RELIABLE ORIENTATIONS
OF EULERIAN GRAPHS

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JOINT WORK WITH ZOLTÁN KIRÁLY
DEFINITIONS:

**$k$-Edge-Connectivity**

**Undirected**

\[ d(x) \geq k \]

**Directed**

\[ \phi(x) \geq k \]

**$k$-Arc-Connectivity**

\[ x \neq \emptyset \quad v-x \neq \emptyset \]

**$k$-Vertex-Connectivity**

\[ |x| < k \quad \text{directed} \]

\[ |V| \geq k + 1 \]
DEF. $G=(V,E)$ is **minimally $k$-edge-connected** if
- $G$ is $k$-edge-connected,
- $G-e$ is not $k$-edge-connected for all $e \in E$.

**EX.**

\[ \begin{array}{c}
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    \bullet \\
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    \bullet \\
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  \begin{array}{c}
    \begin{array}{c}
      e \\
      \bullet
    \end{array} \\
  \end{array}
\end{array} \]

- **Minimally 2-E-C**
- **Not minimally 2-E-C**

**Theorem (Mader)**

Every minimally $k$-edge-connected graph has a vertex of degree $k$. 

DEF. SPLITTING OFF

COMPLETE SPLITTING OFF

THEOREM (LOVA’S 2)

If $G = (V, E)$ is 2$k$-edge-connected, $s \in V$ with $\delta(s)$ even, then there exists a complete splitting off such that $G'$ is 2$k$-edge-connected.
THEOREM (NASH-WILLIAMS)

G has a $k$-arc-connected orientation if and only if $G$ is $2k$-edge-connected.

$\Rightarrow$

$\downarrow$

$\Rightarrow$

$G \xrightarrow{k\text{-A-C}} \overrightarrow{G} \xrightarrow{2k\text{-E-C}} G$


\text{LOVÁSZ: EASY BY SPLITTING OFF}
THEOREM (NASH-WILLIAMS)
If \( G=(V,E) \) is \( 2k \)-EDGE-CONNECTED then there exists a pairing \( M \) of \( T_G \) s.t.
\[
d_M(x) \leq d_G(x) - 2k \quad \forall x \neq \emptyset.
\]
where \( T_G = \{ u \in V : d_G(u) \text{ is odd} \} \) and a set \( M \) of new edges is a pairing if
\[
d_M(u) = \begin{cases} 
1 & \text{if } u \in T_G \\
0 & \text{if } u \in V - T_G 
\end{cases}
\]

REMARKS:

1. KIRÁLY, S2.: EASY BY SPLITTING OFF
2. NASH-WILLIAMS: FOR ALL EULERIAN ORIENTATION \( \vec{G} + \vec{M} \) of \( G + M \), \( \vec{G} \) is \( k \)-ARC-CONNECTED.

\[
\phi_G(x) = \phi_{G+M}(x) - \phi_M(x) \geq \frac{d_{G+M}(x)}{2} - d_M(x) = \frac{d_G(x) - d_M(x)}{2} \geq k \quad \forall x \neq \emptyset.
\]
PROOF: INDUCTION ON $|E|$.

CASE 1.

$\exists d(s)$ EVEN

COMPLETE SPLITTING OFF

(Lovász)

$G^{2k-E-C}$

$G'^{2k-E-C}$

$R: T_{G'} = T_G$

$T_G$

INDUCTION

$\exists M' \text{ PAIRING OF } T_{G'}$

$\exists M \text{ PAIRING OF } T_G$

$d_M(x) \leq d_{G'}(x) - 2k$

$d_M'(x) \leq d_{G'}(x) - 2k \ \forall x$

$d_M(x) \leq d_G(x) - 2k$

$d_M'(x) \leq d_G(x) - 2k \ \forall x$
CASE 2.

\[ T_G = V \]

Let \( G \) be a \( 2k \)-E-C.

\[ \exists e : G^e = G - e \quad \text{where} \quad 2k \text{-E-C} \]

\[ R : T_{G^e} = T_G - u - v \]

\[ \exists M' \text{ PAIRING OF } T_{G^e} \]

\[ M = M' \cup \{uv\} \]

Pairing of \( T_G \)

\[ d_M(x_i) = d_{M'}(x_i) \leq d_{G^e}(x_i) - 2k \]

\[ = d_G(x_i) - 2k \]

OR

\[ d_M(x_2) = d_{M'}(x_2) + 1 \leq d_{G^e}(x_2) + 1 - 2k \]

\[ = d_G(x_2) - 2k \]
**Theorem (Nash-Williams)**

If $G$ is $2k$-edge-connected then there exists a pairing $M$ s.t. for all Eulerian orientation $\tilde{G} + \tilde{M}$ of $G + M$, $\tilde{G}$ is $k$-arc-connected.

