On $(2k, k)$-connected graphs

Zoltán Szigeti

Combinatorial Optimization Group
Laboratoire G-SCOP
INP Grenoble, France

11 septembre 2015

Joint work with : Olivier Durand de Gevigney

Outline

- Results on :
  - Orientation
  - Construction
  - Splitting off
  - Augmentation
- Concerning :
  - Edge-connectivity
  - $(4, 2)$-connectivity
  - $(2k, k)$-connectivity

Orientation : arc-connectivity

Definition

1. A digraph D is called *k-arc-connected* if $\forall\ \emptyset \neq X \subset V$, $|\rho_D(X)| \geq k$.
2. A graph G is called *k-edge-connected* if $\forall\ \emptyset \neq X \subset V$, $d_G(X) \geq k$.

Orientation : arc-connectivity

Definition

1. A digraph D is called *k-arc-connected* if $\forall\, \emptyset \neq X \subset V$, $|\rho_D(X)| \geq k$.
2. A graph G is called *k-edge-connected* if $\forall\, \emptyset \neq X \subset V$, $d_G(X) \geq k$.

Theorem (Nash-Williams)

G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.

Orientation : arc-connectivity

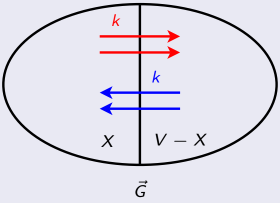
Definition

1. A digraph D is called *k-arc-connected* if $\forall\ \emptyset \neq X \subset V$, $|\rho_D(X)| \geq k$.
2. A graph G is called *k-edge-connected* if $\forall\ \emptyset \neq X \subset V$, $d_G(X) \geq k$.

Theorem (Nash-Williams)

G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.

Necessity :



Orientation : arc-connectivity

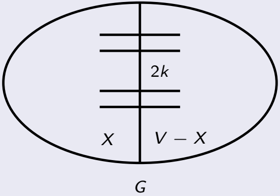
Definition

1. A digraph D is called *k-arc-connected* if $\forall\ \emptyset \neq X \subset V$, $|\rho_D(X)| \geq k$.
2. A graph G is called *k-edge-connected* if $\forall\ \emptyset \neq X \subset V$, $d_G(X) \geq k$.

Theorem (Nash-Williams)

G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.

Necessity :



Orientation : k-vertex-connectivity

Definition

1. A digraph D is called *k-vertex-connected* if $|V| \geq k+1$, $\forall\ X \subset V,\ |X| = k-1,$ $D - X$ is 1-arc-connected.
2. A graph G is called *k-vertex-connected* if $|V| \geq k+1$, $\forall\ X \subset V,\ |X| = k-1,$ $G - X$ is connected.

Orientation : k-vertex-connectivity

Definition

1. A digraph D is called *k-vertex-connected* if $|V| \geq k+1$, $\forall\ X \subset V$, $|X| = k-1$, $D - X$ is 1-arc-connected.
2. A graph G is called *k-vertex-connected* if $|V| \geq k+1$, $\forall\ X \subset V$, $|X| = k-1$, $G - X$ is connected.

Conjecture (Frank)

G has a k-vertex-connected orientation if and only if $|V| \geq k+1$ and $\forall\ X \subset V$, $|X| < k$, $G - X$ is $(2k - 2|X|)$-edge-connected.

Orientation : k-vertex-connectivity

Definition

1. A digraph D is called *k-vertex-connected* if $|V| \geq k+1$, $\forall\ X \subset V$, $|X| = k-1$, $D - X$ is 1-arc-connected.
2. A graph G is called *k-vertex-connected* if $|V| \geq k+1$, $\forall\ X \subset V$, $|X| = k-1$, $G - X$ is connected.

Conjecture (Frank)

G has a k-vertex-connected orientation if and only if $|V| \geq k+1$ and $\forall\ X \subset V$, $|X| < k$, $G - X$ is $(2k - 2|X|)$-edge-connected.

Theorem (Durand de Gevigney) ($k \geq 3$)

1. *This conjecture is false.*

Orientation : k-vertex-connectivity

Definition

1. A digraph D is called *k-vertex-connected* if $|V| \geq k+1$, $\forall\ X \subset V$, $|X| = k-1$, $D - X$ is 1-arc-connected.
2. A graph G is called *k-vertex-connected* if $|V| \geq k+1$, $\forall\ X \subset V$, $|X| = k-1$, $G - X$ is connected.

Conjecture (Frank)

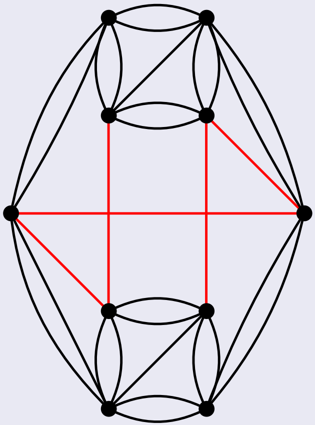
G has a k-vertex-connected orientation if and only if $|V| \geq k+1$ and $\forall\ X \subset V$, $|X| < k$, $G - X$ is $(2k - 2|X|)$-edge-connected.

Theorem (Durand de Gevigney) ($k \geq 3$)

1. *This conjecture is false.*
2. *Deciding whether G has a k-vertex-connected orientation is NP-complete.*

Counter-example for $k = 3$

Example of Durand de Gevigney

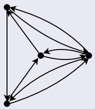


Orientation : 2-vertex-connectivity

Remark (Necessary condition)

If $\vec{G}$ is 2-vertex-connected, then

Example

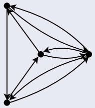


Orientation : 2-vertex-connectivity

Remark (Necessary condition)

If $\vec{G}$ is 2-vertex-connected, then $|V| \geq 3$,

Example



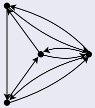
Orientation : 2-vertex-connectivity

Remark (Necessary condition)

If $\vec{G}$ is 2-vertex-connected, then $|V| \geq 3$,

1. G is 4-edge-connected and,

Example



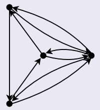
Orientation : 2-vertex-connectivity

Remark (Necessary condition)

