Evolution of Packing Arborescences

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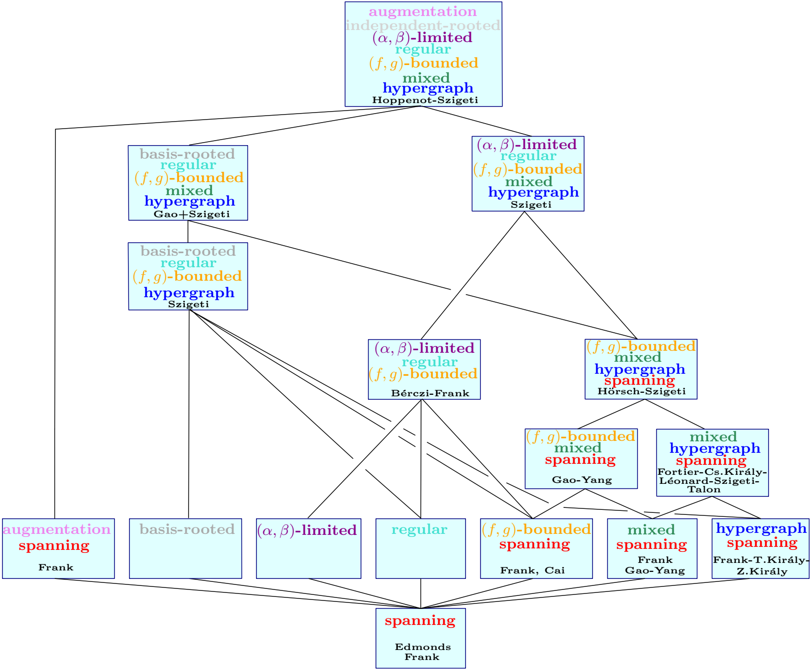
Joint work with Pierre Hoppenot


augmentation
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(α, β)-limited
regular
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mixed
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Hoppenot-Szigeti
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Gao+Szigeti
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Bérczi-Frank
(f, g)-bounded
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Frank, Cai
mixed
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Frank
Gao-Yang
hypergraph
spanning
Frank-T.Király-Z.Király
spanning
Edmonds
Frank

Arborescences

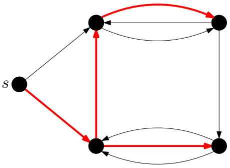
Definition

1. s-arborescence : $\vec{F} = (V + s, A)$,
   1. F is a forest and $d_A^-(v) = 1 \ \forall\, v \in V$, $\Longleftrightarrow$
   2. exactly one directed path exists in $\vec{F}$ from s to every $v \in V$.
2. (U, B) is a spanning subgraph of $D = (V, A)$ if $U = V$ and $B \subseteq A$.

Theorem 1

$D = (V + s, A)$ has a spanning s-arborescence $\Longleftrightarrow$

$$d_A^-(X) \geq 1 \ \ \forall\, \emptyset \neq X \subseteq V.$$



Packing of spanning arborescences with fixed roots

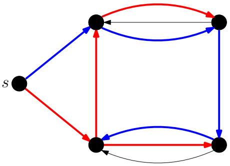
Definition

packing subgraphs in D : arc disjoint subgraphs in D.

Theorem 2 (Edmonds 1973)

$D = (V + s, A)$ *has a packing of k spanning s-arborescences* $\Longleftrightarrow$

$$d_A^-(X) \geq k \quad \forall\, \emptyset \neq X \subseteq V.$$

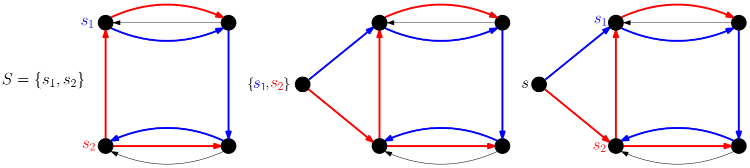


Packing of spanning arborescences with fixed roots

Theorem 3 (Edmonds 1973)

Let $D = (V, A)$ be a digraph and S a multiset of k vertices in V.
There exists a packing of spanning s-arborescences ($s \in S$) in D $\Longleftrightarrow$

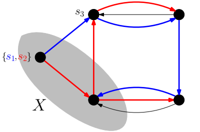
$$|S_X| \;\geq\; k - d_A^-(X) \;\; \forall\, \emptyset \neq X \subseteq V.$$



Theorem 2 and 3 are equivalent.

Definitions

1. multiset S of V : vertex set with multiplicities.
2. S_X for $X \subseteq V$: restriction of S in X.



Packing of spanning arborescences with flexible roots

Theorem 4 (Frank 1978)

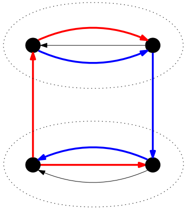
Let $D = (V, A)$ be a digraph and $k \in \mathbb{Z}_+$.
There exists a packing of k spanning arborescences in D $\Longleftrightarrow$

$$k \geq \sum_{X \in \mathcal{P}} (k - d_A^-(X)) \quad \forall\, \mathcal{P} \in \mathrm{P}(V).$$

Theorem 4 immediately implies Theorem 2.
Theorem 2 implies Theorem 4 using submodularity.

