Reachability-based packing of arborescences: Algorithmic aspects

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Outline

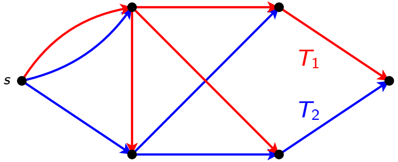
- Packing of arborescences :
  - spanning
  - reachability
  - matroid-based
  - reachability-based
- Algorithmic aspects : weighted case with matroid intersection for
  - matroid-based
  - reachability-based
- Related problems
  - reachability-based matroid-restricted
  - matroid-based spanning
  - polymatroid-based
  - reachability-based hyperarborescences

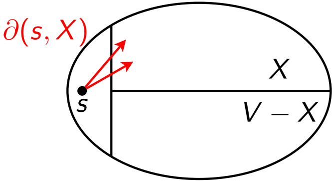
Packing of spanning s-arborescences : Definitions

Definition

Let $D = (V + s, A)$ be a digraph, $X \subseteq V$ and $v \in V$.

1. packing of subgraphs : arc-disjoint subgraphs,
2. spanning subgraph of D : subgraph that contains all the vertices of D,
3. s-arborescence : directed tree, indegree of every vertex except s is 1,
4. root arc : arc leaving s,
5. $\partial(s, X)$: root arcs entering X,
6. $\partial(v)$: set of arcs entering of v.

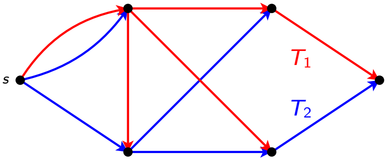




Packing of spanning s-arborescences : Results

Results

1. Characterization (Edmonds 1973).
2. Algorithmic aspects :
   1. Unweighted case : Algorithmic proof (E ; Lovász 1976).
   2. Weighted case : Weighted matroid intersection (Edmonds 1979) + Unweighted case.
      Let $D = (V + s, A)$ and G be the underlying undirected graph of D.
      - $\vec{F} \subseteq A$ is a packing of k spanning s-arborescences of D $\iff$
      - F is a packing of k spanning trees of G and $|\partial_{\vec{F}}(v)| = k \ \forall v \in V$ $\iff$
      - F is a common base of $\mathcal{M}_1 = k$-sum of the graphic matroid of G and $\mathcal{M}_2 = \oplus_{v \in V} U_{|\partial(v)|,k}$.

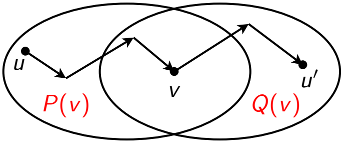


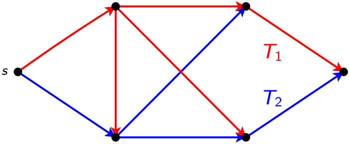
Packing of reachability s-arborescences : Definitions

Definition

Let $D = (V + s, A)$ be a digraph and $v \in V$.

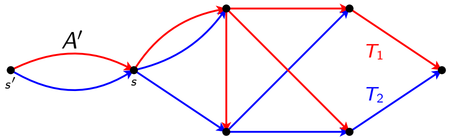
1. $P(v) = \{u \in V : v \text{ is reachable from } u \text{ in } D\}$,
2. $Q(v) = \{u' \in V : u' \text{ is reachable from } v \text{ in } D\}$,
3. reachability s-arborescence T_i for ss_i : $V(T_i) = Q_D(s_i) \cup s$,
4. packing of reachability s-arborescences $\{T_1, \dots, T_t\}$ $(t = |\partial(s, V)|)$:
   - for each root arc ss_i, T_i is a reachability s-arborescence $\Longleftrightarrow$
   - $\{ss_i \in A : s_i \in P_{T_i}(v)\} = \{ss_i \in A : s_i \in P_D(v)\} \;\; \forall v \in V.$

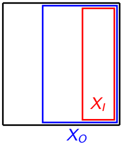




Results

1. Characterization (Kamiyama, Katoh, Takizawa 2009).
2. Short proof using bi-sets (Bérczi, Frank 2008).
3. Algorithmic aspects :
   1. Unweighted case : Algorithmic proof (KKT).
   2. Weighted case : Matroid intersection (Bérczi, Frank 2009).
4. Extension : A packing of reachability s'-arborescences in D' gives a packing of k spanning s-arborescences in D if the condition of Edmonds is satisfied.





