On 2-vertex-connected orientations of graphs

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Joint work with:
Joseph Cheriyan (Waterloo) and Olivier Durand de Gevigney (Grenoble)
Outline

- Definitions
- Orientations with arc-connectivity constraints
- Orientations with vertex-connectivity constraints
  - $k$-vertex-connected orientations
  - 2-vertex-connected orientations
- Packing of special spanning subgraphs
- Matroid theory
On 2-vertex-connected orientations
Connectivity

Definitions

An **undirected** graph $G = (V, E)$ is

- **connected** if there exists a $(u, v)$-path $\forall u, v \in V$,
- **$k$-edge-connected** if $D - X$ is connected $\forall X \subset E, |X| \leq k - 1$,
- **$k$-vertex-connected** if $D - X$ is connected $\forall X \subset V, |X| = k - 1$ and $|V| > k$.

A **directed** graph $D = (V, A)$ is

- **strongly connected** if $\exists$ a directed $(u, v)$-path $\forall (u, v) \in V \times V$,
- **$k$-arc-connected** if $D - X$ is strongly connected $\forall X \subset A, |X| \leq k - 1$,
- **$k$-vertex-conn.** if $D - X$ is strongly connected $\forall X \subset V, |X| = k - 1$ and $|V| > k$. 

Z. Szigeti (G-SCOP, Grenoble)
Theorem (Nash-Williams 1960)

Given an undirected graph \( G \),

- there exists a \( k \)-arc-connected orientation of \( G \)
- \( G \) is \( 2k \)-edge-connected.
Theorem (Nash-Williams 1960)

Given an undirected graph $G$, there exists a $k$-arc-connected orientation of $G$ if and only if $G$ is $2k$-edge-connected.

**necessity:**

$$\vec{G}$$

$X \cup V - X$
Theorem (Nash-Williams 1960)

Given an undirected graph $G$,

- there exists a $k$-arc-connected orientation of $G$  \(\iff\)
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necessity:
Conjecture (Thomassen 1989)

There exists a function $f(k)$ such that every $f(k)$-vertex-connected graph has a $k$-vertex-connected orientation.
**$k$-vertex-connected orientation**

**Conjecture (Thomassen 1989)**

There exists a function $f(k)$ such that every $f(k)$-vertex-connected graph has a $k$-vertex-connected orientation.

**Conjecture (Frank 1995)**

Given an undirected graph $G = (V, E)$ with $|V| > k$,
- there exists a $k$-vertex-connected orientation of $G$ if and only if $G - X$ is $(2k - 2|X|)$-edge-connected for all $X \subseteq V$ with $|X| < k$. 

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On 2-vertex-connected orientations

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**necessity :**

\[ \tilde{G} - X \text{ is } (k - |X|)\text{-vertex-connected} \]
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Remark
Frank’s conjecture would imply that $f(k) \leq 2k$. 

Conjecture (Frank 1995)

Given an undirected graph $G = (V, E)$ with $|V| > 2$,

- there exists a 2-vertex-connected orientation of $G$ \iff
- $G$ is 4-edge-connected and $G - v$ is 2-edge-connected for all $v \in V$. 
Conjecture (Frank 1995)

Given an undirected graph $G = (V, E)$ with $|V| > 2$,
- there exists a 2-vertex-connected orientation of $G$
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Theorem (Berg-Jordán 2006)

Given an Eulerian graph $G = (V, E)$ with $|V| > 2$,
- there exists a 2-vertex-connected Eulerian orientation of $G$
- $G$ is 4-edge-connected and $G - v$ is 2-edge-connected for all $v \in V$. 
Theorem (Jordán 2005)

Every 18-vertex-connected graph has a 2-vertex-connected orientation:
\[ f(2) \leq 18. \]
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### Remark

Frank’s conjecture would imply that $f(2) \leq 4$.**
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Given an Eulerian graph \( G = (V, E) \) with \( |V| > 2 \),

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Given an Eulerian graph $G = (V, E)$ with $|V| > 2$,

- there exists a 2-vertex-connected orientation of $G$ \iff $G - v$ is 2-edge-connected for all $v \in V$.