If $\vec{G}$ is 2-vertex-connected, then $|V| \geq 3$,

1. G is 4-edge-connected and,
2. for all $v \in V$, $G - v$ is 2-edge-connected.

Example



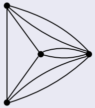
Orientation : 2-vertex-connectivity

Definition

A graph G is called *(4, 2)-connected* if $|V| \geq 3$,

1. G is 4-edge-connected and,
2. for all $v \in V$, $G - v$ is 2-edge-connected.

Example



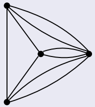
Orientation : 2-vertex-connectivity

Definition

A graph G is called $(4,2)$-*connected* if $|V| \geq 3$,

1. G is 4-edge-connected and,
2. for all $v \in V$, $G - v$ is 2-edge-connected.

Example



Theorem (Sufficent condition)

A graph G has a 2-vertex-connected orientation

1. *if G is $(4,2)$-connected and Eulerian (Berg, Jordán).*

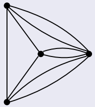
Orientation : 2-vertex-connectivity

Definition

A graph G is called *(4, 2)-connected* if $|V| \geq 3$,

1. G is 4-edge-connected and,
2. for all $v \in V$, $G - v$ is 2-edge-connected.

Example



Theorem (Sufficent condition)

A graph G has a 2-vertex-connected orientation

1. *if G is $(4, 2)$-connected and Eulerian (Berg, Jordán).*
2. *if G is 18-vertex-connected (Jordán).*

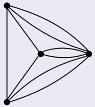
Orientation : 2-vertex-connectivity

Definition

A graph G is called $(4,2)$-*connected* if $|V| \geq 3$,

1. G is 4-edge-connected and,
2. for all $v \in V$, $G - v$ is 2-edge-connected.

Example



Theorem (Sufficent condition)

A graph G has a 2-vertex-connected orientation

1. *if G is $(4,2)$-connected and Eulerian (Berg, Jordán).*
2. *if G is 18-vertex-connected (Jordán).*
3. *if G is 14-vertex-connected (Cheriyan, Durand de Gevigney, Szigeti).*

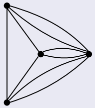
Orientation : 2-vertex-connectivity

Definition

A graph G is called *(4, 2)-connected* if $|V| \geq 3$,

1. G is 4-edge-connected and,
2. for all $v \in V$, $G - v$ is 2-edge-connected.

Example



Theorem (Sufficent condition)

A graph G has a 2-vertex-connected orientation

1. *if G is $(4, 2)$-connected and Eulerian (Berg, Jordán).*
2. *if G is 18-vertex-connected (Jordán).*
3. *if G is 14-vertex-connected (Cheriyan, Durand de Gevigney, Szigeti).*

Theorem (Thomassen)

G has a 2-vertex-connected orientation if and only if G is $(4, 2)$-connected.

Construction : edge-connectivity

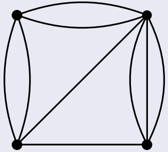
Theorem (Lovász)

A graph is $2k$*-edge-connected if and only if it can be obtained from* K_2^{2k} *by a sequence of the following two operations :*

(a) *adding a new edge,*

(b) *pinching* k *edges.*

Example



Construction : edge-connectivity

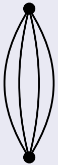
Theorem (Lovász)

A graph is $2k$-edge-connected if and only if it can be obtained from K_2^{2k} by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching k edges.*

Example



Construction : edge-connectivity

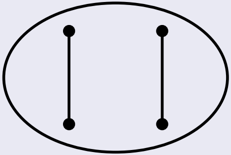
Theorem (Lovász)

A graph is $2k$-edge-connected if and only if it can be obtained from K_2^{2k} by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching k edges.*

Example



Construction : edge-connectivity

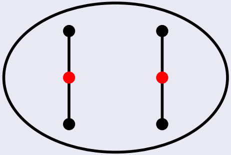
Theorem (Lovász)

A graph is $2k$-edge-connected if and only if it can be obtained from K_2^{2k} by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching k edges.*

Example



Construction : edge-connectivity

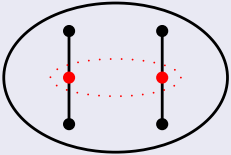
Theorem (Lovász)

A graph is $2k$-edge-connected if and only if it can be obtained from K_2^{2k} by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching k edges.*

Example



Construction : edge-connectivity

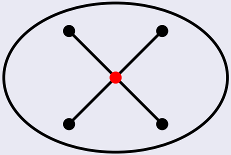
Theorem (Lovász)

A graph is $2k$-edge-connected if and only if it can be obtained from K_2^{2k} by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching k edges.*

Example



Construction : edge-connectivity

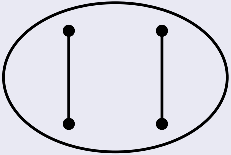
Theorem (Lovász)

A graph is $2k$-edge-connected if and only if it can be obtained from K_2^{2k} by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching k edges.*

Example



Construction : edge-connectivity

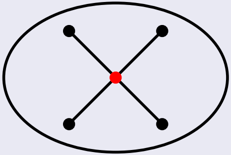
Theorem (Lovász)

A graph is $2k$-edge-connected if and only if it can be obtained from K_2^{2k} by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching k edges.*

Example



Construction : edge-connectivity

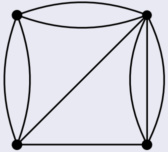
Theorem (Lovász)

A graph is $2k$*-edge-connected if and only if it can be obtained from* K_2^{2k} *by a sequence of the following two operations :*

(a) *adding a new edge,*

(b) *pinching* k *edges.*

Example



Construction : edge-connectivity

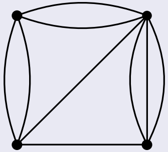
Theorem (Lovász)

A graph is $2k$-edge-connected if and only if it can be obtained from K_2^{2k} by a sequence of the following two operations :

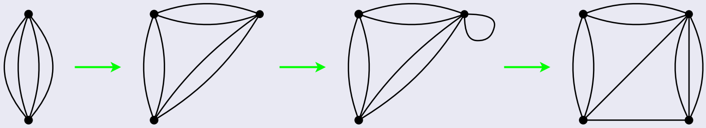
(a) *adding a new edge,*

(b) *pinching k edges.*

Example



Example



Construction : edge-connectivity

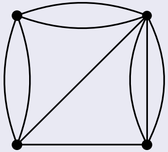
Theorem (Lovász)

A graph is $2k$*-edge-connected if and only if it can be obtained from* K_2^{2k} *by a sequence of the following two operations :*

(a) *adding a new edge,*

(b) *pinching* k *edges.*

Example



Remark

1. These operations preserve $2k$-edge-connectivity.

Construction : edge-connectivity

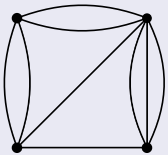
Theorem (Lovász)

A graph is $2k$-edge-connected if and only if it can be obtained from K_2^{2k} by a sequence of the following two operations :

(a) *adding a new edge,*

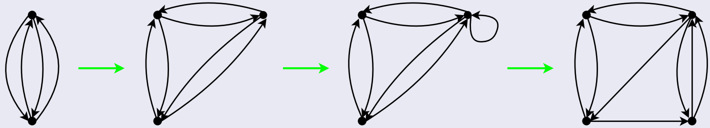
(b) *pinching k edges.*

Example



Remark

1. These operations preserve $2k$-edge-connectivity.
2. It implies Nash-Williams' orientation result on k-arc-connectivity.