Definition

$\mathrm{P}(V)$: the set of subpartitions of V.



Packing of spanning arborescences

spanning

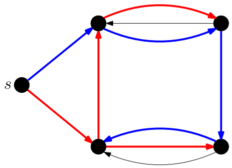
Edmonds
Frank

Matroid tools

Arc set of a packing : Matroid intersection

$\vec{B} \subseteq \vec{E}$ is the arc set of a packing of k spanning s-arborescences in $\vec{G} \Longleftrightarrow$

1. B is the edge set of a packing of k spanning trees in G, (k sum of the graphic matroid of G)
2. $d^-_{\vec{B}}(v) = k \;\; \forall\, v \in V(G) - s.$ (partition matroid)



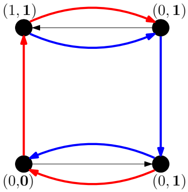
Root sets : Basis of a matroid

If D has a packing of k spanning arborescences, then root sets of packings of k spanning arborescences forms the set of bases of a matroid.

(f,g)-bounded packing of spanning arborescences

Definition

(f,g)-bounded packing of arborescences : number of v-arborescences in the packing is at least $f(v)$ and at most $g(v)$ $\forall v \in V$.



Theorem 5 (Frank 1978, Cai 1983)

Let $D=(V,A)$ be a digraph, $f,g : V \to \mathbb{Z}_+$ functions and $k \in \mathbb{Z}_+$. There exists an (f,g)-bounded packing of k spanning arborescences $\Longleftrightarrow$

1. $g(v) \geq f(v) \quad \forall\ v \in V$,
2. $\min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \geq \sum_{X\in\mathcal{P}}(k - d_A^-(X)) \quad \forall\ \mathcal{P} \in \mathrm{P}(V)$.

Theorem 5 implies Theorem 4.

(f, g)-bounded packing of spanning arborescences

(f, g)-bounded
spanning

Frank, Cai

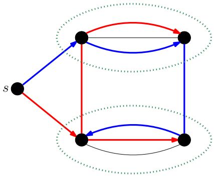
spanning

Edmonds
Frank

Packing spanning mixed arborescences : fixed root

Definition

1. mixed s-arborescence :
   it can be oriented to obtain an s-arborescence.
2. $e_E(\mathcal{P})$: number of edges in E entering at least one member of $\mathcal{P} \in \mathrm{P}(V)$.



Theorem 6 (Frank 1978)

Let $F = (V + s, E \cup A)$ be a mixed graph and $k \in \mathbb{Z}_+$.
There exists a packing of k spanning mixed s-arborescences $\Longleftrightarrow$

$$e_E(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} (k - d_A^-(X)) \ \ \forall\, \mathcal{P} \in \mathrm{P}(V).$$

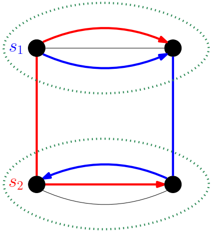
Theorem 6 implies Theorem 2.

Packing spanning mixed arborescences : fixed roots

Theorem 7

Let $F = (V, E \cup A)$ *be a mixed graph and* S *a multiset of vertices in* V*.*
There exists a packing of spanning mixed s-arborescences $(s \in S)$ $\Longleftrightarrow$

$$e_E(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} (|S_{\overline{X}}| - d_A^-(X)) \;\; \forall\, \mathcal{P} \in \mathrm{P}(V).$$



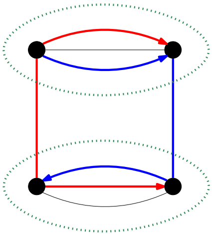
Theorems 6 and 7 are equivalent.

Packing spanning mixed arborescences : flexible roots

Theorem 8 (Gao, Yang 2021)

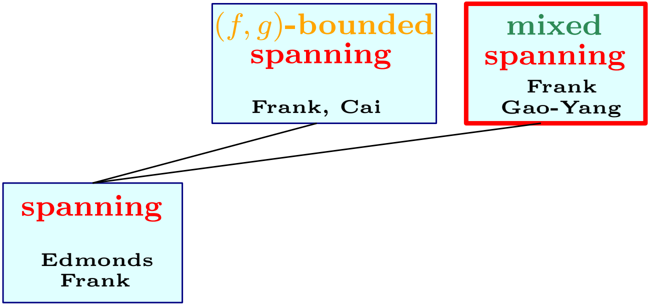
Let $F = (V, E \cup A)$ be a mixed graph and $k \in \mathbb{Z}_+$.
There exists a packing of k spanning mixed arborescences $\Longleftrightarrow$

$$e_E(\mathcal{P}) + k \geq \sum_{X \in \mathcal{P}} (k - d_A^-(X)) \quad \forall\, \mathcal{P} \in \mathrm{P}(V).$$



Theorem 8 implies Theorem 6.