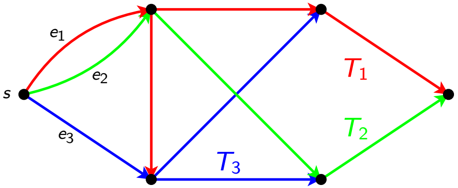
Matroid-based packing of s-arborescences : Definition

Motivation

Motivated by Katoh and Tanigawa's problem on matroid-based packing of rooted trees (introduced to solve a rigidity problem).

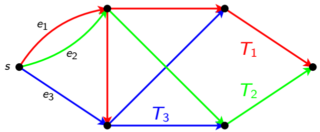
Definition

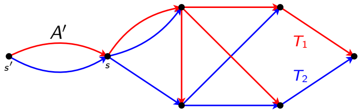
Let $D = (V + s, A)$ be a digraph and $\mathcal{M}$ a matroid on the set of root arcs. Matroid-based packing of s-arborescences $\{T_1, \dots, T_t\}$ $(t = |\partial(s, V)|)$: $\{ss_i \in A : s_i \in P_{T_i}(v)\}$ is a base of $\{ss_i \in A : s_i \in V\}$ $\forall v \in V$.



Results

1. Characterization (Durand de Gevigney, Nguyen, Szigeti 2013).
2. Algorithmic aspects :
   1. Unweighted case : Algorithmic proof (DdGNSz).
   2. Weighted case : Polyhedral description (DdGNSz) + Ellipsoid method (GLS) + submodular function minimization (GLS, S, IFF).
3. Extension : An $\mathcal{M}'$-based packing of s'-arborescences in D' ($\mathcal{M}'$ free matroid on A') gives a packing of k spanning s-arborescences in D.

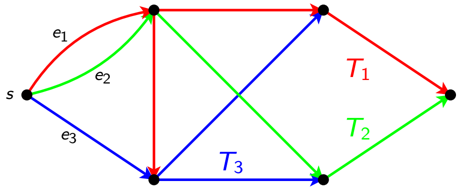




Reachability-based packing of s-arborescences

Definition

Let $D = (V + s, A)$ be a digraph and $\mathcal{M}$ a matroid on the set of root arcs. Reachability-based packing of s-arborescences $\{T_1, \ldots, T_t\}$ $(t = |\partial(s, V)|)$: $\{ss_i \in A : s_i \in P_{T_i}(v)\}$ is a base of $\{ss_i \in A : s_i \in P_D(v)\}$ $\forall v \in V$.



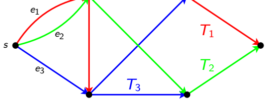
Remark

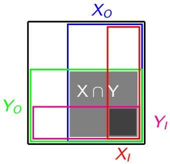
A reachability-based packing of s-arborescences doesn't necessarily contain reachability s-arborescences.

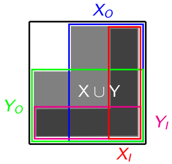
Reachability-based packing of *s*-arborescences

Results

1. Characterization (Cs. Király 2016).
2. Algorithmic aspects :
   1. Unweighted case : Algorithmic proof (K).
   2. Weighted case : Submodular flows defined by an intersecting supermodular bi-set function (Bérczi, T. Király, Kobayashi 2016).
3. Extension :
   1. For free matroid, back to packing of reachability *s*-arborescences.
   2. An $\mathcal{M}$-reachability-based packing of *s*-arborescences is an $\mathcal{M}$-based packing of *s*-arborescences if the condition of DdGNSz is satisfied.







