An excluded minor characterization of Seymour graphs

Zoltán Szigeti

Laboratoire G-SCOP
Grenoble, France

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joint with A. Ageev, Y. Benchetrit, A. Sebő

Outline

1. Motivation
2. Definitions : complete packing of cuts, joins
3. Seymour Graphs
4. Around Seymour graphs
5. Old co-NP characterization of Seymour graphs
6. New co-NP characterization of Seymour graphs
7. Ideas of the proof
8. Algorithmic aspects
9. Open problem

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Edge-disjoint paths problem

Given a graph $H = (V, E)$ and k pairs of vertices $\{s_i, t_i\}$, decide whether there exist k edge-disjoint paths connecting the k pairs s_i, t_i.

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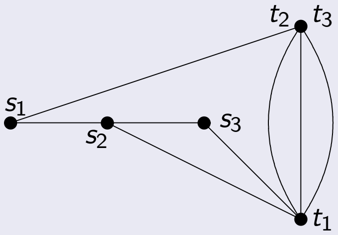
Complete packing of cuts

Given a graph $G = (V', E' + F')$, decide whether there exist $|F'|$ edge-disjoint cuts in G, each containing exactly one edge of F'.

An example

Edge-disjoint paths problem



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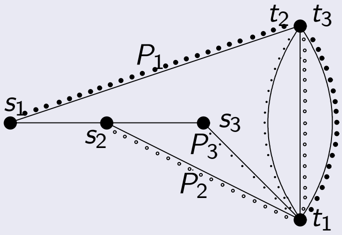

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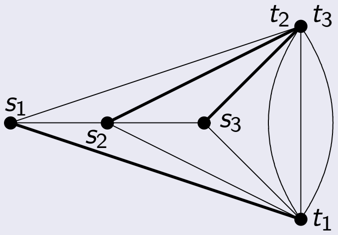
Adding the edges



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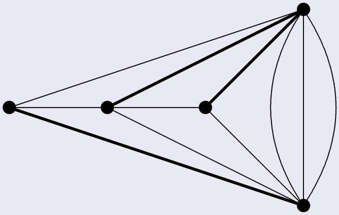
The graph H'



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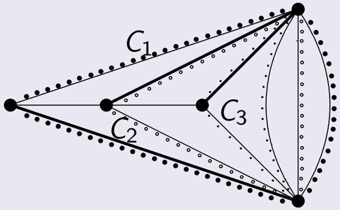

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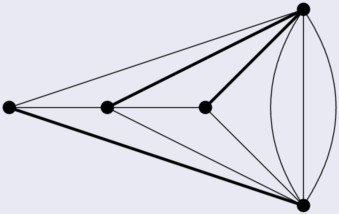

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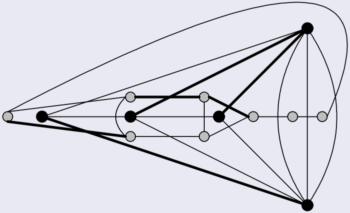
H' and his dual



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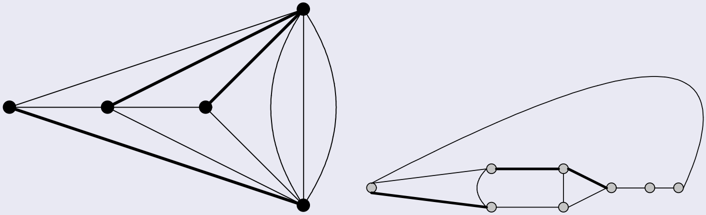

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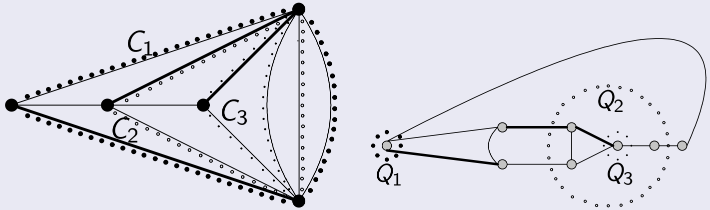
Complete packing of cycles and cuts



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The graphs are not planar anymore !

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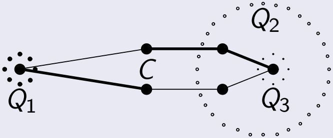

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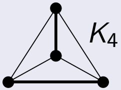

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Theorem (Middendorf, Pfeiffer)

Given a join in a graph, decide whether there exists a complete packing of cuts is an NP-complete problem.