Packing spanning mixed arborescences

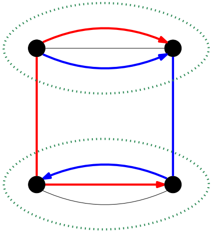


(f,g)-bounded packing spanning mixed arborescences

Theorem 9 (Gao, Yang 2021)

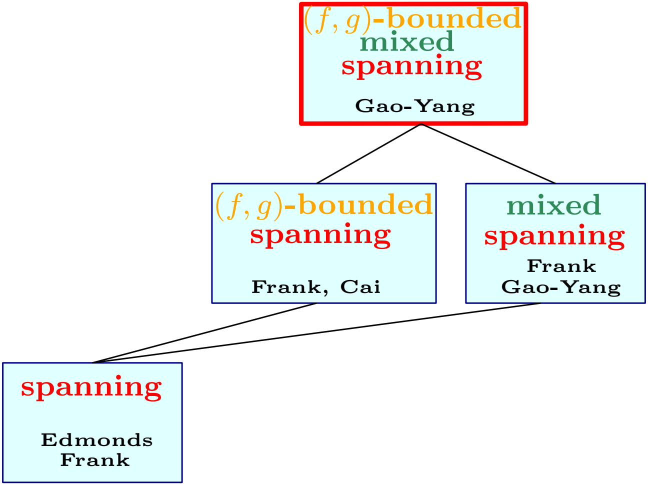
Let $F=(V,E\cup A)$ *be a mixed graph,* $f,g:V\to\mathbb{Z}_+$ *functions,* $k\in\mathbb{Z}_+$*.*
An *(f,g)-bounded packing of k spanning mixed arborescences* *exists* $\Longleftrightarrow$

1. $g(v)\geq f(v)\quad\forall\, v\in V,$
2. $e_E(\mathcal{P})+\min\{k-f(\overline{\cup\mathcal{P}}),g(\cup\mathcal{P})\}\geq\sum_{X\in\mathcal{P}}(k-d_A^-(X))\ \forall\,\mathcal{P}\in\mathrm{P}(V).$



Theorem 9 implies Theorems 5 and 8.

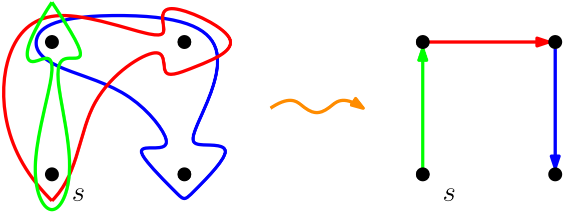
(f,g)-bounded packing spanning mixed arborescences



Packing hyperarborescences

Definition

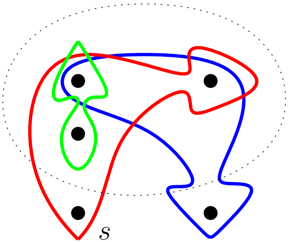
1. directed hypergraph : $\mathcal{D} = (V, \mathcal{A})$.
2. hyperarc : one head, at least one tail.
3. hyperarborescence : directed hypergraph that can be trimmed to an arborescence.
4. spanning s-hyperarborescence : directed hypergraph that can be trimmed to a spanning s-arborescence.



Theorem 10 (Frank, Király, Király 2003)

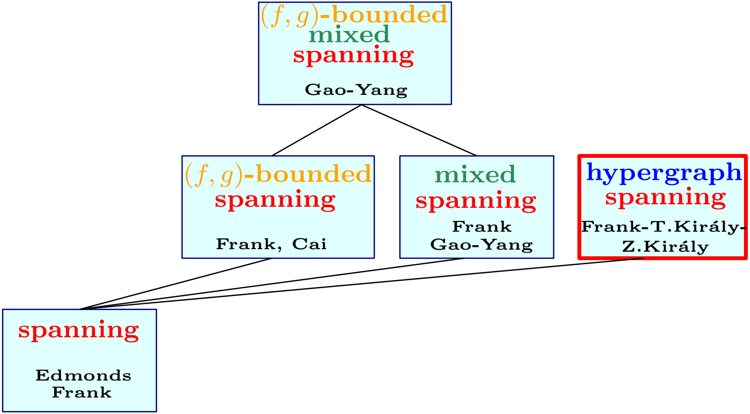
Let $\mathcal{D} = (V, \mathcal{A})$ be a directed hypergraph, S a multiset of k vertices in V. There exists a packing of spanning s-hyperarborescences $(s \in S)$ in $\mathcal{D}$ $\Longleftrightarrow$

$$|S_X| \quad \geq \quad k - d^-_{\mathcal{A}}(X) \quad \forall\, \emptyset \neq X \subseteq V.$$



Theorem 10 implies Theorem 3.

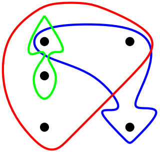
Packing hyperarborescences



Packing mixed hyperarborescences

Definition

1. mixed hypergraph : $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$,
2. hyperedge : vertex set of size at least two,
3. mixed s-hyperarborescence $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$: $\exists$ s-hyperarborescence $\vec{\mathcal{F}} = (V, \vec{\mathcal{E}} \cup \mathcal{A})$,
4. $e_{\mathcal{E}}(\mathcal{P})$: number of hyperedges in $\mathcal{E}$ entering at least one member of $\mathcal{P} \in \mathrm{P}(V)$.



