RELIABLE ORIENTATIONS OF EULERIAN GRAPHS

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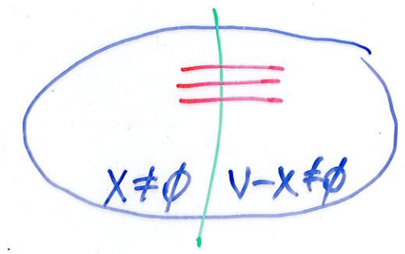
JOINT WORK WITH ZOLTÁN KIRÁLY

DEFINITIONS:

k-EDGE-CONNECTIVITY

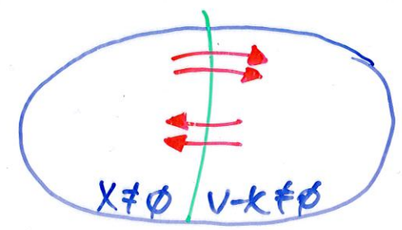
UNDIRECTED



$d(X) \geq k$

k-ARC-CONNECTIVITY

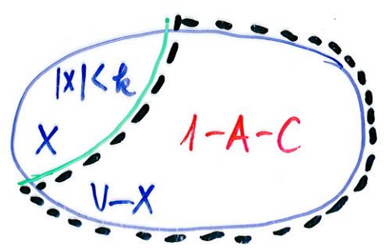
DIRECTED



$\varrho(X) \geq k$

k-VERTEX-CONNECTIVITY

DIRECTED



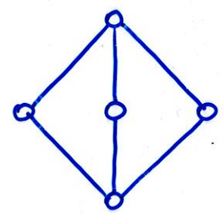
$|V| \geq k+1$

DEF. $G=(V,E)$ IS MINIMALLY k-EDGE-CONNECTED

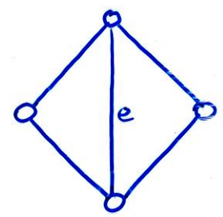
IF • G IS k-EDGE-CONNECTED,

• $G-e$ IS NOT k-EDGE-CONNECTED $\forall e \in E$.

EX.



MINIMALLY 2-E-C

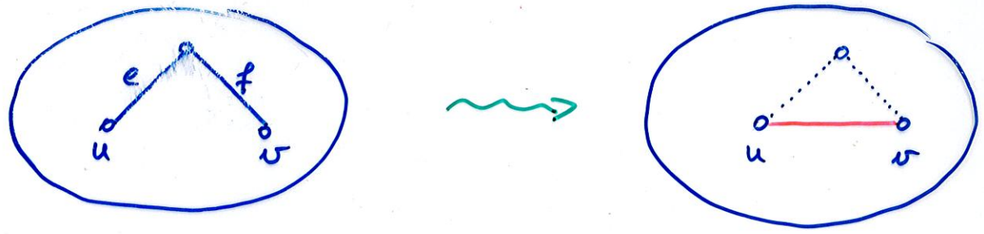


NOT MINIMALLY 2-E-C

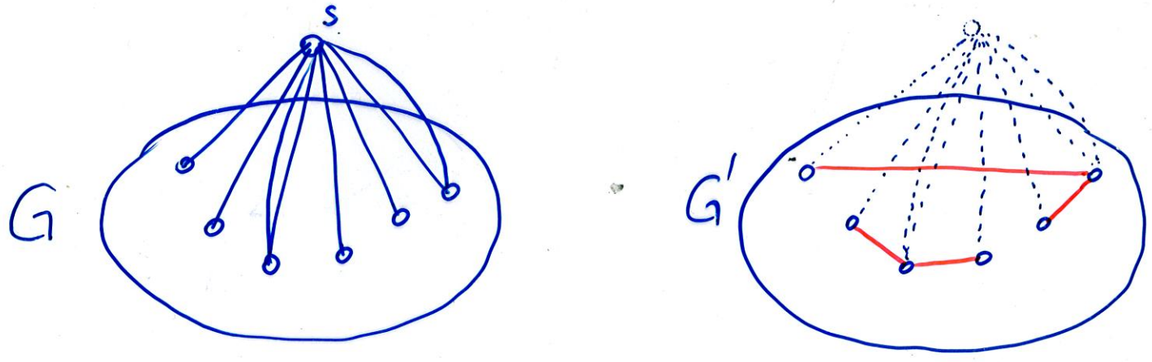
THEOREM (MADER)

EVERY MINIMALLY k-EDGE-CONNECTED GRAPH HAS A VERTEX OF DEGREE k.