Seymour graphs

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Around Seymour graphs

Subclasses

1. Seymour : Graphs without odd cycle,
2. Seymour : Graphs without subdivision of K_4,
3. Gerards : Graphs without odd K_4 and without odd prism,
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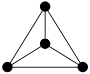

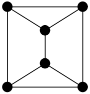

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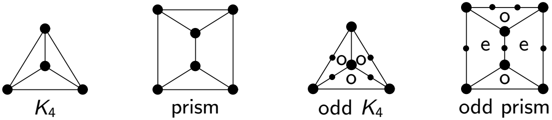

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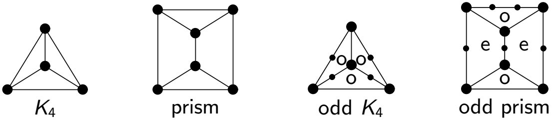

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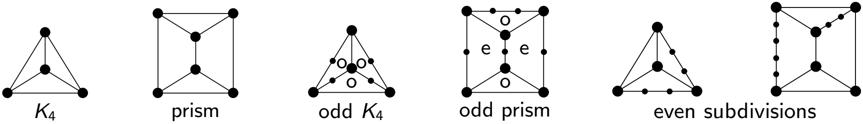
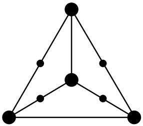


Superclass

Seymour graph $\Longrightarrow$ no even subdivision of K_4 and of prism.



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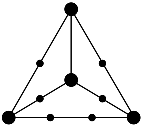

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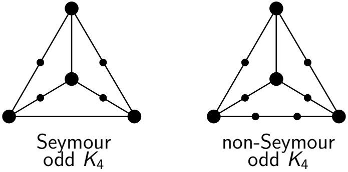

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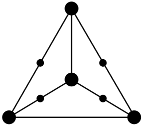

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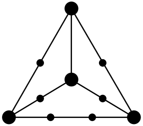

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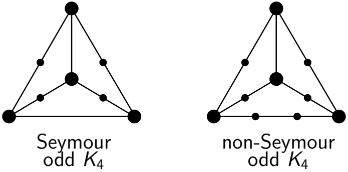

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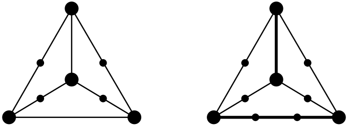

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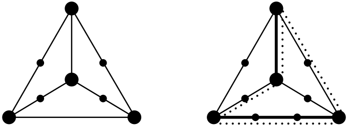

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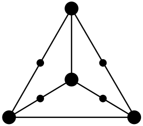

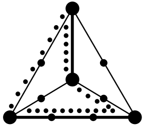

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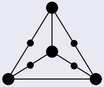
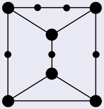


Seymour odd K_4



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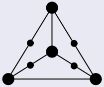

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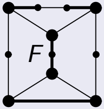

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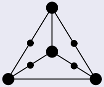

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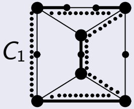

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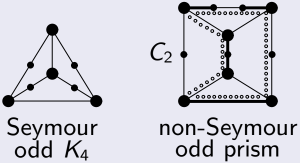

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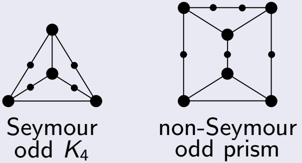

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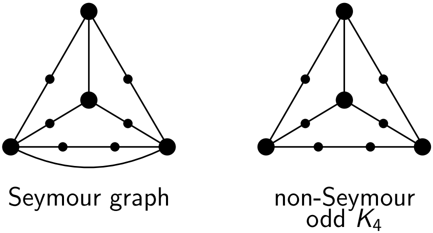
Attention !

1. Seymour property is not inherited to subgraphs.
2. Contraction of an edge does not keep Seymour property.



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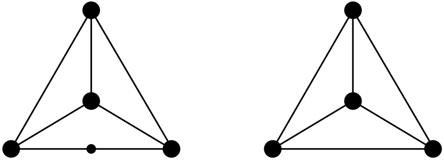

Seymour graph

non-Seymour odd K_4

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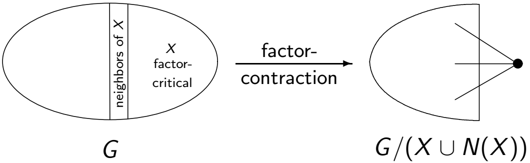

A new notion of contraction

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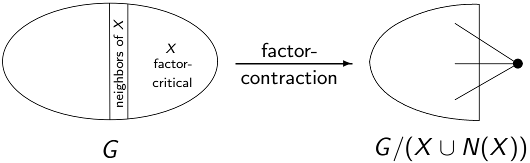

Important lemma

Factor-contraction keeps the Seymour property !