Theorem 11 (Fortier, Király, Léonard, Szigeti, Talon 2018)

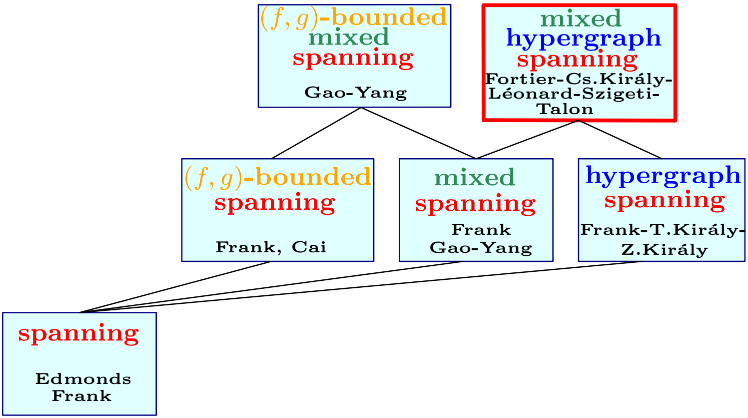
Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, S a multiset of vertices in V. A packing of spanning mixed s-hyperarborescences $(s \in S)$ in $\mathcal{F}$ exists $\Longleftrightarrow$

$$e_{\mathcal{E}}(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} (|S_{\overline{X}}| - d^-_{\mathcal{A}}(X)) \quad \forall\, \mathcal{P} \in \mathrm{P}(V).$$

If $\mathcal{F}$ is a mixed graph, then Theorem 11 reduces to Theorem 7.

If $\mathcal{F}$ is a directed hypergraph, then Theorem 11 reduces to Theorem 10.

Packing mixed hyperarborescences



(f,g)-bounded packing mixed hyperarborescences

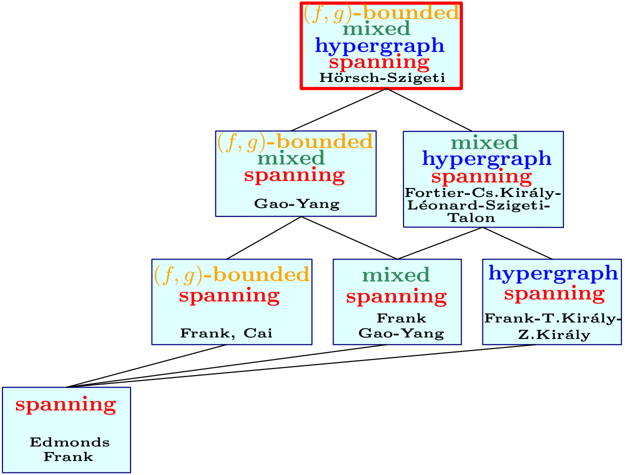
Theorem 12 (Hörsch, Szigeti 2021)

*Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be mixed hypergraph, $f, g : V \to \mathbb{Z}_+$ functions, $k \in \mathbb{Z}_+$.
(f,g)-bounded packing of k spanning mixed hyperarborescences exists in $\mathcal{F}$ $\Longleftrightarrow$*

1. $g(v) \geq f(v) \quad \forall\, v \in V$,
2. $e_{\mathcal{E}}(\mathcal{P}) + \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \geq \sum_{X\in\mathcal{P}}(k - d_{\mathcal{A}}^{-}(X)) \quad \forall\, \mathcal{P} \in \mathrm{P}(V)$.

If $\mathcal{F}$ is a mixed graph, then Theorem 12 reduces to Theorem 9.
If $k = |S|$, and $f(v) = g(v) = |S_v|$ for all $v \in V$, then Theorem 12 reduces to Theorem 11.

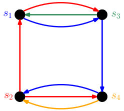
(f, g)-bounded packing mixed hyperarborescences



(f,g)-bounded regular (α,β)-limited packing arborescences

Definition

1. *h-regular* packing of arborescences :
   each vertex belongs to h arborescences in the packing.
2. *(α,β)-limited packing of arborescences* :
   packing of at least α and at most β arborescences.