DEF. SPLITTING OFF



COMPLETE SPLITTING OFF



THEOREM (LOVÁSZ)

IF $G=(V,E)$ IS $2k$-EDGE-CONNECTED, $s \in V$ WITH $d(s)$ EVEN THEN THERE EXISTS A COMPLETE SPLITTING OFF SUCH THAT G' IS $2k$-EDGE-CONNECTED.

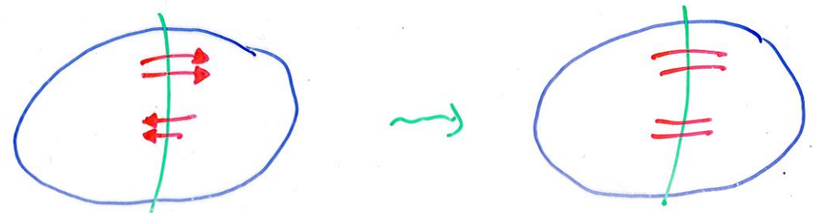
THEOREM (NASH-WILLIAMS)

G HAS A k-ARC-CONNECTED ORIENTATION

$\Updownarrow$

G IS $2k$-EDGE-CONNECTED

$\Downarrow$



$\Uparrow$ LOVÁSZ: EASY BY SPLITTING OFF

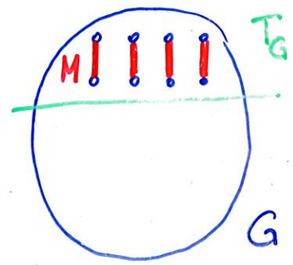
THEOREM (NASH-WILLIAMS)

IF $G=(V,E)$ IS $2k$-EDGE-CONNECTED THEN THERE EXISTS A PAIRING M OF T_G S.T.

$$d_M(X) \leq d_G(X) - 2k \quad \forall\, X \subsetneq V,\ X \neq \emptyset .$$

WHERE $T_G = \{v \in V : d_G(v) \text{ IS ODD}\}$ AND A SET M OF NEW EDGES IS A PAIRING IF

$$d_M(v) = \begin{cases} 1 & \text{IF } v \in T_G \\ 0 & \text{IF } v \in V - T_G \end{cases} .$$



REMARKS:

(1) **KIRÁLY, Sz.**: EASY BY SPLITTING OFF

(2) **NASH-WILLIAMS**: FOR ALL EULERIAN ORIENTATION $\vec{G}+\vec{M}$ OF $G+M$, $\vec{G}$ IS k-ARC-CONNECTED.

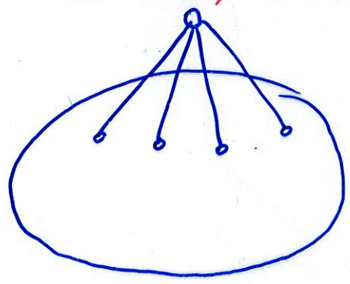
$$\varrho_{\vec{G}}(X) = \varrho_{\vec{G}+\vec{M}}(X) - \varrho_{\vec{M}}(X) \geq \frac{d_{G+M}(X)}{2} - d_M(X) =$$

$$= \frac{d_G(X) - d_M(X)}{2} \geq k \quad \forall\, X \subsetneq V,\ X \neq \emptyset .$$

PROOF: INDUCTION ON $|E|$.

CASE 1.