A new notion of minor

Definitions

1. Star-contraction : contraction of all edges incident to a vertex.
2. Odd cycle-contraction : contraction of an odd cycle.
3. STOC-minor : Graph obtained by a series of star and odd cycle-contractions.

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Remark

If G can be factor-contracted to H then H is a STOC-minor of G !

New co-NP characterizations of Seymour graphs

Theorem (Ageev, Benchetrit, Sebő, Szigeti)

The following conditions are equivalent :

1. G is not Seymour,
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- G/C is not Seymour,
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1. $2|F|$ cuts so that
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Example : If $\mathcal{Q}$ is a CPC, then $2\mathcal{Q}$ is a C2PC.

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Theorem (Edmonds-Johnson, Lovász)

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- For $d_H(c) = 3$: $\mathcal{Q}$ can be chosen so that it contains $\delta(c)$.
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Graphs

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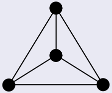
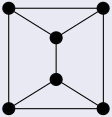
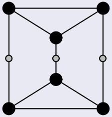


K_4



prism



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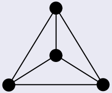

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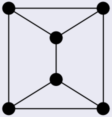

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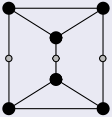
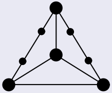
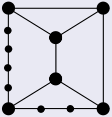
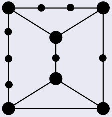


bi-prism

and their even subdivisions







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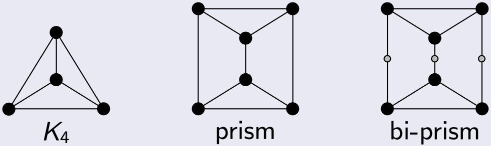

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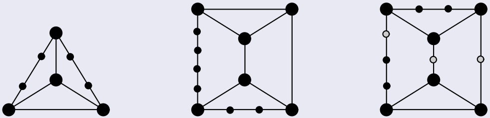

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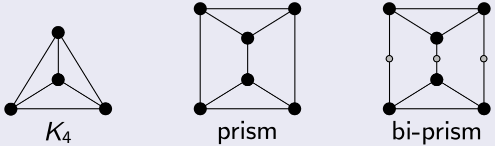

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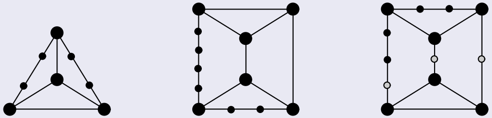

K_4-obstruction

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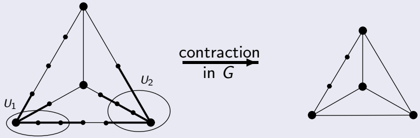
An odd K_4 subgraph H of G with disjoint sets $U_i \subseteq V(H)$ such that

1. $H[U_i \cup N_H(U_i)]$ is an even subdivision of a 3-star,
2. contracting each $U_i \cup N_G(U_i)$, H transforms into an even subdivision of K_4.



Prism-obstruction

An odd prism subgraph H of G with disjoint sets $U_i \subseteq V(H)$ such that

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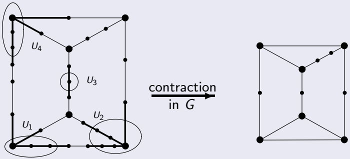
2. contracting each $U_i \cup N_G(U_i)$, H transforms into an even subdivision of the prism or of the biprism (no edge of G connects the two connected components of the biprism minus its separator).



And some other co-NP characterizations of Seymour graphs

Theorem (Ageev, Benchetrit, Sebő, Szigeti)

The following conditions are equivalent :

1. G is not Seymour,
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Proof of sufficiency :

Lemma

If C is an odd cycle in G and G/C is not Seymour then neither is G.

Ideas of the Proof

- G/C is not Seymour,
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1. Given a graph G, decide whether it is a Seymour graph.
2. Given a graph G and a join F in G, decide whether there exists an F-complete packing of cuts.

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Open problem

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Find a construction for Seymour graphs !

Thanks !