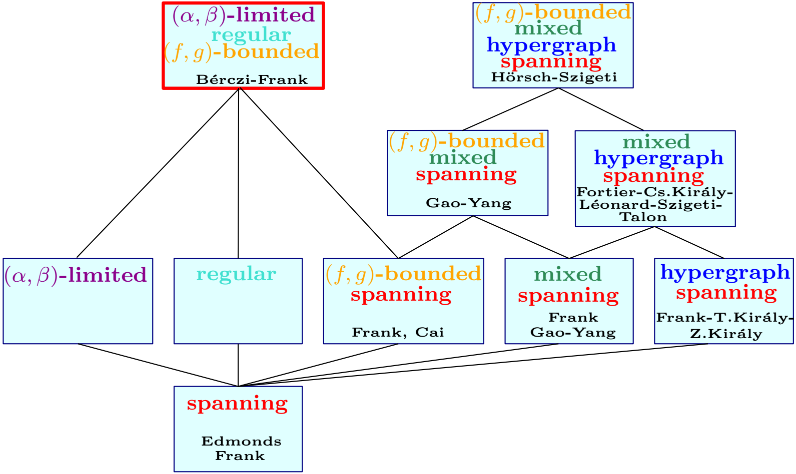
Theorem 13 (Bérczi, Frank 2018)

Let $D=(V,A)$ be a digraph, $f,g : V \to \mathbb{Z}_+$ functions and $h,\alpha,\beta \in \mathbb{Z}_+$.
An (f,g)-bounded h-regular (α,β)-limited packing of arborescences exists $\Longleftrightarrow$

1. $g_h(v) := \min\{g(v), h\} \geq f(v) \;\; \forall\, v \subseteq V,$
2. $\min\{g_h(V), \beta\} \geq \alpha,$
3. $\min\{\beta - f(\overline{\cup\mathcal{P}}), g_h(\cup\mathcal{P})\} \geq \sum_{X\in\mathcal{P}}(h - d_A^-(X)) \;\; \forall\, \mathcal{P} \in \mathrm{P}(V).$

Theorem 13 implies Theorem 5.

(f, g)-bounded regular (α, β)-limited packing arborescences



(f,g)-bounded regular (α,β)-limited packing mixed hyperarborescences

Theorem 14 (Szigeti 2023)

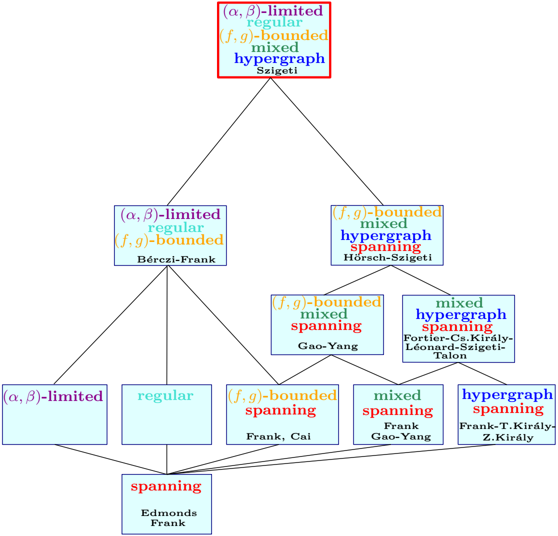
Let $\mathcal{F}=(V,\mathcal{E}\cup\mathcal{A})$ *be mixed hypergraph,* $f,g:V\to\mathbb{Z}_+$ *functions,* $h,\alpha,\beta\in\mathbb{Z}_+$.
$\exists$ *(f,g)-bounded h-regular (α,β)-limited packing of mixed hyperarborescens* $\Longleftrightarrow$

1. $g_h(v)\geq f(v)\;\;\forall\, v\in V$,
2. $\min\{g_h(V),\beta\}\geq\alpha$,
3. $e_{\mathcal{E}}(\mathcal{P})+\min\{\beta-f(\overline{\cup\mathcal{P}}),g_h(\cup\mathcal{P})\}\geq\sum_{X\in\mathcal{P}}(h-d_{\mathcal{A}}^-(X))\;\;\forall\,\mathcal{P}\in\mathrm{P}(V)$.

Theorem 14 implies Theorems 12 and 13.

Definition

1. *h*-regular packing of hyperarborescences : each vertex is a head or the root in h hyperarborescences in the packing.



(α,β)-limited regular (f,g)-bounded mixed hypergraph
Szigeti
(α,β)-limited regular (f,g)-bounded
Bérczi-Frank
(f,g)-bounded mixed hypergraph spanning
Hörsch-Szigeti
(f,g)-bounded mixed spanning
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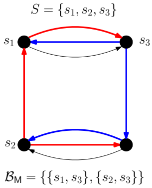
Frank

Matroid-basis-rooted packing of spanning arborescences



Definition

M-basis-rooted packing of arborescences :
roots of the arborescences in the packing form a basis of a given matroid M.

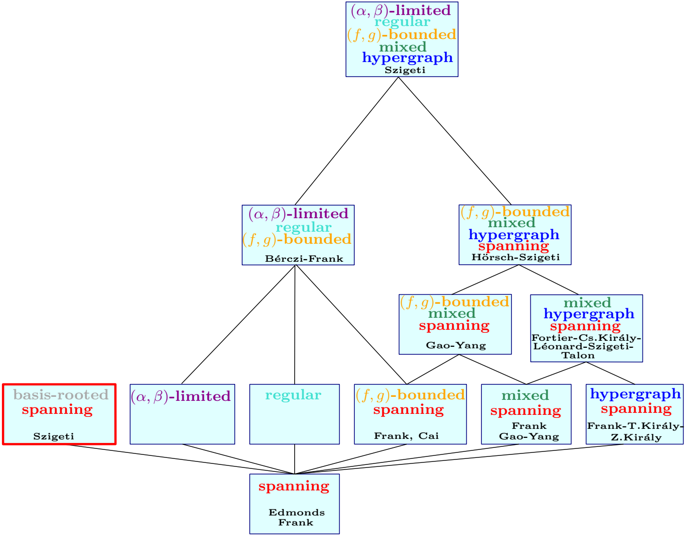
Theorem 15 (Szigeti 2024)

