# Edge-connectivity augmentations of graphs and hypergraphs

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# Edge-connectivity augmentation problems

*Graphs*

- global edge-connectivity augmentation [Watanabe, Nakamura],
- global edge-connectivity augmentation over symmetric parity families [Sz],
- node to area global edge-connectivity augmentation [Ishii, Hagiwara],
- global edge-connectivity augmentation by attaching stars [B. Fleiner],
- global edge-connectivity augmentation with partition constraint [Bang-Jensen, Gabow, Jordán, Sz].
- local edge-connectivity augmentation [Frank],
- local edge-connectivity augmentation by attaching stars [Jordán, Sz],

# Edge-connectivity augmentation problems

### *Hypergraphs*

- global edge-connectivity augmentation in hypergraphs by adding graph edges [Bang-Jensen, Jackson],
- global edge-connectivity augmentation in hypergraphs by adding uniform hyperedges [T. Király],
- local edge-connectivity augmentation in hypergraphs by adding graph edges (NP-complete) [Cosh, Jackson, Z. Király],
- local edge-connectivity augmentation in hypergraphs by adding a hypergraph of minimum total size [Sz].

# Edge-connectivity augmentation problems

## *Set functions*

- covering a symmetric crossing supermodular set function by a graph [Benczúr, Frank],
- covering a symmetric crossing supermodular set function by a uniform hypergraph [T. Király],
- covering a symmetric crossing supermodular set function $p \neq 1$ by a graph with partition constraint [Grappe, Sz],
- covering a symmetric skew-supermodular set function by a graph (NP-complete) [Z. Király],
- covering a symmetric semi-monotone set function by a graph [Ishii ; Grappe, Sz],
- covering a symmetric skew-supermodular set function by a hypergraph of minimum total size [Sz].

# Graphs : Basic Problem

## Global edge-connectivity augmentation of a graph

- Given a graph $G = (V, E)$ and an integer $k$, what is the minimum number $\gamma$ of new edges whose addition results in a $k$-edge-connected graph ?
- $\gamma := \min\{|F| : d_{G+F}(X) \geq k \ \forall \emptyset \neq X \subset V\}$
  $= \min\{|F| : d_{(V,F)}(X) \geq k - d_G(X) \ \forall \emptyset \neq X \subset V\}$.
- $p_1(X) = k$ and $p_2(X) = k - d_G(X)$ are symmetric, crossing supermodular.

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# Graphs : Problem with partition constraint

## Global edge-connectivity augmentation of a graph with partition constraint

- Given a bipartite graph $G = (V_1, V_2; E)$ and an integer $k$, what is the minimum number $\gamma$ of new edges whose addition results in a $k$-edge-connected bipartite graph ?
- Given a graph $G = (V, E)$, a partition $\mathcal{P}$ of $V$ and an integer $k$, what is the minimum number $\gamma$ of new edges between different members of $\mathcal{P}$ whose addition results in a $k$-edge-connected graph ?
  - $(G = (V_1, V_2; E), \mathcal{P} = \{V_1, V_2\})$ = Bipartite graph Problem
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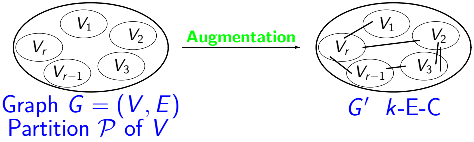

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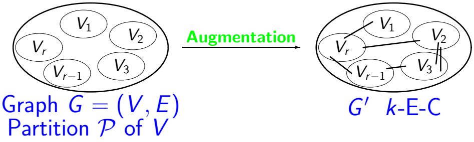

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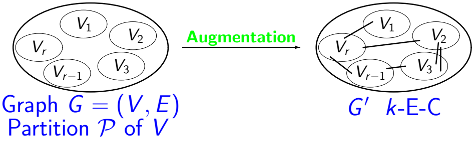

# Connectivity functions

## Symmetric function

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## Crossing supermodular function

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$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y).$$

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## Well-known examples

1. $p(X) = k$,
2. $p(X) = -d_G(X)$, (degree function of a graph)
3. $p(X) = k - d_G(X)$,
4. $p(X) = p'(X) - d_G(X)$, ($p'(X)$ is a symmetric crossing supermodular function).