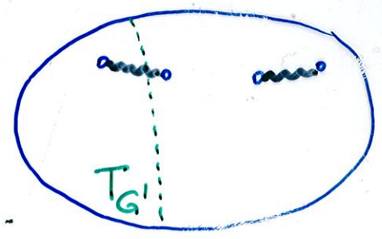
$\exists\ d(s)$ EVEN



COMPLETE SPLITTING OFF

$\Rightarrow$

(LOVÁSZ)



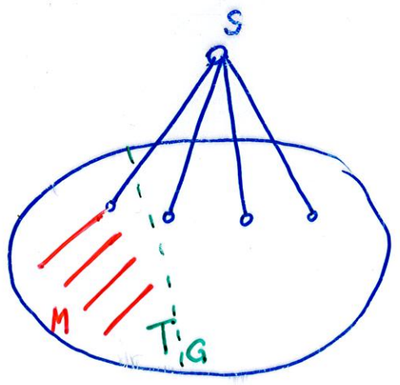
G $2k$-E-C

G' $2k$-E-C

R.: $T_{G'} = T_G$

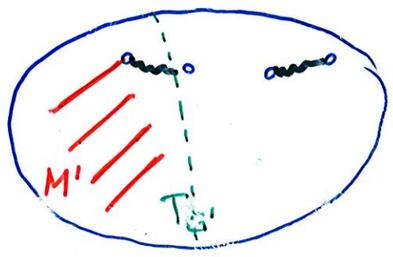
INDUCTION

$\exists\ M'$ PAIRING OF $T_{G'}$



$M := M'$

PAIRING OF T_G



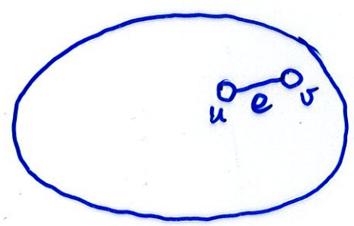
$d_M(X) \leq d_{G'}(X) - 2k$

$\leq d_G(X) - 2k \quad \forall X$

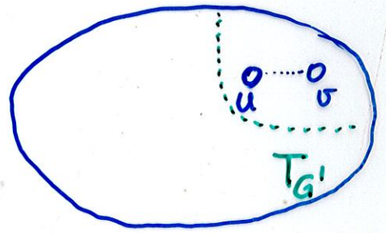
$d_{M'}(X) \leq d_{G'}(X) - 2k \quad \forall X$

CASE 2.

$T_G = V$



(MADER)



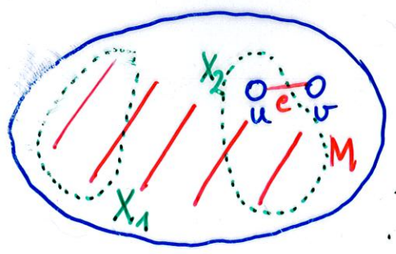
G $2k$-E-C

$\exists e : G' := G - e$ $2k$-E-C

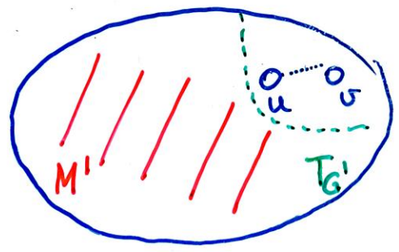
R.: $T_{G'} = T_G - u - v$

INDUCTION

$\exists M'$ PAIRING OF $T_{G'}$



$M := M' \cup \{uv\}$
PAIRING OF T_G



$d_M(X_1) = d_{M'}(X_1) \leq d_{G'}(X_1) - 2k$

$= d_G(X_1) - 2k$

OR

$d_M(X_2) = d_{M'}(X_2) + 1 \leq d_{G'}(X_2) + 1 - 2k$

$= d_G(X_2) - 2k$

$d_{M'}(X) \leq d_{G'}(X) - 2k \quad \forall X$

THEOREM (NASH-WILLIAMS)

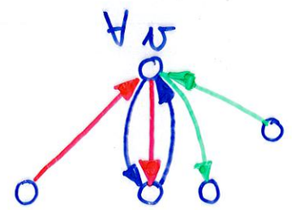
IF G IS $2k$-EDGE-CONNECTED THEN THERE EXISTS A PAIRING M S.T. FOR ALL EULERIAN ORIENTATION $\vec{G}+\vec{M}$ OF $G+M$, $\vec{G}$ IS k-ARC-CONNECTED.