Algorithmic aspects : matroid constructions

Theorem (Edmonds-Rota 1966)

- *$D := (V, A)$ a digraph,*
- *$f : 2^A \to \mathbb{Z}_+$ a monotone intersecting submodular set function,*
- *$\mathcal{I} := \{B \subseteq A : |H| \leq f(H) \ \forall H \subseteq B\}$.*

Then $\mathcal{I}$ forms the family of independent sets of a matroid on A.

Theorem (Frank 2009 ; Cs. Király, Szigeti, Tanigawa)

- $D := (V, A)$ a digraph,
- $\mathcal{F}$ an intersecting bi-set family on V,
- $b : \mathcal{F} \to \mathbb{Z}_+$ an intersecting submodular bi-set function,
- $\mathcal{I} := \{B \subseteq A : i_B(\mathsf{X}) \leq b(\mathsf{X}) \ \forall \mathsf{X} \in \mathcal{F}\}$.

Then $\mathcal{I}$ forms the family of independent sets of a matroid on A.

Algorithmic aspects : matroid intersection

Theorem (Cs. Király, Szigeti, Tanigawa)

The arc sets of matroid-based/reachability-based packings of s-arborescences can be written as common bases of $\mathcal{M}'$ and $\mathcal{M}''$, where

1. matroid-based : $\mathcal{M}'$ by $f(H) = k|V(H) - s| - k + r(H \cap \partial(s, V))$, $\mathcal{M}'' = \oplus_{v \in V} U_{|\partial(v)|,k}$.
2. reachability-based : $\mathcal{M}'$ by $b(\mathrm{X}) = m(X_I) - p(\mathrm{X})$, $\mathcal{M}'' = \oplus_{v \in V} U_{|\partial(v)|,r(\partial(s,P(v)))}$.

Corollary : in polynomial time one can

- decide if an instance has a solution,
- find a minimum weight arc set that can be decomposed into a reachability-based packing of s-arborescences,
- find a minimum weight reachability-based packing of s-arborescences.

Matroid-restricted packing of spanning s-arborescences

Definition

Let $D = (V + s, A)$ be a digraph and $\mathcal{M} = (A, \mathcal{I})$ a matroid.
Matroid-restricted packing of s-arborescences $T_1, \ldots, T_k$: $\cup_1^k A(T_i) \in \mathcal{I}$.

Results

1. For general matroid $\mathcal{M}$, the problem is NP-complete, even for $k = 1$.
2. For $\mathcal{M} = \oplus_{v \in V} \mathcal{M}_v$, where $\mathcal{M}_v$ is a matroid on $\partial(v)$,
   1. Characterization (Frank 2009 ; Bernáth, T. Király 2016).
   2. Algorithmic aspects : Weighted case : weighted matroid intersection.
   3. Extension : For free matroid, packing of spanning s-arborescences.

Theorem (Cs. Király, Szigeti, Tanigawa)

For $\mathcal{M} = \oplus_{v \in V} \mathcal{M}_v$, where $\mathcal{M}_v$ is a matroid on $\partial(v)$, the results on matroid-based/reachability-based packings can be extended to matroid-based/reachability-based matroid-restricted packings.

Other related problems

Theorem (Fortier, Cs. Király, Szigeti, Tanigawa 2016+)

Matroid-based packing of spanning s-arborescences :

1. *NP-complete for general matroids,*
2. *solvable for rank 2/graphic/transversal matroids.*

Theorem (Matsuoka, Szigeti 2017+)

Polymatroid-based packing of s-arborescences :

1. *Characterization,*
2. *Algorithmic aspects : unweighted capacitated case.*

Theorem (Fortier, Cs. Király, Léonard, Szigeti, Talon 2018)

Reachability-based packing of s-hyperarborescences :

1. *Characterization,*
2. *Algorithmic aspects : weighted case.*

Thank you for your attention !