# Covering a function

## Covering

A graph $H = (V, F)$ covers a function $p : 2^V \rightarrow \mathbb{Z}$ if

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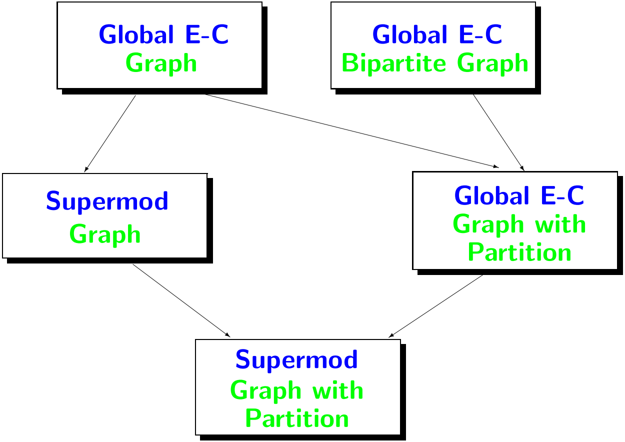
## Minimization problem 2

Given a symmetric crossing supermodular function $p$ on $V$ and a graph $G = (V, E)$, what is the minimum number of new edges such that the graph $H = (V, E + F)$ covers $p$?

# Relations among these problems

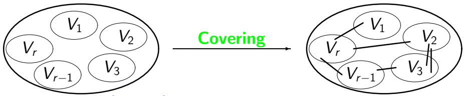


# Covering a function : Problem with partition constraint

## Covering a function by a graph with partition constraint

Given a graph $G = (V, E)$, a partition $\mathcal{P}$ of $V$ and a symmetric crossing supermodular function $p$, what is the minimum number $\gamma$ of new edges between different members of $\mathcal{P}$ whose addition results in a graph that covers $p$?

- $(G = (V, E), \mathcal{P} = \{\{v\} : v \in V\}$ and $p) =$ Covering of $p$
- $(G = (V, E), \mathcal{P}$ and $p = k) =$ Global edge-connectivity augmentation of a graph with partition constraint




Graph $G = (V, E)$
Partition $\mathcal{P}$ of $V$
Function $p$

$G'$ covers $p$

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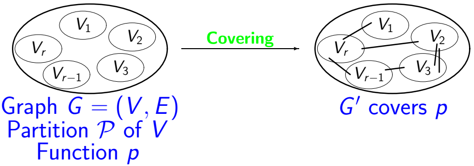

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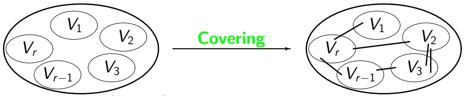


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## Theorem (Watanabe, Nakamura)

*Let $G = (V, E)$ be a graph and $k \geq 2$. Then the minimum number $\gamma$ of new edges whose addition results in a $k$-edge-connected graph is*

$$\gamma = \alpha.$$

# Results : Graph problem with partition constraint

## Lowerbound

Let $\Phi := \max\{\alpha, \beta_1, \ldots, \beta_r\}$ where

$$\alpha := \max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} (k - d(X)) \rceil : \mathcal{X} \in \mathcal{S}(V)\},$$

$$\beta_j := \max\{\sum_{Y \in \mathcal{Y}} (k - d(Y)) : \mathcal{Y} \in \mathcal{S}(V_j)\} \quad \forall 1 \leq j \leq r.$$

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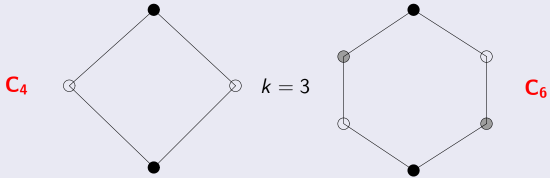
## Attention !