Construction : $(4, 2)$-connectivity

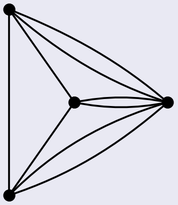
Theorem (Jordán)

A graph is $(4, 2)$-connected if and only if it can be obtained from K_3^2 by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching 2 edges so that if one of them is a loop then the other one is not adjacent to it.*

Example



Construction : $(4,2)$-connectivity

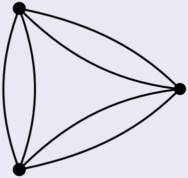
Theorem (Jordán)

A graph is $(4,2)$-connected if and only if it can be obtained from K_3^2 by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching 2 edges so that if one of them is a loop then the other one is not adjacent to it.*

Example



Construction : $(4,2)$-connectivity

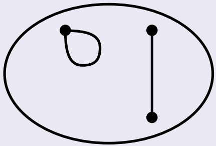
Theorem (Jordán)

A graph is $(4,2)$-connected if and only if it can be obtained from K_3^2 by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching 2 edges so that if one of them is a loop then the other one is not adjacent to it.*

Example



Construction : $(4, 2)$-connectivity

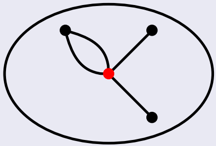
Theorem (Jordán)

A graph is $(4, 2)$-connected if and only if it can be obtained from K_3^2 by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching 2 edges so that if one of them is a loop then the other one is not adjacent to it.*

Example



Construction : $(4, 2)$-connectivity

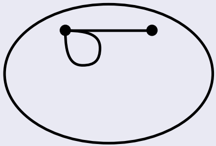
Theorem (Jordán)

A graph is $(4, 2)$-connected if and only if it can be obtained from K_3^2 by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching 2 edges so that if one of them is a loop then the other one is not adjacent to it.*

Example



Construction : $(4, 2)$-connectivity

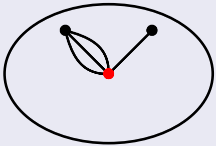
Theorem (Jordán)

A graph is $(4, 2)$-connected if and only if it can be obtained from K_3^2 by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching 2 edges so that if one of them is a loop then the other one is not adjacent to it.*

Example



Construction : $(4,2)$-connectivity

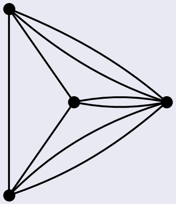
Theorem (Jordán)

A graph is $(4,2)$-connected if and only if it can be obtained from K_3^2 by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching 2 edges so that if one of them is a loop then the other one is not adjacent to it.*

Example



Construction : $(4,2)$-connectivity

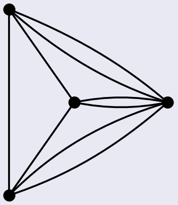
Theorem (Jordán)

A graph is $(4,2)$-connected if and only if it can be obtained from K_3^2 by a sequence of the following two operations :

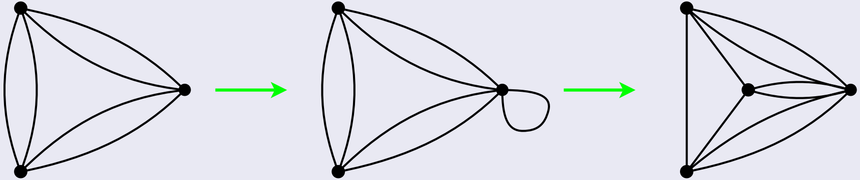
(a) *adding a new edge,*

(b) *pinching 2 edges so that if one of them is a loop then the other one is not adjacent to it.*

Example



Example



Construction : $(4,2)$-connectivity

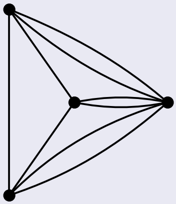
Theorem (Jordán)

A graph is $(4,2)$-connected if and only if it can be obtained from K_3^2 by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching 2 edges so that if one of them is a loop then the other one is not adjacent to it.*

Example



Remark

1. These operations preserve $(4,2)$-connectivity.

Construction : $(4, 2)$-connectivity

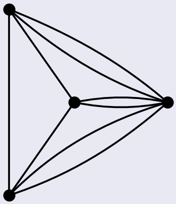
Theorem (Jordán)

A graph is $(4, 2)$-connected if and only if it can be obtained from K_3^2 by a sequence of the following two operations :

(a) *adding a new edge,*

(b) *pinching 2 edges so that if one of them is a loop then the other one is not adjacent to it.*

Example

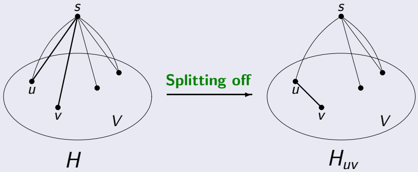


Remark

1. These operations preserve $(4, 2)$-connectivity.
2. Jordán's result does not imply Thomassen's result on 2-vertex-connectivity orientation.

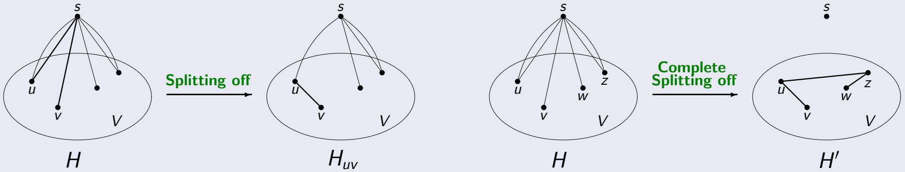
Splitting off : edge-connectivity

Definitions



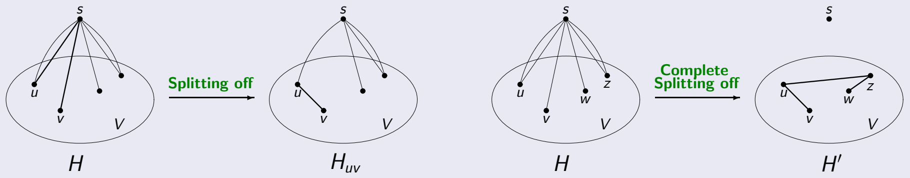
Splitting off : edge-connectivity

Definitions



Splitting off : edge-connectivity

Definitions



Theorem (Lovász)

Let $H = (V + s, E)$ be an ℓ-edge-connected graph in V, $\ell \geq 2$, $d_H(s)$ even. There exists a complete splitting off at s preserving ℓ-edge-connectivity.