Find a spanning subgraph $G$ such that

- $G - v$ is 2-edge-connected for all $v \in V$,
- $G - v$ contains 2 edge-disjoint connected spanning subgraphs for all $v \in V$,
- $G$ contains 2 edge-disjoint 2-vertex-connected spanning subgraphs.
Proof

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Theorem (Jordán 2005) (Tool 2)

Every 6$k$-vertex-connected graph contains $k$ edge-disjoint 2-vertex-connected spanning subgraphs.
Proof

(Jordán 2005)

- Let $H$ be a 18-vertex-connected graph.
- By Tool 2, $H$ contains 3 edge-disjoint 2-vertex-connected spanning subgraphs: $G_1$, $G_2$, and $G_3$.
- Let $G' := G_1 \cup G_2$. Then $G' - v$ is 2-edge-connected for every vertex $v$.
- Let $T$ be the set of odd degree vertices in $G'$.

- By Tool 1, $G$, and hence $H$, has a 2-vertex-connected orientation.
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- Let $T$ be the set of odd degree vertices in $G'$. Since $G_3$ is connected, it contains a $T$-join $F$. Then $G := G' + F$ is Eulerian.
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Proof of the new upperbound

**Theorem**

Every *14*-vertex-connected graph has a *2*-vertex-connected orientation.

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Theorem (Cheriyan, Durand de Gevigney, Szigeti 2011) (Tool 2’)

Every \((6k + 2\ell)\)-vertex-connected graph contains \( k \) 2-vertex-connected and \( \ell \) connected edge-disjoint spanning subgraphs.
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How to prove them?

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Rigidity Matroid

Definition

Given a graph $G = (V, E)$ with $n = |V|$. 

Rigidity Matroid:

- independent sets: $\mathcal{R}(G) = \{F \subseteq E : i_F(X) \leq 2|X| - 3 \quad \forall X \subseteq V\}$ (Crapo 1979).
- rank function $r_{\mathcal{R}}(F) = \min \{\sum_{X \in \mathcal{H}} (2|X| - 3) : \mathcal{H} \text{ set of subsets of } V \text{ covering } F\}$ (Lovász-Yemini 1982).
- $G$ is rigid if $r_{\mathcal{R}}(E) = 2n - 3$ (Laman 1970).
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![a rigid graph](image)
Remark

Every rigid graph is 2-vertex-connected.
Remark

Every **rigid** graph is 2-vertex-connected.

Proof

- Suppose $G$ is not 2-vertex-connected.
- Then there exists a covering \{X, Y\} of $E$ such that $|X \cap Y| \leq 1$.
- $r_R(E) \leq 2|X| - 3 + 2|Y| - 3 = 2|X \cup Y| + 2|X \cap Y| - 6 \leq 2n - 4$.
- $G$ is not **rigid**.
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Every rigid graph is 2-vertex-connected.

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Theorem (Lovász-Yemini 1982)

Every 6-vertex-connected graph is rigid.
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Theorem (Lovász-Yemini 1982)
Every 6-vertex-connected graph is rigid.

Theorem (Jordán 2005)
Every $6k$-vertex-connected graph contains $k$ rigid edge-disjoint spanning subgraphs.
Circuit Matroid

Definition

Given a graph $G = (V, E)$ with $n = |V|$.

Circuit Matroid:

- independent sets: $\mathcal{C}(G) =$ the edge sets of the forests of $G$.
- rank function $r_C(F) = n - c(F)$.
- $G$ is connected if $\exists$ a spanning tree ($r_C(E) = n - 1$).
Theorem (Tutte 1961)

Given an undirected graph $G$ and an integer $\ell \geq 1$,

- there exist $\ell$ edge-disjoint spanning trees of $G$ \iff
- for every partition $\mathcal{P}$ of $V$, $G, \ell = 2$
Theorem (Tutte 1961)

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Packing of connected spanning subgraphs

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- for every partition $\mathcal{P}$ of $V$, $|E(\mathcal{P})| \geq \ell(|\mathcal{P}| - 1)$.

