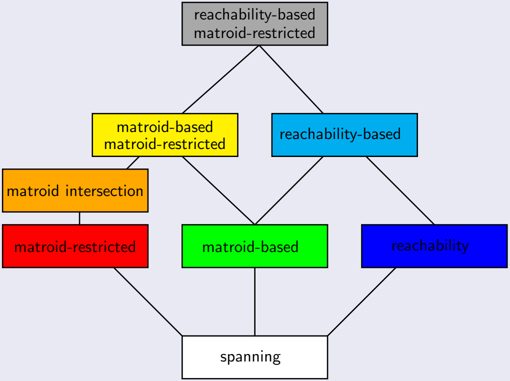
Outline

- Packing of arborescences
  - spanning
  - matroid-restricted
  - matroid-based
  - reachability
  - reachability-based
- New result
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted
- Algorithmic aspects, weighted case for
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted



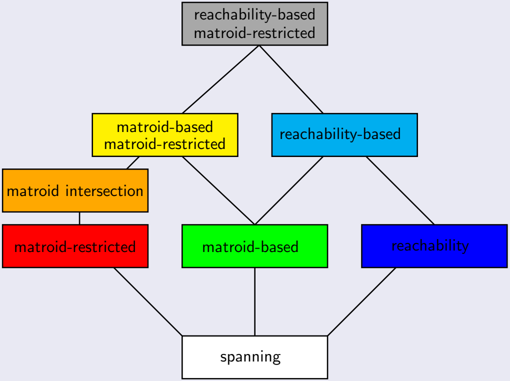
Outline

- Packing of arborescences
  - spanning
  - matroid-restricted
  - matroid-based
  - reachability
  - reachability-based
- New result
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted
- Algorithmic aspects, weighted case for
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted



Outline

- Packing of arborescences
  - spanning
  - matroid-restricted
  - matroid-based
  - reachability
  - reachability-based
- New result
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted
- Algorithmic aspects, weighted case for
  - matroid-based matroid-restricted
  - reachability-based matroid-restricted



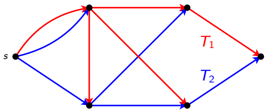
Packing of spanning s-arborescences

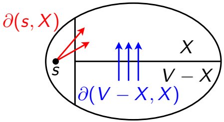
Definition

1. s-arborescence : directed tree, indegree of every vertex except s is 1,
2. spanning subgraph of D : subgraph that contains all the vertices of D,
3. packing of arborescences : arc-disjoint arborescences,
4. $\partial(Z, X)$: set of arcs from Z to X, for $Z \subseteq V(D) - X$,
5. $|\partial(X)|$: indegree of X.

Theorem (Edmonds 1973)

- *$D = (V + s, A)$ has a packing of k spanning s-arborescences* $\iff$
- $|\partial(X)| \geq k \quad \forall\, \emptyset \neq X \subseteq V$.



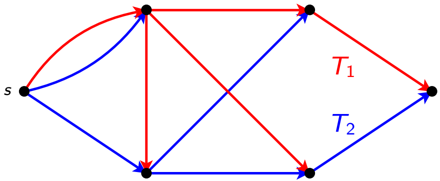


Packing spanning arborescences with matroid intersection

Remark

Let $D = (V + s, A)$ and G be the underlying undirected graph of D.

1. $\vec{F} \subseteq A$ is a packing of k spanning s-arborescences of $D \iff$
2. F is a packing of k spanning trees of G, $|\partial^{\vec{F}}(v)| = k \ \forall v \in V \iff$
3. F is a common base of $\mathcal{M}_1 =$ k-sum of the graphic matroid of G and $\mathcal{M}_2 = \oplus_{v \in V} U_{|\partial(v)|,k}$.



Matroid-restricted packing of spanning s-arborescences

Definition

Given a digraph $D = (V + s, A)$ and a matroid $\mathcal{M} = (A, \mathcal{I})$, a packing of spanning s-arborescences $T_1, \dots, T_k$ is matroid-restricted if $\cup_1^k A(T_i) \in \mathcal{I}$.

Theorem

Given a digraph $D = (V + s, A)$, $k \in \mathbb{Z}_+$ and a matroid $\mathcal{M} = (A, r)$ which is the direct sum of the matroids $\mathcal{M}_v = (\partial(v), r_v)$ $\forall v \in V$.

- *D has an $\mathcal{M}$-restricted packing of k spanning s-arborescences $\Longleftrightarrow$*
- $r(\partial(X)) \geq k \ \forall \ \emptyset \neq X \subseteq V$.

Remarks

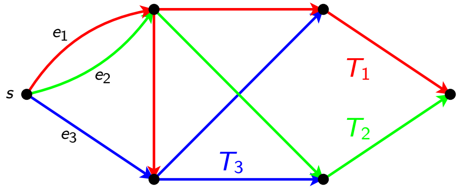
1. For free matroid, we are back to packing of k spanning s-arborescen.
2. This problem can also be formulated as matroid intersection.
3. For general matroid $\mathcal{M}$, the problem is NP-complete, even for $k = 1$.