Splitting off : edge-connectivity

Theorem (Lovász)

Let $H = (V + s, E)$ be an ℓ-edge-connected graph in V, $\ell \geq 2$, $d_H(s)$ even. There exists a complete splitting off at s preserving ℓ-edge-connectivity.

Remark

It implies the construction of $2k$-edge-connected graphs G :

1. G can be obtained from K_2^{2k} by the operations :
   (a) adding a new edge, (b) pinching k edges.

Splitting off : edge-connectivity

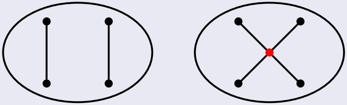
Theorem (Lovász)

Let $H = (V + s, E)$ be an ℓ-edge-connected graph in V, $\ell \geq 2$, $d_H(s)$ even. There exists a complete splitting off at s preserving ℓ-edge-connectivity.

Remark

It implies the construction of $2k$-edge-connected graphs G :

1. G can be obtained from K_2^{2k} by the operations :
   (a) adding a new edge, (b) pinching k edges.
2. G must be reduced to K_2^{2k} by the inverse operations :
   (a) deleting an edge, (b) complete splitting off at a vertex of degree $2k$.



Splitting off : edge-connectivity

Theorem (Lovász)

Let $H = (V + s, E)$ be an ℓ-edge-connected graph in V, $\ell \geq 2$, $d_H(s)$ even. There exists a complete splitting off at s preserving ℓ-edge-connectivity.

Remark

It implies the construction of $2k$-edge-connected graphs G :

1. G can be obtained from K_2^{2k} by the operations :
   (a) adding a new edge, (b) pinching k edges.
2. G must be reduced to K_2^{2k} by the inverse operations :
   (a) deleting an edge, (b) complete splitting off at a vertex of degree $2k$.
   This can be done by Mader's result on minimally $2k$-edge-connected graphs and by Lovász' splitting off result.

Splitting off : $(4, 2)$-connectivity

Theorem (Jordán)

Let $H = (V + s, E)$ *be a* $(4, 2)$*-connected graph with* $d_H(s) = 4$. *There exists a complete splitting-off at* s *preserving* $(4, 2)$*-connectivity if and only if there exists no obstacle at* s.

Splitting off : $(4,2)$-connectivity

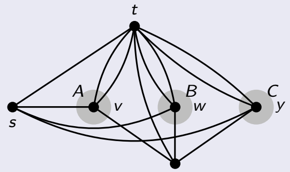
Theorem (Jordán)

Let $H = (V + s, E)$ be a $(4,2)$-connected graph with $d_H(s) = 4$. There exists a complete splitting-off at s preserving $(4,2)$-connectivity if and only if there exists no obstacle at s.

Definition

For the set $\{t, v, w, y\}$ of neighbors of s, the pair $(t, \{A, B, C\})$ is called an *obstacle* at s if $\{A, B, C\}$ is a subpartition of $V - t$ such that its elements are of degree 2 in $H - t$ and $v \in A, w \in B, y \in C$.

Example



Splitting off : $(4,2)$-connectivity

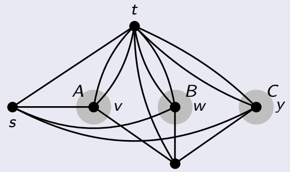
Theorem (Jordán)

Let $H = (V + s, E)$ be a $(4,2)$-connected graph with $d_H(s) = 4$. There exists a complete splitting-off at s preserving $(4,2)$-connectivity if and only if there exists no obstacle at s.

Definition

For the set $\{t, v, w, y\}$ of neighbors of s, the pair $(t, \{A, B, C\})$ is called an *obstacle* at s if $\{A, B, C\}$ is a subpartition of $V - t$ such that its elements are of degree 2 in $H - t$ and $v \in A, w \in B, y \in C$.

Example



Remark

It implies the construction of $(4,2)$-connected graphs.

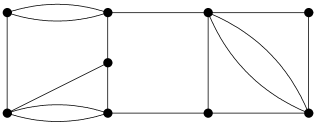
Augmentation

Theorem (Watanabe-Nakamura)

Let $G = (V, E)$ be a graph and $\ell \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is ℓ-edge-connected is equal to

$$\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}} (\ell - d_G(X)) \right\} \right\rceil,$$

where $\mathcal{X}$ is a subpartition of V.



Graph G and $\ell = 4$

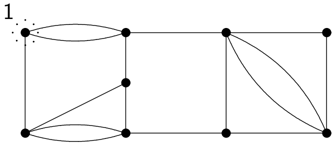
Augmentation

Theorem (Watanabe-Nakamura)

Let $G = (V, E)$ be a graph and $\ell \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is ℓ-edge-connected is equal to

$$\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}} (\ell - d_G(X)) \right\} \right\rceil,$$

where $\mathcal{X}$ is a subpartition of V.



Graph G and $\ell = 4$

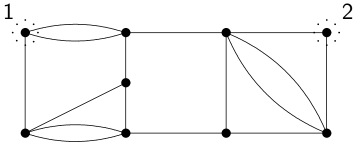
Augmentation

Theorem (Watanabe-Nakamura)

Let $G = (V, E)$ be a graph and $\ell \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is ℓ-edge-connected is equal to

$$\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}} (\ell - d_G(X)) \right\} \right\rceil,$$

where $\mathcal{X}$ is a subpartition of V.



Graph G and $\ell = 4$

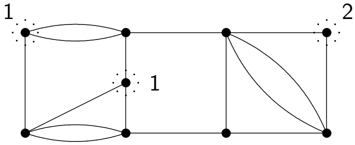
Augmentation

Theorem (Watanabe-Nakamura)

Let $G = (V, E)$ be a graph and $\ell \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is ℓ-edge-connected is equal to

$$\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}} (\ell - d_G(X)) \right\} \right\rceil,$$

where $\mathcal{X}$ is a subpartition of V.