(FEASIBLE PAIRING)

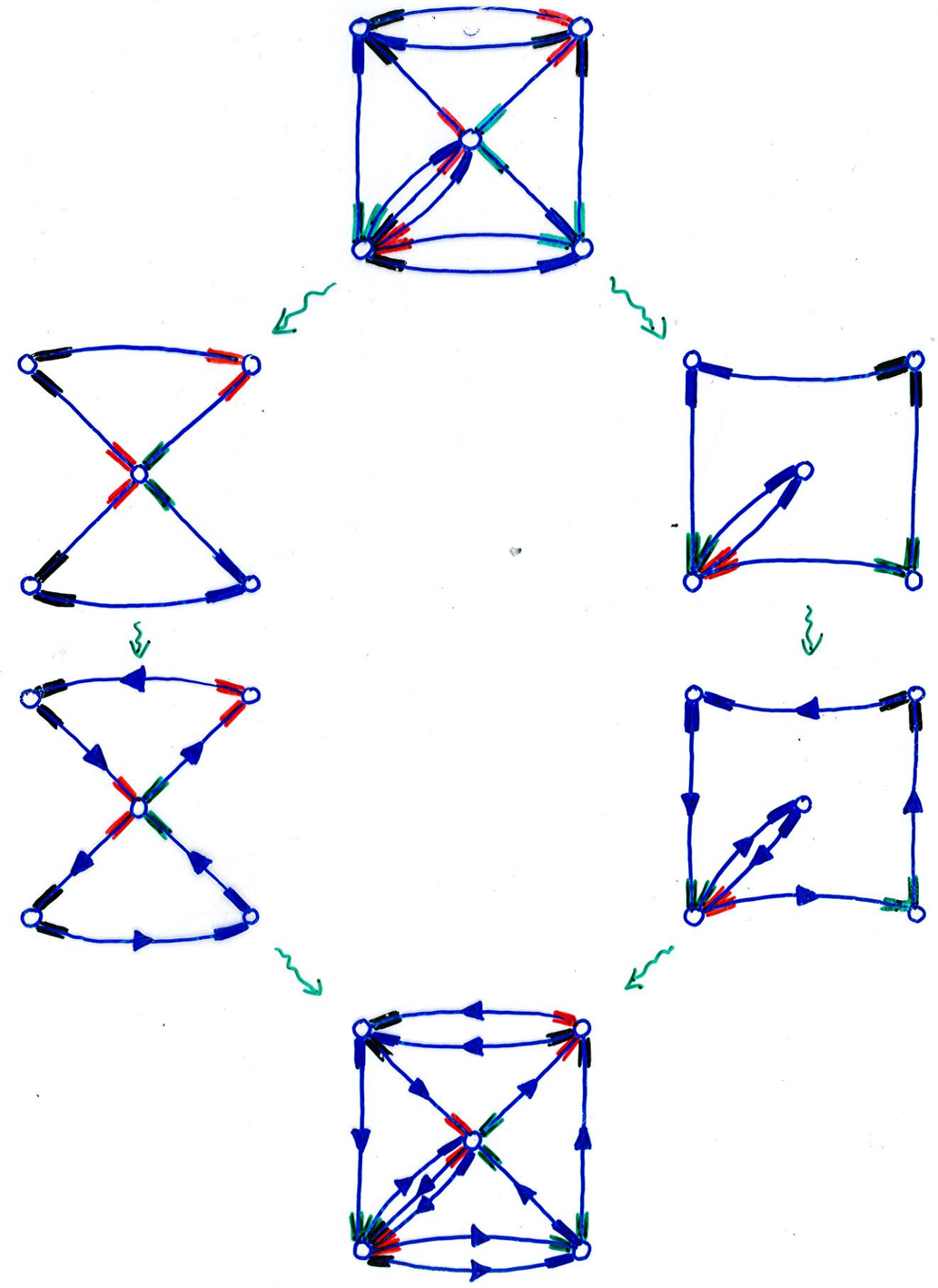
OBSERVATION

GIVEN
- AN EULERIAN GRAPH $G=(V,E)$,
- A PARTITION $\mathcal{P}_v$ OF THE EDGES INCIDENT TO v INTO PAIRS $\forall v \in V$,

THERE EXISTS AN EULERIAN ORIENTATION $\vec{G}$ OF G COMPATIBLE WITH ALL $\mathcal{P}_v$



EXAMPLE:



CONJECTURE (THOMASSEN, FRANK)

$G=(V,E)$ HAS A k-VERTEX-CONNECTED ORIENTATION

$(|V| \geq k+1)$

$\Updownarrow$

$G-X$ IS $2(k-|X|)$-EDGE-CONNECTED FOR ALL $X \subset V$, $|X| < k$.