Let $D = (V, A)$ be a digraph, S a multiset in V, $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. There exists an M-basis-rooted packing of spanning arborescences $\Longleftrightarrow$

$$r_{\mathsf{M}}(S_{\cup\mathcal{P}}) \geq \sum_{X \in \mathcal{P}} (r_{\mathsf{M}}(S) - d_A^-(X)) \quad \forall\, \mathcal{P} \in \mathrm{P}(V).$$

Theorem 15 implies Theorems 3 and 4.
Theorem 15 also implies Theorem 5.

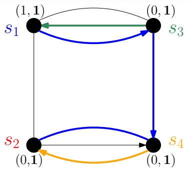
Matroid-basis-rooted packing of spanning arborescences



Theorem 16 (Szigeti 2024)

Let $F = (V, E \cup A)$ be a mixed graph, $h \in \mathbb{Z}_+, f, g : V \to \mathbb{Z}$ functions, S a multiset of vertices in V and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists an M-basis-rooted (f, g)-bounded h-regular packing of mixed arborescences $\Longleftrightarrow$

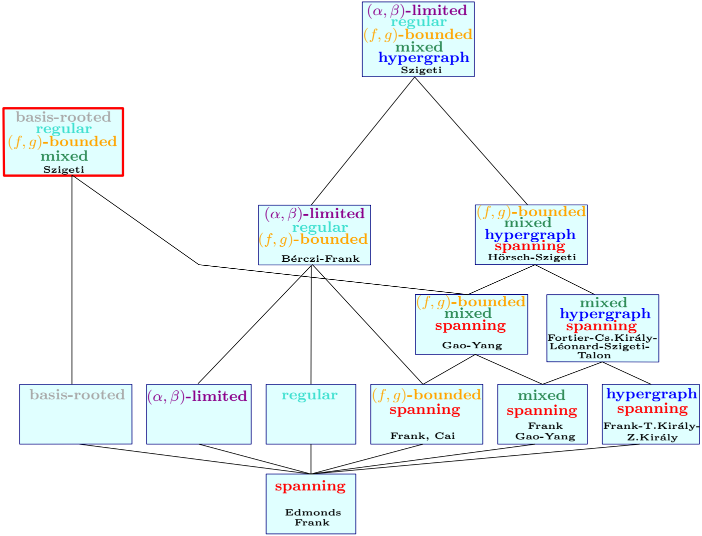
1. $g_h(v) \geq f(v) \quad \forall\, v \in V$,
2. $r_\mathsf{M}(S_X) + g_h(\overline{X}) \geq r_\mathsf{M}(S) \quad \forall\, X \subseteq V$.
3. $e_E(\mathcal{P}) + r_\mathsf{M}(S_X) + g_h(Y - X) \geq \sum_{Z \in \mathcal{P}} (h - d_A^-(Z)) + f(X - Y)$ $\forall\, X, Y \subseteq V, \mathcal{P} \in \mathsf{P}(Y)$,



$\mathcal{B}_\mathsf{M} = \{\{s_1, s_2, s_3\}, \{s_1, s_3, s_4\}\}$

Theorem 16 implies Theorems 9 and 15.

M-basis-rooted (f, g)-bounded regular packing of mixed arborescences



M-basis-rooted (f, g)-bounded regular packing of hyperarborescences

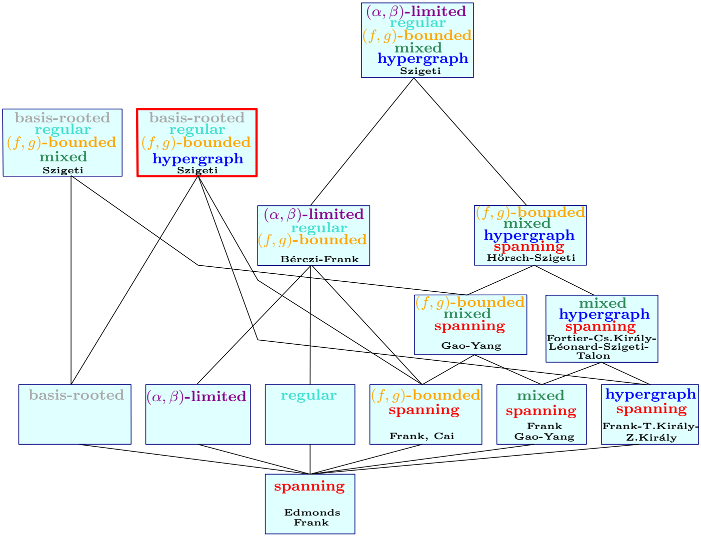
Theorem 17 (Szigeti 2024)