Graph G and $\ell = 4$

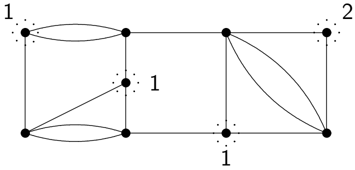
Augmentation

Theorem (Watanabe-Nakamura)

Let $G = (V, E)$ be a graph and $\ell \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is ℓ-edge-connected is equal to

$$\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}} (\ell - d_G(X)) \right\} \right\rceil,$$

where $\mathcal{X}$ is a subpartition of V.



Graph G and $\ell = 4$

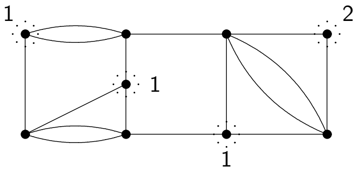
Augmentation

Theorem (Watanabe-Nakamura)

Let $G = (V, E)$ be a graph and $\ell \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is ℓ-edge-connected is equal to

$$\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}} (\ell - d_G(X)) \right\} \right\rceil,$$

where $\mathcal{X}$ is a subpartition of V.



Opt$\geq \lceil \frac{5}{2} \rceil = 3$

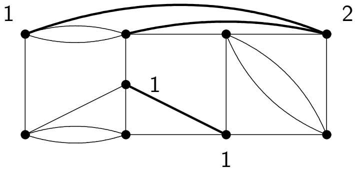
Augmentation

Theorem (Watanabe-Nakamura)

Let $G = (V, E)$ be a graph and $\ell \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is ℓ-edge-connected is equal to

$$\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}} (\ell - d_G(X)) \right\} \right\rceil,$$

where $\mathcal{X}$ is a subpartition of V.



Graph $G + F$ is 4-edge-connected and $|F| = 3$

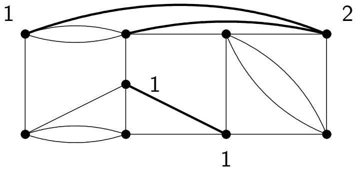
Augmentation

Theorem (Watanabe-Nakamura)

Let $G = (V, E)$ be a graph and $\ell \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is ℓ-edge-connected is equal to

$$\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}} (\ell - d_G(X)) \right\} \right\rceil,$$

where $\mathcal{X}$ is a subpartition of V.



Opt$= \lceil \frac{1}{2}$maximum deficiency of a subpartition of $V\rceil$

General method

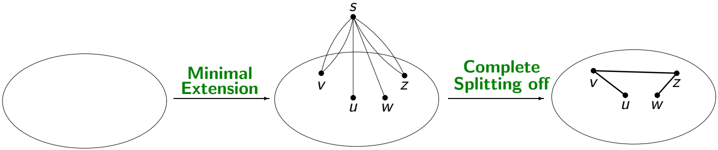
Frank's algorithm

General method

Frank's algorithm

1. Minimal extension,

2. Complete splitting off preserving the edge-connectivity requirements.

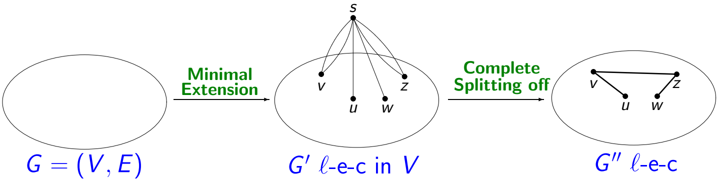


$G = (V, E)$ $\qquad$ G' ℓ-e-c in V $\qquad$ G'' ℓ-e-c

General method

Frank's algorithm

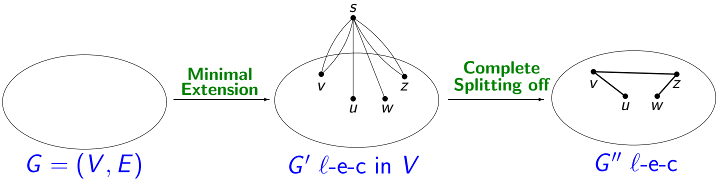
1. Minimal extension,
   1. Add a new vertex s,
2. Complete splitting off preserving the edge-connectivity requirements.



General method

Frank's algorithm

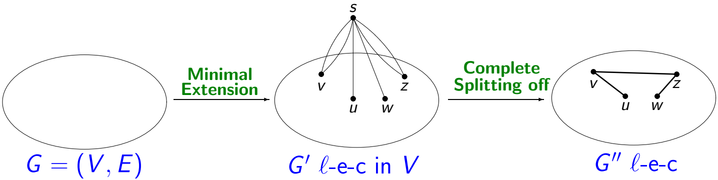
1. Minimal extension,
   1. Add a new vertex s,
   2. Add a minimum number of new edges incident to s to satisfy the edge-connectivity requirements,
2. Complete splitting off preserving the edge-connectivity requirements.



General method

Frank's algorithm

1. Minimal extension,
   1. Add a new vertex s,
   2. Add a minimum number of new edges incident to s to satisfy the edge-connectivity requirements,
   3. If the degree of s is odd, then add an arbitrary edge incident to s.
2. Complete splitting off preserving the edge-connectivity requirements.



General method

Frank's algorithm

1. Minimal extension,
   1. Add a new vertex s,
   2. Add a minimum number of new edges incident to s to satisfy the edge-connectivity requirements,
   3. If the degree of s is odd, then add an arbitrary edge incident to s.
2. Complete splitting off preserving the edge-connectivity requirements.

Remark

1. Minimal extension works for symmetric skew supermodular functions.

General method

Frank's algorithm

1. Minimal extension,
   1. Add a new vertex s,
   2. Add a minimum number of new edges incident to s to satisfy the edge-connectivity requirements,
   3. If the degree of s is odd, then add an arbitrary edge incident to s.
2. Complete splitting off preserving the edge-connectivity requirements.

Remark

1. Minimal extension works for symmetric skew supermodular functions.
2. For a new edge-connectivity augmentation problem a new complete splitting off result (preserving the edge-connectivity requirement) must be proven.

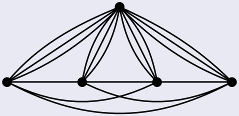
$(2k, k)$-connected graph

Definition

G is called *$(2k, k)$-connected* if $|V| \geq 3$,

1. G is $2k$-edge-connected and,
2. for all $v \in V$, $G - v$ is k-edge-connected.

Example



A $(6, 3)$-connected graph.