(Feasible pairing)

**Observation**

Given • an Eulerian graph $G = (V,E)$,
• a partition $P_V$ of the edges incident to $V$ into pairs $A \cup B$, $V$,

there exists an Eulerian orientation $\tilde{G}$ of $G$ compatible with all $P_V$.
EXAMPLE:
CONJECTURE (THOMASSEN, FRANK)

\[ G = (V,E) \] has a \( k \)-vertex-connected \((|V| \geq k+1)\) orientation

\[ G - X \text{ is } 2(k-|X|)\text{-edge-connected} \]

for all \( X \subset V, |X| < k \).

\[ \Downarrow \]

\[ \tilde{G} \text{ is } k\text{-V-C} \Rightarrow \]

\[ \tilde{G} - X \text{ is } (k-|X|)\text{-V-C} \Rightarrow \]

\[ \tilde{G} - X \text{ is } (k-|X|)\text{-A-C} \Rightarrow \]

\[ G - X \text{ is } 2(k-|X|)\text{-E-C} \quad \forall X \subset V, |X| < k. \]

\[ \Uparrow \text{ OPEN EVEN FOR } k = 2. \]
CONJECTURE \((k=2)\)
\[G = (V,E) \text{ has a } 2\text{-vertex-connected (} |V| \geq 3) \text{ orientation} \iff G \text{ is 4-edge-connected}\]
\[G - u \text{ is 2-edge-connected } \forall u \in V.\]

THEOREM (BERG-JORDÁN)
AN EULERIAN GRAPH \(G = (V,E)\) HAS AN ORIENTATION \(\tilde{G}\) SUCH THAT \(\tilde{G} - u\) IS 1-ARC-CONNECTED \(\forall u \in V\)
\[\iff G - u \text{ is 2-edge-connected } \forall u \in V.\]

CONJECTURE (FRANK)
AN EULERIAN GRAPH \(G = (V,E)\) HAS AN ORIENTATION \(\tilde{G}\) SUCH THAT \(\tilde{G} - u\) IS \(k\)-ARC-CONNECTED \(\forall u \in V\)
\[\iff G - u \text{ is } 2^k\text{-edge-connected } \forall u \in V.\]

THEOREM (KIRÁLY-SZ.) TRUE
SHORT, EASY PROOF
PROOF:

G EULERIAN
G-U 2k-E-C \( \forall \in V \)

\( M_\sigma \) FEASIBLE
PAIRING OF G-U

\( \tilde{G} \) EULERIAN
\( \tilde{G} - U + M_\sigma \) EULERIAN
\( \forall \in V \)

\( \tilde{G} - U \) R-A-C \( \forall \in V \)

COMPATIBLE EULERIAN ORIENTATION G

\( P_\sigma \)
Remark: Not true if $G$ is not Eulerian

$(G-u)$ is 2E-C $\forall u \in V \Rightarrow \nexists \tilde{G}: \tilde{G}-u$ is K-A-C $\forall u \in V$

Ex.

$G$

$G-u$ is 2-E-C $\forall u$

$\# \tilde{G}: \tilde{G}-u$ is 1-A-C $\forall u$:

WLOG $\delta(u) \leq 1$

$\downarrow$

$\tilde{G}-u$ is NOT 1-A-C

Remark: Not counter-example for conjecture of Frank, Thomassen:

$G$ is NOT 4-E-C.
**Remark:** Not true if \( G \) is not Eulerian

\[
\begin{align*}
G \text{ is } 2k-E-C & \quad \exists \tilde{G}: \tilde{G} \text{ is } k-A-C \\
G - u \text{ is } 2(k-1)-E-C \forall u \in V & \quad \tilde{G} - u \text{ is } (k-1)-A-C \forall \tilde{u} \in V
\end{align*}
\]

**Example:**

\[
\begin{align*}
G & \quad G \text{ is } 8-E-C \\
G - u \text{ is } 6-E-C \forall u \in V & \quad \tilde{G} \text{ is } 4-A-C, \tilde{G} - u \text{ is } 3-A-C \forall \tilde{u} \in \tilde{V}
\end{align*}
\]

\[
\begin{align*}
\text{WLOG } \delta(u) &= 4 \\
\exists u &\n \quad \tilde{G} - u \text{ is NOT } 3-A-C
\end{align*}
\]

**Remark:** Not counter-example for Conjecture of Frank, Thomassen:

\( G - u \text{ is NOT } 4-E-C \).