Let $\mathcal{D} = (V, \mathcal{A})$ be a directed hypergraph, $h \in \mathbb{Z}_+, f, g : V \to \mathbb{Z}$ functions, S a multiset of vertices in V and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. There exists an M-basis-rooted (f, g)-bounded h-regular packing of hyperarborescences $\Longleftrightarrow$

1. $g_h(v) \geq f(v) \ \ \forall\ v \in V$,
2. $r_{\mathsf{M}}(S_X) + g_h(\overline{X}) \geq r_{\mathsf{M}}(S) \ \ \forall\ X \subseteq V$.
3. $r_{\mathsf{M}}(S_X) + g_h(Y - X) \geq \sum_{Z \in \mathcal{P}}(h - d^-_{\mathcal{A}}(Z)) + f(X - Y)$ $\forall\ X, Y \subseteq V, \mathcal{P} \in \mathsf{P}(Y)$,

Theorem 17 implies Theorems 5, 10, and 15.

M-basis-rooted (f,g)-bounded regular packing of hyperarborescences



Theorem 18 (Gao+Szigeti)

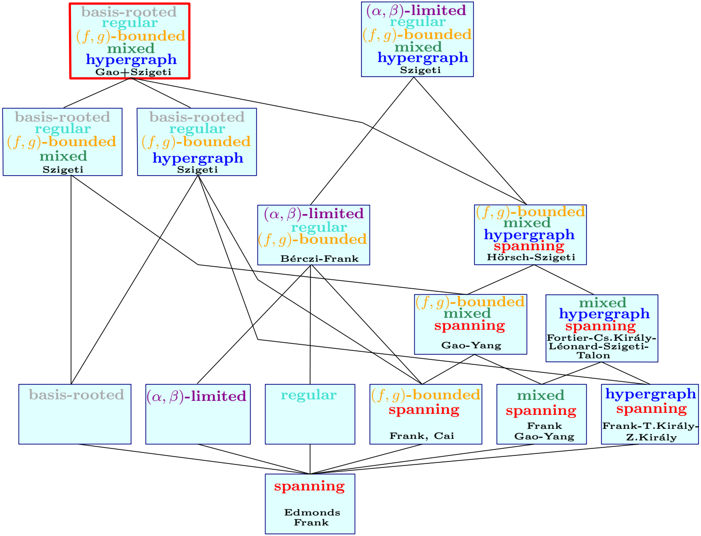
Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ *be a mixed hypergraph,* $h \in \mathbb{Z}_+, f, g : V \to \mathbb{Z}$ *functions,* S *a multiset of vertices in* V *and* $\mathrm{M} = (S, r_{\mathrm{M}})$ *a matroid.*
$\exists$ M-*basis-rooted* (f,g)*-bounded h-regular packing of mixed hyperarborescences*

$$\Longleftrightarrow$$

1. $g_h(v) \geq f(v) \ \ \forall\ v \in V,$
2. $r_{\mathrm{M}}(S_X) + g_h(\overline{X}) \geq r_{\mathrm{M}}(S) \ \ \forall\ X \subseteq V.$
3. $e_{\mathcal{E}}(\mathcal{P}) + r_{\mathrm{M}}(S_X) + g_h(Y - X) \geq \sum_{Z \in \mathcal{P}}(h - d_{\mathcal{A}}^{-}(Z)) + f(X - Y)$
$\forall\ X, Y \subseteq V, \mathcal{P} \in \mathrm{P}(Y),$

Theorem 18 implies Theorems 12, 16 and 17.

M-basis-rooted (f, g)-bounded regular packing of mixed hyperarborescences



Augmentation for packing spanning arborescences

Theorem 19 (Frank 2011)

Let $D = (V, A)$ be a digraph, $k, \gamma \in \mathbb{Z}_+$, and S a multiset of vertices in V.

1. *We can add γ arcs to D to have a packing of spanning s-arborescences $(s \in S)$ $\Longleftrightarrow$*

$$\gamma \geq \sum_{X \in \mathcal{P}} (|S_{\overline{X}}| - d_A^-(X)) \quad \forall \mathcal{P} \in \mathrm{P}(V).$$

2. *We can add γ arcs to D to have a packing of k spanning arborescences $\Longleftrightarrow$*

$$\gamma + k \geq \sum_{X \in \mathcal{P}} (k - d_A^-(X)) \quad \forall \mathcal{P} \in \mathrm{P}(V).$$

Theorem 19 implies Theorems 3 and 4.