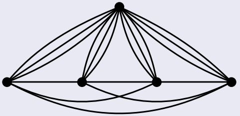
$(2k, k)$-connected graph

Definition

G is called *$(2k, k)$-connected* if $|V| \geq 3$,

1. G is $2k$-edge-connected and,
2. for all $v \in V$, $G - v$ is k-edge-connected.

Example

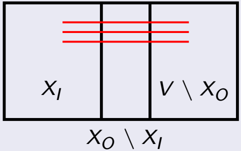


A $(6, 3)$-connected graph.

Definition

1. Bi-set : $\mathsf{X} = (X_O, X_I)$, with $X_I \subseteq X_O$,

Example



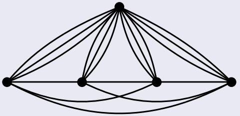
$(2k, k)$-connected graph

Definition

G is called $(2k, k)$*-connected* if $|V| \geq 3$,

1. G is $2k$-edge-connected and,
2. for all $v \in V$, $G - v$ is k-edge-connected.

Example

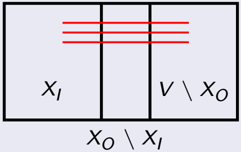


A $(6, 3)$-connected graph.

Definition

1. Bi-set : $\mathsf{X} = (X_O, X_I)$, with $X_I \subseteq X_O$,
2. $d_G^b(\mathsf{X})$: number of edges between X_I and $V \setminus X_O$,

Example



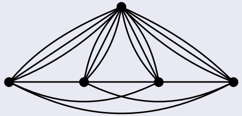
$(2k, k)$-connected graph

Definition

G is called *$(2k, k)$-connected* if $|V| \geq 3$,

1. G is $2k$-edge-connected and,
2. for all $v \in V$, $G - v$ is k-edge-connected.

Example

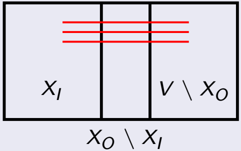


A $(6, 3)$-connected graph.

Definition

1. Bi-set : $\mathsf{X} = (X_O, X_I)$, with $X_I \subseteq X_O$,
2. $d_G^b(\mathsf{X})$: number of edges between X_I and $V \setminus X_O$,
3. $f_G^b(\mathsf{X})$: $d_G^b(\mathsf{X}) + k|X_O \setminus X_I|$.

Example



$(2k, k)$-connected graph

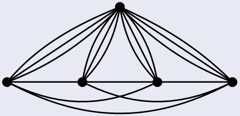
Definition

G is called *$(2k, k)$-connected* if $|V| \geq 3$,

1. G is $2k$-edge-connected and,
2. for all $v \in V$, $G - v$ is k-edge-connected.

$\Leftrightarrow$ for all non-trivial bi-sets X of V, $f_G^b(\mathsf{X}) \geq 2k$.

Example

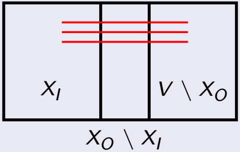


A $(6, 3)$-connected graph.

Definition

1. Bi-set : $\mathsf{X} = (X_O, X_I)$, with $X_I \subseteq X_O$,
2. $d_G^b(\mathsf{X})$: number of edges between X_I and $V \setminus X_O$,
3. $f_G^b(\mathsf{X})$: $d_G^b(\mathsf{X}) + k|X_O \setminus X_I|$.

Example



Splitting off : $(2k, k)$-connectivity

Theorem (Durand de Gevigney, Szigeti)

Let $H = (V + s, E)$ be a $(2k, k)$-connected graph in V with $k \geq 2$ and $d_H(s)$ even. There exists a complete splitting-off at s preserving $(2k, k)$-connectivity if and only if there exists no obstacle at s.

Splitting off : $(2k, k)$-connectivity

Theorem (Durand de Gevigney, Szigeti)

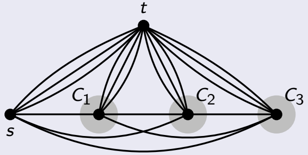
Let $H = (V + s, E)$ be a $(2k, k)$-connected graph in V with $k \geq 2$ and $d_H(s)$ even. There exists a complete splitting-off at s preserving $(2k, k)$-connectivity if and only if there exists no obstacle at s.

Definition

The pair $(t, \mathcal{C})$ is called an *obstacle* at s if

1. t is a neighbor of s with $d_H(s, t)$ odd,
2. $\mathcal{C}$ is a subpartition of $V - t$ such that its elements are of degree k in $H - t$ and cover all neighbors of s but t.

Example



Splitting off : $(2k, k)$-connectivity

Theorem (Durand de Gevigney, Szigeti)

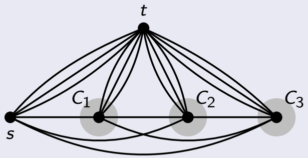
Let $H = (V + s, E)$ be a $(2k, k)$-connected graph in V with $k \geq 2$ and $d_H(s)$ even. There exists a complete splitting-off at s preserving $(2k, k)$-connectivity if and only if there exists no obstacle at s.

Definition

The pair $(t, \mathcal{C})$ is called an *obstacle* at s if

1. t is a neighbor of s with $d_H(s, t)$ odd,
2. $\mathcal{C}$ is a subpartition of $V - t$ such that its elements are of degree k in $H - t$ and cover all neighbors of s but t.

Example



Remark

1. It implies Jordán's splitting off result on $(4, 2)$-connected graphs.

Splitting off : $(2k,k)$-connectivity

Theorem (Durand de Gevigney, Szigeti)

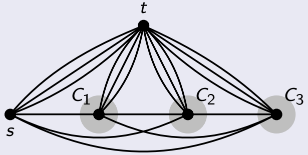
Let $H=(V+s,E)$ be a $(2k,k)$-connected graph in V with $k\geq 2$ and $d_H(s)$ even. There exists a complete splitting-off at s preserving $(2k,k)$-connectivity if and only if there exists no obstacle at s.

Definition

The pair $(t,\mathcal{C})$ is called an *obstacle* at s if

1. t is a neighbor of s with $d_H(s,t)$ odd,
2. $\mathcal{C}$ is a subpartition of $V-t$ such that its elements are of degree k in $H-t$ and cover all neighbors of s but t.