$\Downarrow$ $\vec{G}$ IS k-V-C $\Rightarrow$

$\vec{G}-X$ IS $(k-|X|)$-V-C $\Rightarrow$

$\vec{G}-X$ IS $(k-|X|)$-A-C $\Rightarrow$

$G-X$ IS $2(k-|X|)$-E-C $\quad \forall X \subset V, |X| < k.$

$\Uparrow$ OPEN EVEN FOR $k=2$.

CONJECTURE ($k=2$)

$G=(V,E)$ HAS A 2-VERTEX-CONNECTED ORIENTATION

$(|V| \geq 3)$

$\Leftrightarrow$

G IS 4-EDGE-CONNECTED

$G-v$ IS 2-EDGE-CONNECTED $\forall v \in V$.

THEOREM (BERG-JORDÁN)

AN EULERIAN GRAPH $G=(V,E)$ HAS AN ORIENTATION $\vec{G}$ SUCH THAT $\vec{G}-v$ IS 1-ARC-CONNECTED $\forall v \in V$

$\Leftrightarrow$

$G-v$ IS 2-EDGE-CONNECTED $\forall v \in V$.

CONJECTURE (FRANK)

AN EULERIAN GRAPH $G=(V,E)$ HAS AN ORIENTATION $\vec{G}$ SUCH THAT $\vec{G}-v$ IS k-ARC-CONNECTED $\forall v \in V$

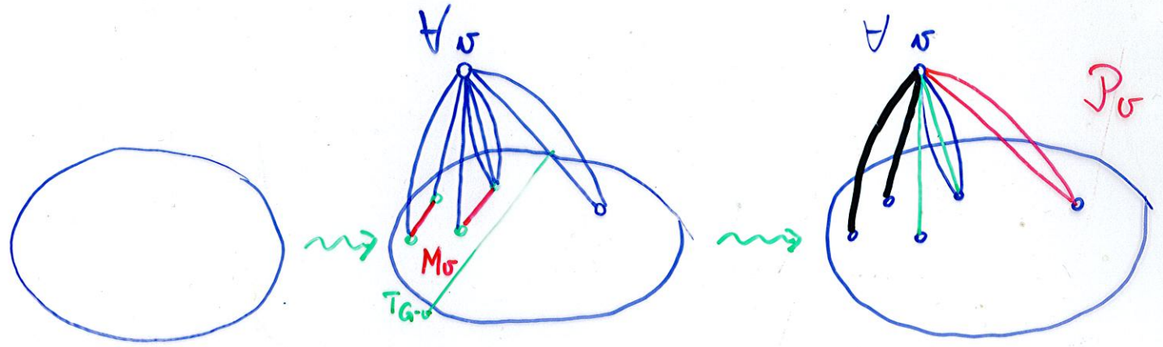
$\Leftrightarrow$

$G-v$ IS $2k$-EDGE-CONNECTED $\forall v \in V$.