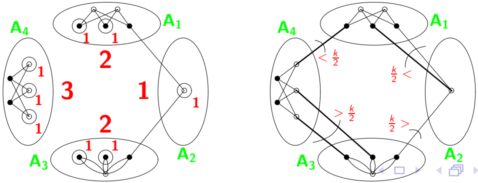


## $C_4$-configuration

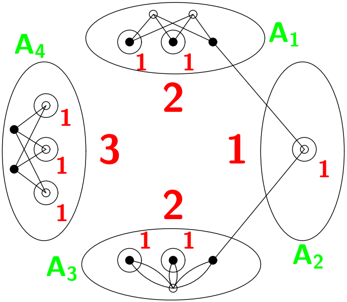
A partition $\{A_1, A_2, A_3, A_4\}$ of $V$ is a $C_4$-configuration of $G$ if $k$ is odd and

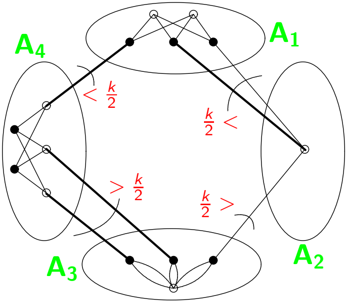
$$
\begin{array}{rcll}
k - d(A_i) & > & 0 & \forall 1 \leq i \leq 4, \\
d(A_i, A_{i+2}) & = & 0 & \forall 1 \leq i \leq 2, \\
\sum_{X \in \mathcal{X}_i} (k - d(X)) & = & k - d(A_i) & \exists \mathcal{X}_i \in \mathcal{S}(A_i)\ \forall 1 \leq i \leq 4, \\
\mathcal{X}_j \cup \mathcal{X}_{j+2} & \in & \mathcal{S}(V_l) & \exists 1 \leq l \leq r\ \exists 1 \leq j \leq 2, \\
k - d(A_i) + k - d(A_{i+2}) & = & \Phi & \forall 1 \leq i \leq 2.
\end{array}
$$

$C_4$-configuration



$C_4$-**configuration**



## $C_6$-configuration

A partition $\{A_1, A_2, \ldots, A_6\}$ of $V$ is a $C_6$-configuration of $G$ if $k$ is odd,

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\begin{aligned}
k - d(A_i) &= 1 \quad \forall 1 \leq i \leq 6, \\
k - d(A_i \cup A_{i+1}) &= 1 \quad \forall 1 \leq i \leq 6, (A_7 = A_1) \\
\Phi &= 3, \\
k - d(A'_i) &= 1 \quad \exists 1 \leq j_1, j_2, j_3 \leq r,\ \forall 1 \leq i \leq 6,\ \exists A'_i \subseteq A_i \cap V_{j_{i-3\lfloor}}
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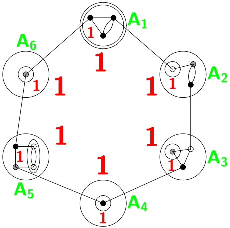
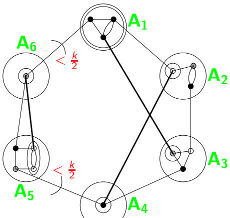
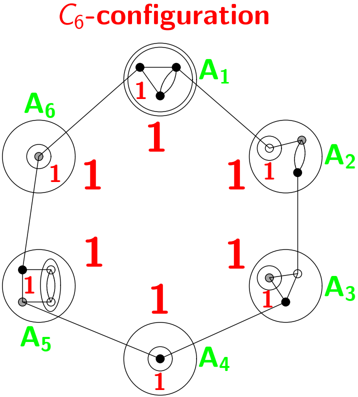
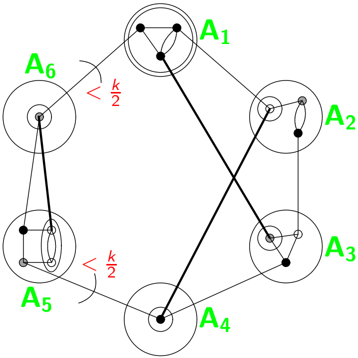
$C_6$-configuration




$C_6$-configuration
A1
A2
A3
A4
A5
A6
1


A1
A2
A3
A4
A5
A6
$< \frac{k}{2}$
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# Results : Graph problem with partition constraint