Example



Remark

1. It implies Jordán's splitting off result on $(4,2)$-connected graphs.
2. $H-su$ is $(2k,k)$-connected graph in V if and only if $u=t$.

Construction : (2k,k)-connectivity

Theorem (Durand de Gevigney, Szigeti)

A graph G is $(2k, k)$-connected with k even if and only if G can be obtained from K_3^k by a sequence of the following two operations :

1. *adding a new edge,*
2. *pinching a set F of k edges such that, for all vertices v, $d_F(v) \leq k$.*

Construction : (2k,k)-connectivity

Theorem (Durand de Gevigney, Szigeti)

A graph G is $(2k, k)$-connected with k even if and only if G can be obtained from K_3^k by a sequence of the following two operations :

1. *adding a new edge,*
2. *pinching a set F of k edges such that, for all vertices v, $d_F(v) \leq k$.*

Remark

1. These operations preserve $(2k, k)$-connectivity.

Construction : (2k,k)-connectivity

Theorem (Durand de Gevigney, Szigeti)

A graph G is $(2k, k)$-connected with k even if and only if G can be obtained from K_3^k by a sequence of the following two operations :

1. *adding a new edge,*
2. *pinching a set F of k edges such that, for all vertices v, $d_F(v) \leq k$.*

Remark

1. These operations preserve $(2k, k)$-connectivity.
2. It implies Jordán's construction result on $(4, 2)$-connected graphs.

Construction : (2k,k)-connectivity

Theorem (Durand de Gevigney, Szigeti)

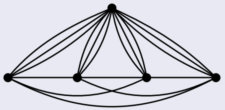
A graph G is $(2k, k)$-connected with k even if and only if G can be obtained from K_3^k by a sequence of the following two operations :

1. *adding a new edge,*
2. *pinching a set F of k edges such that, for all vertices v, $d_F(v) \leq k$.*

Remark

1. These operations preserve $(2k, k)$-connectivity.
2. It implies Jordán's construction result on $(4, 2)$-connected graphs.
3. It is not true for k odd.

Example ($k = 3$)



Augmentation : (2k,k)-connectivity

Theorem (Durand de Gevigney, Szigeti)

Let $G = (V, E)$ *be a graph* ($|V| \geq 3$) *and* $k \geq 2$ *an integer. The minimum cardinality of a set* F *of edges such that* $(V, E \cup F)$ *is* $(2k, k)$*-connected is equal to* $\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}_1} (2k - d_G(X)) + \sum_{X \in \mathcal{X}_2} (k - d_{G - v_X}(X)) \right\} \right\rceil$, *where* $\mathcal{X}_1 \cup \mathcal{X}_2$ *is a subpartition of* V *and* $v_X \in V \setminus X$.

Augmentation : (2k,k)-connectivity

Theorem (Durand de Gevigney, Szigeti)

Let $G = (V, E)$ be a graph ($|V| \geq 3$) and $k \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is $(2k, k)$-connected is equal to $\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}_1} (2k - d_G(X)) + \sum_{X \in \mathcal{X}_2} (k - d_{G - v_X}(X)) \right\} \right\rceil$, where $\mathcal{X}_1 \cup \mathcal{X}_2$ is a subpartition of V and $v_X \in V \setminus X$.

Proof

1. Minimal extension works (because $f_G^\flat$ is submodular on bi-sets),

Augmentation : (2k,k)-connectivity

Theorem (Durand de Gevigney, Szigeti)

Let $G = (V, E)$ *be a graph* ($|V| \geq 3$) *and* $k \geq 2$ *an integer. The minimum cardinality of a set* F *of edges such that* $(V, E \cup F)$ *is* $(2k, k)$*-connected is equal to* $\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}_1} (2k - d_G(X)) + \sum_{X \in \mathcal{X}_2} (k - d_{G-v_X}(X)) \right\} \right\rceil$, *where* $\mathcal{X}_1 \cup \mathcal{X}_2$ *is a subpartition of* V *and* $v_X \in V \setminus X$.

Proof

1. Minimal extension works (because f_G^{b} is submodular on bi-sets), and in case of parity step u can be chosen with $d_H(s, u)$ even.

Augmentation : (2k,k)-connectivity

Theorem (Durand de Gevigney, Szigeti)

Let $G = (V, E)$ be a graph ($|V| \geq 3$) and $k \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is $(2k, k)$-connected is equal to $\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}_1} (2k - d_G(X)) + \sum_{X \in \mathcal{X}_2} (k - d_{G - v_X}(X)) \right\} \right\rceil$, where $\mathcal{X}_1 \cup \mathcal{X}_2$ is a subpartition of V and $v_X \in V \setminus X$.

Proof

1. Minimal extension works (because $f_G^{\flat}$ is submodular on bi-sets), and in case of parity step u can be chosen with $d_H(s, u)$ even.
2. No obstacle exists in H, otherwise :

Augmentation : (2k,k)-connectivity

Theorem (Durand de Gevigney, Szigeti)

Let $G = (V, E)$ be a graph ($|V| \geq 3$) and $k \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is $(2k, k)$-connected is equal to $\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}_1} (2k - d_G(X)) + \sum_{X \in \mathcal{X}_2} (k - d_{G - v_X}(X)) \right\} \right\rceil$,
where $\mathcal{X}_1 \cup \mathcal{X}_2$ is a subpartition of V and $v_X \in V \setminus X$.

Proof

1. Minimal extension works (because $f_G^{\flat}$ is submodular on bi-sets), and in case of parity step u can be chosen with $d_H(s, u)$ even.
2. No obstacle exists in H, otherwise :
   1. by $H - st$ is $(2k, k)$-connected in V, $t = u$ and,

Augmentation : (2k,k)-connectivity

Theorem (Durand de Gevigney, Szigeti)

Let $G = (V, E)$ be a graph ($|V| \geq 3$) and $k \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is $(2k, k)$-connected is equal to $\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}_1} (2k - d_G(X)) + \sum_{X \in \mathcal{X}_2} (k - d_{G-v_X}(X)) \right\} \right\rceil$, where $\mathcal{X}_1 \cup \mathcal{X}_2$ is a subpartition of V and $v_X \in V \setminus X$.

Proof

1. Minimal extension works (because $f_G^\flat$ is submodular on bi-sets), and in case of parity step u can be chosen with $d_H(s, u)$ even.
2. No obstacle exists in H, otherwise :
   1. by $H - st$ is $(2k, k)$-connected in V, $t = u$ and,
   2. by $d_H(s, t)$ is odd, $t \neq u$.