Matroid-based packing of s-arborescences

Definition

Let $D = (V + s, A)$ be a digraph and $\mathcal{M}$ a matroid on $\partial(s, V)$.

1. A packing of s-arborescences $\{T_1, \ldots, T_t\}$ is matroid-based if $\{\text{root arc of } T_i[s, v] : v \in V(T_i)\}$ is a base of $\mathcal{M}$ $\forall v \in V$.

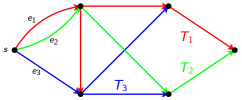


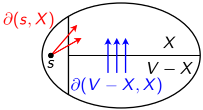
Matroid-based packing of s-arborescences

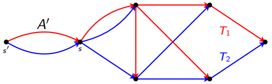
Theorem (Durand de Gevigney, Nguyen, Szigeti 2013)

Let $D = (V + s, A)$ be a digraph and $\mathcal{M} = (\partial(s, V), r)$ a matroid.

- There exists an $\mathcal{M}$-based packing of s-arborescences in $D \Longleftrightarrow$
- $r(\partial(s, X)) + |\partial(V - X, X)| \geq r(\partial(s, V)) \ \forall X \subseteq V$.







Remark

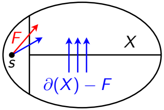
A packing of k spanning s-arborescences in $D = (V + s, A)$ can be obtained as an $\mathcal{M}$-based packing of s'-arborescences in $D' = (V + s + s', A \cup A')$, where $A' = \{k \times s's\}$ and free matroid $\mathcal{M}$ on A'.

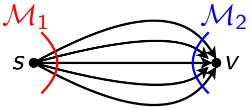
$\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-arborescences

Theorem

Let $D = (V + s, A)$, $\mathcal{M}_1 = (\partial(s, V), r_1)$, $\mathcal{M}_2 = (A, r_2) = \oplus_{v \in V} \mathcal{M}_v$.

- D has an $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-arborescences $\Longleftrightarrow$
- $r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial(s, V)) \;\; \forall X \subseteq V, F \subseteq \partial(s, X)$.





Remarks

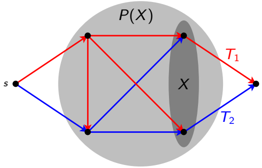
1. It contains matroid-restricted packing of spanning s-arborescences, even matroid intersection. For matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on S, our problem on $(D = (\{s, v\}, \{|S| \times sv\}), \mathcal{M}_1, \mathcal{M}_2)$ reduces to it.
2. For free $\mathcal{M}_2$, we are back to $\mathcal{M}_1$-based packing of s-arborescences.

Packing of reachability s-arborescences

Definition

Let $D = (V + s, A)$ be a digraph.

1. For $X \subseteq V$, $P(X)$ denotes the set of vertices in V from which X is reachable by a directed path in D.
2. packing of reachability s-arborescences $\{T_1, \ldots, T_t\}$:
   $\{\text{root arc of } T_i[s, v] : v \in V(T_i)\} = \partial(s, P(v)) \ \ \forall v \in V$.



Packing of reachability s-arborescences

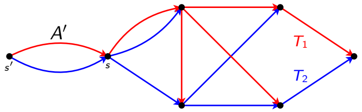
Theorem (Kamiyama, Katoh, Takizawa 2009)

Let $D = (V + s, A)$ be a digraph.

- *There exists a packing of reachability s-arborescences* $\iff$
- $|\partial(X)| \geq |\partial(s, P(X))| \ \forall X \subseteq V$.

Remark

Packing of k spanning s-arborescences in $D = (V + s, A)$ can be obtained as packing of reachability s'-arborescences in $D' = (V + s + s', A \cup A')$ where $A' = \{k \times s's\}$, because $|\partial(X)| \geq k \ \forall \emptyset \neq X \subseteq V$ implies the above condition in D' and that each vertex is reachable from s in D.

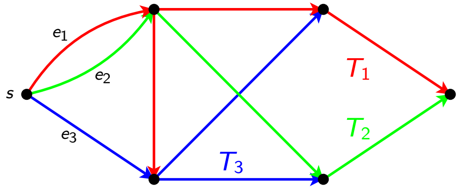


Reachability-based packing of s-arborescences

Definition

Let $D = (V + s, A)$ be a digraph and $(\mathcal{M} = (\partial(s, V), \mathcal{I}), r)$ a matroid.

1. A packing of s-arborescences $\{T_1, \dots, T_t\}$ is reachability-based if $\{\text{root arc of } T_i[s, v] : v \in V(T_i)\} \in \mathcal{I}$ of size $r(\partial(s, P(v)))$ $\forall v \in V$.



Reachability-based packing of s-arborescences

Theorem (Cs. Király 2016)

Let $D = (V + s, A)$ be a digraph and $\mathcal{M} = (\partial(s, V), r)$ a matroid.

