On packing of arborescences with matroid constraints

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Joint work with:
Olivier Durand de Gevigney and Viet Hang Nguyen (Grenoble)
Outline

- **Motivations**
  - Undirected = Orientation + Directed
  - Rigidity

- **Results**
  - Undirected: Matroid-based packing of rooted-trees
  - Directed: Matroid-based packing of rooted-arborescences
  - Orientation: Supermodular function

- **Further results**
  - Algorithmic aspects
  - Generalization

- **Conclusion**
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Theorem (Edmonds 1973)

Let $D$ be a directed graph, $r$ a vertex of $D$ and $k$ a positive integer.

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Body-Bar Framework
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Theorem (Katoh, Tanigawa 2012)
"Rigidity" of a Body-Bar Framework with Bar-Boundary can be characterized by the existence of a matroid-based rooted-tree decomposition.
Matroids

**Definition**

For $\mathcal{I} \subseteq 2^S$, $\mathcal{M} = (S, \mathcal{I})$ is a matroid if

1. $\mathcal{I} \neq \emptyset$,
2. If $X \subseteq Y \in \mathcal{I}$ then $X \in \mathcal{I}$,
3. If $X, Y \in \mathcal{I}$ with $|X| < |Y|$ then $\exists \ y \in Y \setminus X$ such that $X \cup y \in \mathcal{I}$.
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**Examples**

1. Sets of linearly independent vectors in a vector space,
2. Edge-sets of forests of a graph,
3. $U_{n,k} = \{X \subseteq S : |X| \leq k\}$ where $|S| = n$,
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Notion

1. **independent sets** \( \mathcal{I} \),
   1. any subset of an independent set is independent,

2. **base** = maximal independent set,
   1. all basis are of the same size,

3. **rank function** : \( r(X) = \max\{|Y| : Y \in \mathcal{I}, Y \subseteq X\} \).
   1. non-decreasing,
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A matroid-based rooted-graph is a quadruple $(G, \mathcal{M}, S, \pi)$:

1. $G = (V, E)$ undirected graph,
2. $\mathcal{M}$ a matroid on a set $S = \{s_1, \ldots, s_t\}$.
3. $\pi$ a placement of the elements of $S$ at vertices of $V$. 

\[ U_3,2 \]

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\[\pi(s_1), \pi(s_2), \pi(s_3)\]

Definition

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\section*{\textit{M}-based packing of rooted-trees}

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\textbf{Remark}

For the \textbf{free matroid} \( \textit{M} \) with all \( k \) roots at a vertex \( r \),

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**Theorem (Katoh, Tanigawa 2012)**

Let $(G, \mathcal{M}, S, \pi)$ be a matroid-based rooted-graph. Then

- There is a matroid-based packing of rooted-trees in $(G, \mathcal{M}, S, \pi)$ if and only if $\pi$ is $\mathcal{M}$-independent and $(G, \mathcal{M}, S, \pi)$ is partition-connected.
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1. \(T\) is an \(r\)-arborescence,

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\[\begin{align*}
\pi(s_1) & \quad \text{\textcolor{red}{\quad T_1}} \\
\pi(s_2) & \quad \text{\textcolor{blue}{\quad T_2}} \\
\pi(s_3) & \quad \text{\textcolor{green}{\quad T_3}}
\end{align*}\]
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Proof of necessity

Let \( \{(T_1, s_1), \ldots, (T_{|S|}, s_{|S|})\} \) be a matroid-based packing of rooted-arborescences in \((D, \mathcal{M}, S, \pi)\) and \( v \in X \subseteq V \).

Let \( B = \{s_i \in S : v \in V(T_i)\} \), \( B_1 = B \cap S_X \) and \( B_2 = B \setminus B_1 \).

Since \( S_v \subseteq B_1 \subseteq B \) is a base of \( \mathcal{M} \), \( \pi \) is \( \mathcal{M} \)-independent.

Since, for each root \( s_i \) in \( B_2 \), there exists an arc of \( T_i \) that enters \( X \) and the arborescences are arc-disjoint, \( \rho_D(X) \geq |B_2| = |B| - |B_1| = r_M(S) - r_M(B_1) \geq r_M(S) - r_M(S_X) \) that is \((D, \mathcal{M}, S, \pi)\) is rooted-connected.
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Let $G = (V, E)$ be an undirected graph and $h : 2^V \rightarrow \mathbb{Z}_+$ an intersecting supermodular non-increasing set-function.

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Z. Szigeti (G-SCOP, Grenoble)
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Orientation results

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Plan executed

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- \((G, \mathcal{M}, S, \pi)\) is partition-connected.
Theorem (Katoh, Tanigawa 2012)

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Theorem (Durand de Gevigney, Nguyen, Szigeti 2012)

1. A matroid-based packing of rooted-arborescences can be found in polynomial time,
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Open problem

Combinatorial algorithm for finding a matroid-based packing of rooted-arborescences of minimum weight?
Thank you for your attention!