THEOREM (KIRÁLY-Sz.) TRUE

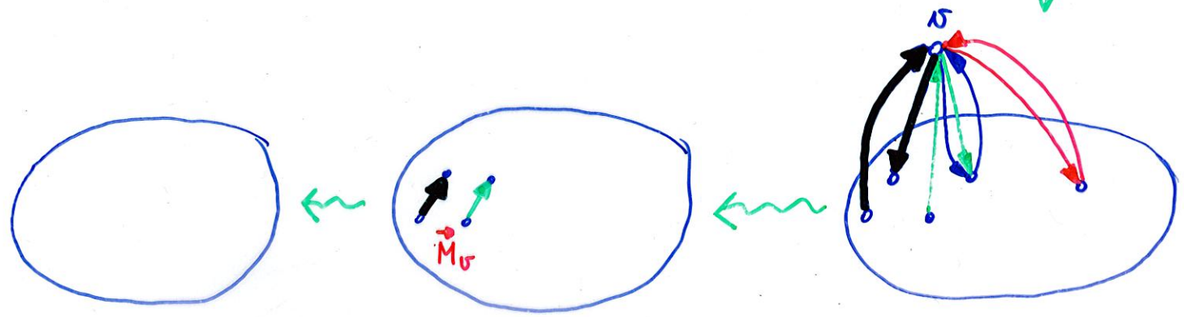
SHORT, EASY PROOF

PROOF :



G EULERIAN $\quad M_v$ FEASIBLE

$G-v$ 2k-E-C $\forall v \in V$ $\quad$ PAIRING OF $G-v$



$\vec{G}$ EULERIAN $\quad \vec{G}-v+\vec{M}_v$ EULERIAN $\quad$ COMPATIBLE EULERIAN ORIENTATION $\vec{G}$

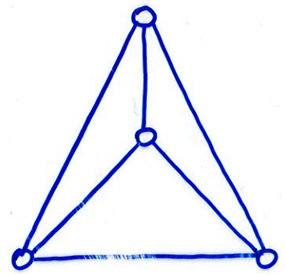
$\vec{G}-v$ k-A-C $\forall v \in V$ $\quad \forall v \in V$

REMARK: NOT TRUE IF G IS NOT EULERIAN

($G-v$ IS $2k$-E-C $\forall v \in V \not\Rightarrow \exists \vec{G}: \vec{G}-v$ IS k-A-C $\forall v \in V$)

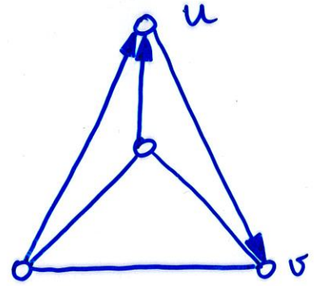
EX.

G



$G-v$ IS 2-E-C $\forall v$

$\nexists \vec{G}: \vec{G}-v$ IS 1-A-C $\forall v$:



WLOG $\delta(u) \leq 1$

$\Downarrow$

$\vec{G}-v$ IS NOT 1-A-C

REMARK: NOT COUNTER-EXAMPLE FOR CONJECTURE OF FRANK, THOMASSEN:

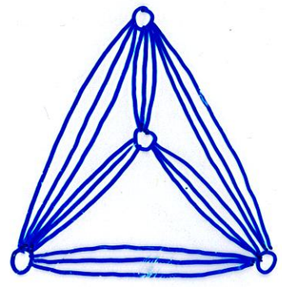
G IS NOT 4-E-C.

REMARK: NOT TRUE IF G IS NOT EULERIAN

$$\left(\begin{array}{l} G \text{ is } 2k\text{-E-C} \\ G-v \text{ is } 2(k-1)\text{-E-C } \forall v \in V \end{array}\right) \not\Rightarrow \left(\begin{array}{l} \exists \vec{G}: \vec{G} \text{ is } k\text{-A-C} \\ \vec{G}-v \text{ is } (k-1)\text{-A-C } \forall v \in V \end{array}\right)$$

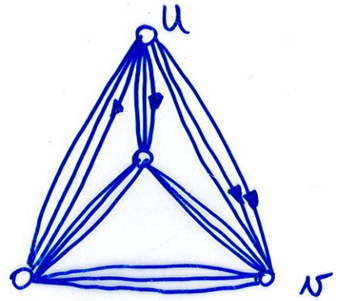
EX.

G



G is 8-E-C
G-v is 6-E-C $\forall v \in V$

$\nexists \vec{G}$ 4-A-C, $\vec{G}-v$ 3-A-C $\forall v \in V$



- WLOG $\delta(u) = 4$
- $\exists v$
- $\vec{G}-v$ is NOT 3-A-C

REMARK: NOT COUNTER-EXAMPLE FOR CONJECTURE OF FRANK, THOMASSEN:

$G-u-v$ is NOT 4-E-C.