- D has an $\mathcal{M}$-reachability-based packing of s-arborescences $\Longleftrightarrow$
- $r(\partial(s, X)) + |\partial(V - X, X)| \geq r(\partial(s, P(X)))\ \forall X \subseteq V$.

Remarks

1. For free matroid, back to packing of reachability s-arborescences.
2. An $\mathcal{M}$-based packing of s-arborescences can be obtained as an $\mathcal{M}$-reachability-based packing of s-arborescences, because $r(\partial(s, X)) + |\partial(V - X, X)| \geq r(\partial(s, V))\ \forall X \subseteq V$ implies the above condition and that $r(\partial(s, P(v))) = r(\partial(s, V))$.

Reachability-based matroid-restricted packing of s-arborescences

Theorem

Let $D = (V + s, A)$, $\mathcal{M}_1 = (\partial(s, V), r_1)$, $\mathcal{M}_2 = (A, r_2) = \oplus_{v \in V} \mathcal{M}_v$.

- $\exists$ $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescen. $\Longleftrightarrow$
- $r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial(s, P(X)))$ $\forall X \subseteq V, F \subseteq \partial(s, X)$.

Remarks

1. An $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-arborescences can be obtained as an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences, because $r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial(s, V))$ $\forall X \subseteq V, F \subseteq \partial(s, X)$ implies the above condition and that $r_1(\partial(s, P(v))) = r_1(\partial(s, V))$.
2. For free matroid $\mathcal{M}_2$, we are back to $\mathcal{M}_1$-reachability-based packing of s-arborescences.

Algorithmic aspects : matroid constructions

Theorem (Edmonds-Rota, +Dilworth truncation)

- *$D := (V, A)$ a digraph,*
- *$f : 2^A \to \mathbb{Z}_+$ a monotone intersecting submodular function,*
- $\mathcal{I} := \{B \subseteq A : |H| \leq f(H)\ \forall H \subseteq B\}$.

Then $\mathcal{I}$ forms the family of independent sets of a matroid on A.

Theorem

- $D := (V, A)$ a digraph,
- $\mathcal{F}$ an intersecting bi-set family on V,
- $b : \mathcal{F} \to \mathbb{Z}_+$ an intersecting submodular bi-set function,
- $\mathcal{I} := \{B \subseteq A : i_B(\mathsf{X}) \leq b(\mathsf{X})\ \forall \mathsf{X} \in \mathcal{F}\}$.

Then $\mathcal{I}$ forms the family of independent sets of a matroid on A.

Algorithmic aspects : matroid intersection

Theorem

Let $D = (V + s, A)$, $\mathcal{M}_1 = (\partial(s, V), r_1)$, $\mathcal{M}_2 = (A, r_2) = \oplus_{v \in V} \mathcal{M}_v$.
The arc sets of the $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packings of s-arborescences can be written as common bases of $\mathcal{M}_1'$ and $\mathcal{M}_2$.

Remark

- $\mathcal{M}_1$-based : $\mathcal{M}_1'$ by $f(H) = k|V(H) - s| - k + r_1(H \cap \partial(s, V))$,
- $\mathcal{M}_1$-reachability-based : $\mathcal{M}_1'$ by $b(X) = m(X_I) - p(X)$.

Corollary

- One can decide in polynomial time if an instance has a solution.
- One can find in polynomial time an arc set of minimum weight that can be decomposed into an $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences.

Algorithmic aspects : unweighted case

Algorithm

INPUT : $(D, \mathcal{M}_1, \mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v)$.
OUTPUT : Either the required packing or a pair violating the condition.

1. If $(D, \mathcal{M}_1, \mathcal{M}_2)$ has no solution then stop with the pair violating the condition.
2. If $\mathcal{M}_2$ is the free matroid then use Cs. Király's algorithm for $\mathcal{M}_1$-reachability-based packing of s-arborescences and stop with the packing.
3. Otherwise, let e be a non-bridge edge in $\mathcal{M}_2$.
4. If $(D - e, \mathcal{M}_1 - e, \mathcal{M}_2 - e)$ has a solution then use recursively our algorithm for it and stop with the packing.
5. Otherwise, $(D, \mathcal{M}_1, \mathcal{M}_2' = (\mathcal{M}_2/e) \oplus e)$ has a solution. Use recursively our algorithm for $(D, \mathcal{M}_1, \mathcal{M}_2')$ and stop with the packing.

Algorithmic aspects : weighted case

Theorem

For $(D = (V + s, A), c : A \rightarrow \mathbb{R}, \mathcal{M}_1, \mathcal{M}_2 = \oplus_{v \in V} \mathcal{M}_v)$, a minimum weight $\mathcal{M}_1$-reachability-based $\mathcal{M}_2$-restricted packing of s-arborescences can be found in polynomial time.

Thank you for your attention !