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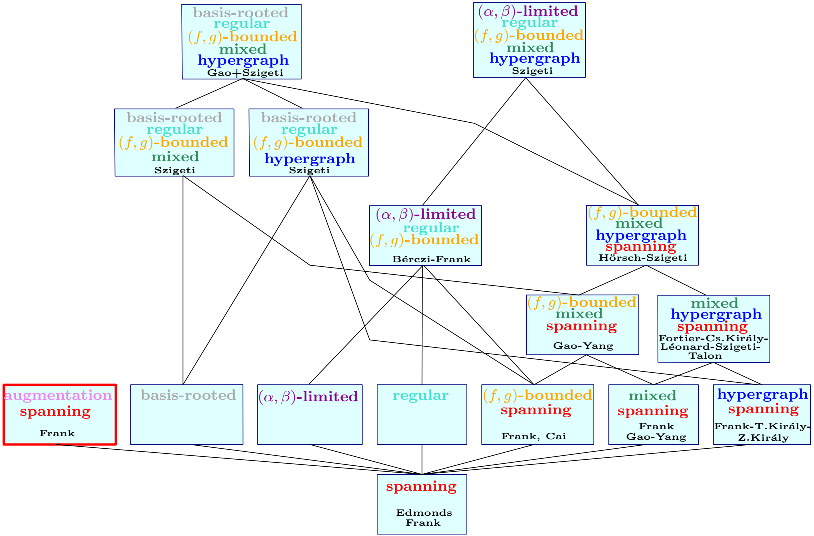


Augmentation for packing mixed hyperarborescences

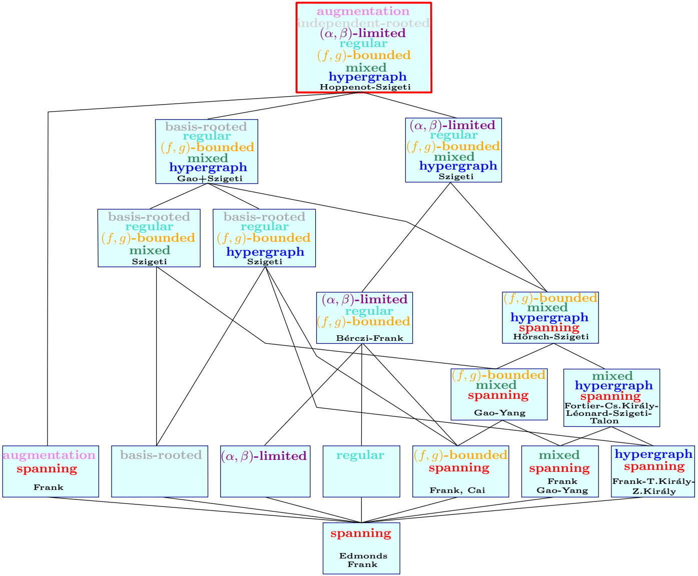
Theorem 20 (Hoppenot, Szigeti)

Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}_+$ functions, $h, \alpha, \beta, \gamma \in \mathbb{Z}_+$, S a multiset of vertices in V, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. We can add an arc set F of size at most γ to $\mathcal{F}$ to have an h-regular M-independent-rooted (f, g)-bounded (α, β)-limited packing of mixed hyperarborescences that contains F $\Longleftrightarrow$ $\forall\, X, Y \subseteq V, \mathcal{P} \in \mathsf{P}(Y)$,

1. $g_h(v) \geq f(v) \;\; \forall\, v \in V$,
2. $\beta \geq \max\{\alpha, f(V)\}$,
3. $r_{\mathsf{M}}(S_X) \geq f(X)$,
4. $\max\{h, \alpha\} - r_{\mathsf{M}}(S_{\overline{X}}) + f(Y - X) - g_h(X - Y) \leq h|Y|$,
5. $\gamma + e_{\mathcal{E}}(\mathcal{P}) + \beta - f(\overline{Y}) \geq \sum_{Z \in \mathcal{P}}(h - d^-_{\mathcal{A}}(Z))$,
6. $\gamma + e_{\mathcal{E}}(\mathcal{P}) + r_{\mathsf{M}}(S_X) - f(X - Y) + g_h(Y - X) \geq \sum_{Z \in \mathcal{P}}(h - d^-_{\mathcal{A}}(Z))$.

Theorem 20 implies Theorems 14, 18, and 19, and hence all the others.

Augmentation for packing mixed hyperarborescences



Proof

Sketch of the proof

1. $Q_1 = K(0,\gamma)$,
   $Q_2 = Q(-\infty_0, r_{\mathsf{M}}(S_{(\cdot)})) \cap K(\max\{h,\alpha\},\beta) \cap T(f, g_h)$,
   $Q_3 = Q(\hat{p}, \infty_0) \cap T(0,h)$,
   $Q_4 = (Q_1 + Q_2) \cap Q_3$,
   where $\hat{p}(Z) = \max\{-e_{\mathcal{E}}(\mathcal{P}) + \sum_{X\in\mathcal{P}}(h - d^-_{\mathcal{A}}(X)) : \mathcal{P} \in \mathsf{P}(Z)\}$ is a supermodular function.
2. An integral vector m is in $Q_4 \iff$ there exist a new arc set F of size at most γ and an h-regular M-independent-rooted (f,g)-bounded (α,β)-limited packing of mixed hyperarborescences in $\mathcal{F} + F$ that contains F with root set R such that $m(v) = d^-_F(v) + |R_v| \;\; \forall\, v \in V$.
3. Frank's theorem on intersection of two generalized polymatroids provides the result.