Remark

Every $2\ell$-edge-connected graph contains $\ell$ edge-disjoint spanning trees.

$|E(\mathcal{P})| = \frac{1}{2} \sum_{P \in \mathcal{P}} d(P) \geq \frac{1}{2} 2\ell|\mathcal{P}| > \ell(|\mathcal{P}| - 1)$. 
Theorem (Cheriyan, Durand de Gevigney, Szigeti 2011)

Every \((6k + 2\ell)\)-vertex-connected graph contains \(k\) rigid and \(\ell\) connected edge-disjoint spanning subgraphs.
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Every \((6k + 2\ell)\)-vertex-connected graph contains \(k\) rigid and \(\ell\) connected edge-disjoint spanning subgraphs.

Tool

\[ M_{k,\ell}(G) = \text{matroid union of } k \text{ copies of } \mathcal{R}(G) \text{ and } \ell \text{ copies of } \mathcal{C}(G). \]

- independent sets are the union of \(k\) independent sets of \(\mathcal{R}(G)\) and \(\ell\) independent sets of \(\mathcal{C}(G)\).
- rank \(r_{M_{k,\ell}}(E) = \min_{F \subseteq E} kr_{\mathcal{R}}(F) + \ell r_{\mathcal{C}}(F) + |E \setminus F|\). (Edmonds 1968)
- \(G\) contains \(k\) rigid and \(\ell\) connected edge-disjoint spanning subgraphs \(\iff r_{M_{k,\ell}}(E) = k(2n - 3) + \ell(n - 1).\)
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Packing of rigid and connected spanning subgraphs

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Packing of rigid and connected spanning subgraphs

Theorem (Cheriyan, Durand de Gevigney, Szigeti 2011)

Every \((6k + 2\ell)\)-vertex-connected graph contains \(k\) rigid and \(\ell\) connected edge-disjoint spanning subgraphs.

Tool

\(\mathcal{M}_{k,\ell}(G) = \text{matroid union of } k \text{ copies of } \mathcal{R}(G) \text{ and } \ell \text{ copies of } \mathcal{C}(G).\)

- independent sets are the union of \(k\) independent sets of \(\mathcal{R}(G)\) and \(\ell\) independent sets of \(\mathcal{C}(G)\).
- rank \(r_{\mathcal{M}_{k,\ell}}(E) = \min_{F \subseteq E} kr_{\mathcal{R}}(F) + \ell r_{\mathcal{C}}(F) + |E \setminus F|\). (Edmonds 1968)
- \(G\) contains \(k\) rigid and \(\ell\) connected edge-disjoint spanning subgraphs \(\iff\) \(r_{\mathcal{M}_{k,\ell}}(E) = k(2n - 3) + \ell(n - 1)\).

Our proof is completely different.

It provides a transparent proof for the theorem of Lovász-Yemini 1982.

It enabled us to weaken the condition: instead of \((6k + 2\ell)\)-vertex-connectivity we used \((6k + 2\ell, 2k)\)-connectivity.
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Definition

$G$ is $(a, b)$-connected if $G - X$ is $(a - b|X|)$-edge-conn. $\forall X \subseteq V$. 

Z. Szigeti (G-SCOP, Grenoble)

On 2-vertex-connected orientations

12 January 2012

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Conjecture (Frank 1995)

Given an undirected graph \(G = (V, E)\) with \(|V| > 2\),
- there exists a 2-vertex-connected orientation of \(G\)
- \(G\) is \((4, 2)\)-connected.
Theorem (Lovász-Yemini 1982)
Every 6-vertex-connected graph is rigid.

Theorem (Jordán 2005)
Every $6k$-vertex-connected graph contains $k$ rigid edge-disjoint spanning subgraphs.

Theorem (Jackson et Jordán 2009)
Every simple $(6, 2)$-connected graph is rigid.

Theorem (Cheriyan, Durand de Gevigney, Szigeti 2011)
Every simple $(6k + 2\ell, 2k)$-connected graph contains $k \geq 1$ rigid (2-vertex-connected) and $\ell$ connected edge-disjoint spanning subgraphs.
Thank you for your attention!