Augmentation : (2k,k)-connectivity

Theorem (Durand de Gevigney, Szigeti)

Let $G = (V, E)$ be a graph ($|V| \geq 3$) and $k \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is $(2k, k)$-connected is equal to $\left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}_1} (2k - d_G(X)) + \sum_{X \in \mathcal{X}_2} (k - d_{G-v_X}(X)) \right\} \right\rceil$, *where $\mathcal{X}_1 \cup \mathcal{X}_2$ is a subpartition of V and $v_X \in V \setminus X$.*

Proof

1. Minimal extension works (because f_G^b is submodular on bi-sets), and in case of parity step u can be chosen with $d_H(s, u)$ even.
2. No obstacle exists in H, otherwise :
   1. by $H - st$ is $(2k, k)$-connected in V, $t = u$ and,
   2. by $d_H(s, t)$ is odd, $t \neq u$.
3. Hence a complete splitting off exists.

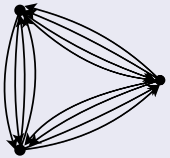
Orientation : (2k,k)-connectivity

Definition

A digraph D is called *$(2k, k)$-connected* if $|V| \geq 3$,

1. D is $2k$-arc-connected and,
2. for all $v \in V$, $D - v$ is k-arc-connected.

Example



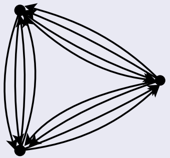
Orientation : (2k,k)-connectivity

Definition

A digraph D is called *$(2k, k)$-connected* if $|V| \geq 3$,

1. D is $2k$-arc-connected and,
2. for all $v \in V$, $D - v$ is k-arc-connected.

Example



Theorem (Z.Király, Szigeti)

An Eulerian graph G has a $(2k, k)$-connected orientation if and only if G is $(4k, 2k)$-connected.

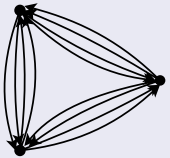
Orientation : (2k,k)-connectivity

Definition

A digraph D is called *$(2k,k)$-connected* if $|V| \geq 3$,

1. D is $2k$-arc-connected and,
2. for all $v \in V$, $D - v$ is k-arc-connected.

Example



Theorem (Z.Király, Szigeti)

An Eulerian graph G has a $(2k,k)$-connected orientation if and only if G is $(4k,2k)$-connected.

Open problem

Is it true for non Eulerian graphs ?

Orientation : Proof

Theorem (Nash-Williams' pairing for global edge-connectivity)

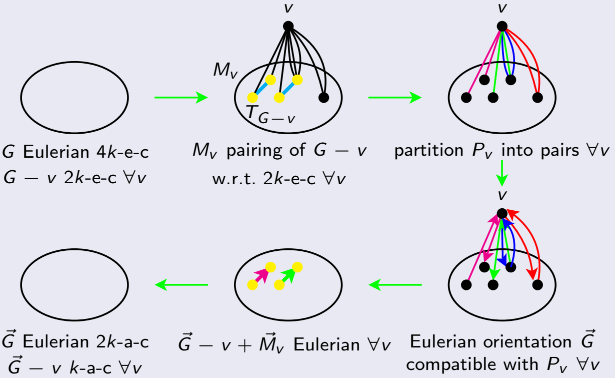
$\forall$ $2k$-edge-connected graph G, $\exists$ a pairing M of the odd degree vertices T_G of G s. t. for every Eulerian orientation $\vec{G} + \vec{M}$, $\vec{G}$ is k-arc-connected.

Orientation : Proof

Theorem (Nash-Williams' pairing for global edge-connectivity)

$\forall$ $2k$-edge-connected graph G, $\exists$ a pairing M of the odd degree vertices T_G of G s. t. for *every* Eulerian orientation $\vec{G} + \vec{M}$, $\vec{G}$ is k-arc-connected.

Proof



Conclusion

What we have seen :

1. Complete splitting off theorem on $(2k, k)$-connectivity,

Conclusion

What we have seen :

1. Complete splitting off theorem on $(2k, k)$-connectivity,
2. Min-max theorem for $(2k, k)$-connectivity augmentation problem,

Conclusion

What we have seen :

1. Complete splitting off theorem on $(2k, k)$-connectivity,
2. Min-max theorem for $(2k, k)$-connectivity augmentation problem,
3. Construction for $(2k, k)$-connectivity when k is even,

Conclusion

What we have seen :

1. Complete splitting off theorem on $(2k, k)$-connectivity,
2. Min-max theorem for $(2k, k)$-connectivity augmentation problem,
3. Construction for $(2k, k)$-connectivity when k is even,
4. Orientation theorem for $(2k, k)$-connectivity when G is Eulerian.

Conclusion

What we have seen :

1. Complete splitting off theorem on $(2k, k)$-connectivity,
2. Min-max theorem for $(2k, k)$-connectivity augmentation problem,
3. Construction for $(2k, k)$-connectivity when k is even,
4. Orientation theorem for $(2k, k)$-connectivity when G is Eulerian.

What we haven't seen :

Conclusion

What we have seen :

1. Complete splitting off theorem on $(2k, k)$-connectivity,
2. Min-max theorem for $(2k, k)$-connectivity augmentation problem,
3. Construction for $(2k, k)$-connectivity when k is even,
4. Orientation theorem for $(2k, k)$-connectivity when G is Eulerian.

What we haven't seen :

1. Algorithm for $(2k, k)$-connectivity augmentation problem,

Conclusion

What we have seen :

1. Complete splitting off theorem on $(2k, k)$-connectivity,
2. Min-max theorem for $(2k, k)$-connectivity augmentation problem,
3. Construction for $(2k, k)$-connectivity when k is even,
4. Orientation theorem for $(2k, k)$-connectivity when G is Eulerian.

What we haven't seen :

1. Algorithm for $(2k, k)$-connectivity augmentation problem,
2. Construction for $(2k, k)$-connectivity when k is odd,

Conclusion

What we have seen :

1. Complete splitting off theorem on $(2k, k)$-connectivity,
2. Min-max theorem for $(2k, k)$-connectivity augmentation problem,
3. Construction for $(2k, k)$-connectivity when k is even,
4. Orientation theorem for $(2k, k)$-connectivity when G is Eulerian.

What we haven't seen :

1. Algorithm for $(2k, k)$-connectivity augmentation problem,
2. Construction for $(2k, k)$-connectivity when k is odd,
3. Orientation theorem for $(2k, k)$-connectivity when G is arbitrary.

Thank you for your attention !