**Theorem (Bang-Jensen, Gabow, Jordán, Sz)**

*Let* $G = (V, E)$ *be a graph,* $\mathcal{P}$ *a partition of* $V$ *and* $k \geq 2$*. Then the minimum number* $\gamma$ *of new edges between different members of* $\mathcal{P}$ *whose addition results in a* $k$*-edge-connected graph is*

$$\gamma = \begin{cases} \Phi & \text{if } G \text{ contains no } C_4\text{- and no } C_6\text{-configuration,} \\ \Phi + 1 & \text{otherwise.} \end{cases}$$

# Results : Covering crossing supermodular functions

## Lowerbound

Let $\Psi := \max\{\alpha_p, L-1\}$ where

$$\alpha_p := \max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} p(X) \rceil : \mathcal{X} \in \mathcal{S}(V)\},$$

$$L := \max\{l : \{Q_1, \ldots, Q_l\} \text{ partition of } V, \; p(\bigcup_{i \in I} Q_i) \geq 1 \; \forall I, p(Q_j) = 1 \; \exists j\}.$$

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## Theorem (Benczúr, Frank)

*Let $p : 2^V \to \mathbb{Z}_+$ be a symmetric crossing supermodular set function. Then the minimum number $\gamma$ of edges of a graph $H = (V, F)$ that covers $p$ is*

$$\gamma = \Psi.$$

# Results : Covering a symmetric crossing supermodular function by a graph with partition constraint

## Lowerbound

Let $\Phi := \max\{\alpha_p, \beta_1, \ldots, \beta_r\}$ where $q(X) = p(X) - d_G(X)$ and

$$\alpha_p := \max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} q(X) \rceil : \mathcal{X} \in \mathcal{S}(V)\},$$

$$\beta_j := \max\{\sum_{Y \in \mathcal{Y}} q(Y) : \mathcal{Y} \in \mathcal{S}(V_j)\} \quad \forall 1 \leq j \leq r.$$

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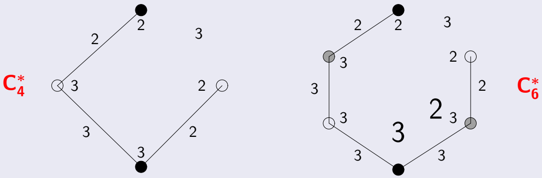

# Results : Covering a symmetric crossing supermodular function by a graph with partition constraint

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A partition $\{A_1, A_2, A_3, A_4\}$ of $V$ is a $C_4^*$-configuration of $G$ if $\forall 1 \leq i \leq 4$

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q(A_i) & > & 0, \\
d(A_i, A_{i+2}) & = & 0, \\
\sum_{X \in \mathcal{X}_i} q(X) & = & q(A_i) \quad \exists \mathcal{X}_i \in \mathcal{S}(A_i), \\
\mathcal{X}_j \cup \mathcal{X}_{j+2} & \in & \mathcal{S}(V_l) \quad \exists 1 \leq l \leq r \ \exists 1 \leq j \leq 2, \\
q(A_i) + q(A_{i+2}) & = & \Phi, \\
p(A_i) + p(A_{i+1}) & - & p(A_i \cup A_{i+1}) \text{ is odd}, \\
p(A_i \cup A_{i-1}) + p(A_i \cup A_{i+1}) & = & p(A_{i-1}) + p(A_{i+1}).
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$$\begin{aligned} q(A_i) &= 1, \\ q(A_i \cup A_{i+1}) &= 1, \ (A_7 = A_1) \\ \Phi &= 3, \\ q(A'_i) &= 1 \qquad \exists 1 \leq j_1, j_2, j_3 \leq r, \ \exists A'_i \subseteq A_i \cap V_{j_{i-3\lfloor \frac{(i-1)}{3} \rfloor}}. \end{aligned}$$

# Results : Covering a symmetric crossing supermodular function by a graph with partition constraint

## Theorem (Grappe, Sz)

*Let $G = (V, E)$ be a graph, $\mathcal{P}$ a partition of $V$ and $p : 2^V \to \mathbb{Z}_+$ a symmetric crossing supermodular set function with $p \neq 1$.*
*Then the minimum number $\gamma$ of new edges between different members of $\mathcal{P}$ whose addition results in a graph that covers $p$ is*

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