Vertex-connectivity orientation

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Orientation : edge-connectivity

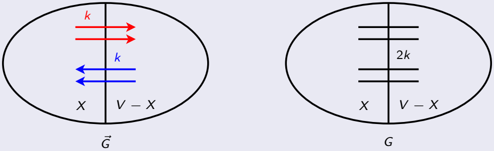
Definition

A graph/digraph G is called *k-edge-connected* if $\forall\ F \subset E$, $|F| = k - 1$, $G - X$ is connected/strongly-connected.

Theorem (Nash-Williams)

G has a k-edge-connected orientation $\iff$ G is $2k$-edge-connected.

Necessity :



Remark

An Eulerian orientation of a $2k$-edge-connected graph is k-edge-connected.

Orientation : k-vertex-connectivity

Definition

A graph/digraph G is called *k-vertex-connected* if $|V| \geq k+1$ and $\forall\ X \subset V, |X| = k-1,$ $G - X$ is connected/strongly-connected.

Conjecture (Frank)

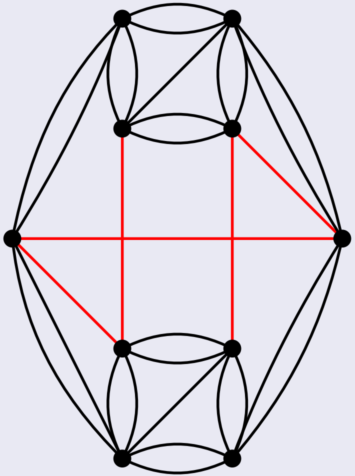
G has a k-vertex-connected orientation if and only if $|V| \geq k+1$ and $\forall\ X \subset V, |X| \leq k-1,$ $G - X$ is $(2k - 2|X|)$-edge-connected.

Theorem

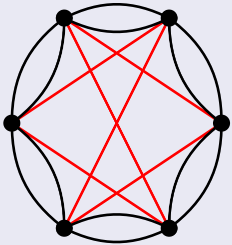
1. Thomassen $k = 2$:
    1. *This conjecture is true.*
2. Durand de Gevigney $k \geq 3$:
    1. *This conjecture is false.*
    2. *Decision problem is NP-complete.*

Counter-examples for $k = 3$

Durand de Gevigney

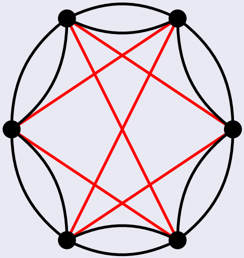


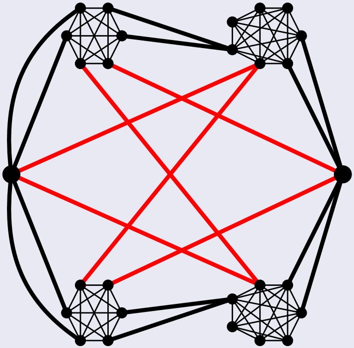
Szigeti



Counter-example for $k = 3$ simple graph

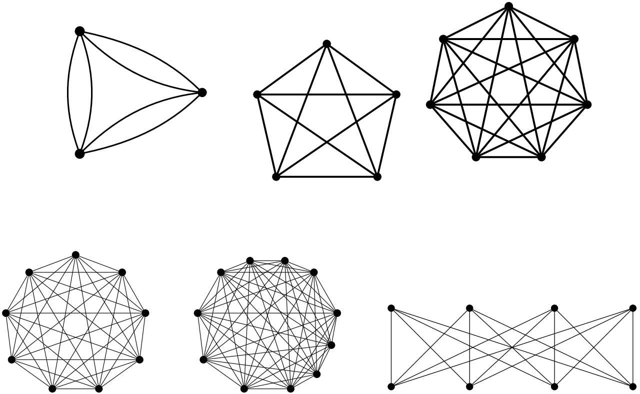
Example (Szigeti)





Vertex-connectivity orientation of $2k$-regular graphs

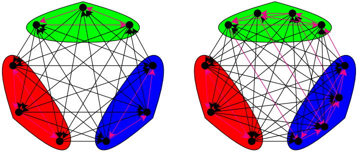
Joseph Cheriyan looked for good graphs : $2k$-regular for which every Eulerian orientation is k-vertex-connected.



Complete graphs

Remark

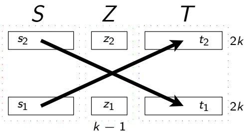
For $k \geq 4$, K_{2k+1} is a bad graph.



Complete bipartite graphs

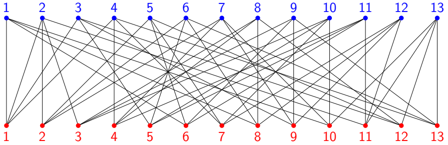
Theorem (Cheriyan)

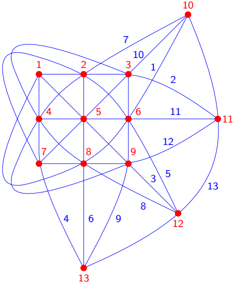
For $k \geq 1$, $K_{2k,2k}$ is a good graph.



Proof

1. $s_1 + s_2 + t_1 + t_2 = 3k + 1$, $1 \leq s_1, s_2, t_1, t_2 \leq k$.
2. $s_1 t_2 + s_2 t_1 \leq k^2 - k$.
   - $s_1 t_2 + s_2 t_1 \leq \min\{d^+(S), d^-(T)\} = \min\{d^-(S), d^+(T)\} \leq \min\{s_1 z_2 + s_2 z_1, t_1 z_2 + t_2 z_1\} \leq \frac{1}{2}(s_1 z_2 + s_2 z_1 + t_1 z_2 + t_2 z_1) = \frac{1}{2}(s_2 + t_2) z_1 + \frac{1}{2}(s_1 + t_1) z_2 \leq \frac{1}{2} 2k z_1 + \frac{1}{2} 2k z_2 = k(z_1 + z_2) = k^2 - k$.
3. $\min\{s_1 t_2 + s_2 t_1 \ s.t.\ (1)\} \geq k^2 + k$.
   - Suppose that the minimum is attained at $(\overline{s}_1, \overline{s}_2, \overline{t}_1, \overline{t}_2)$.
   - If $k > \overline{s}_1 \geq \overline{t}_2 > 1$ then $(\overline{s}_1 + 1)(\overline{t}_2 - 1) + \overline{s}_2 \overline{t}_1 < \overline{s}_1 \overline{t}_2 + \overline{s}_2 \overline{t}_1$, $\hookleftarrow$.
   - Either $\max\{\overline{s}_1, \overline{t}_2\} = k$ or $\min\{\overline{s}_1, \overline{t}_2\} = 1$. By (1), $\max\{\overline{s}_1, \overline{t}_2\} = k$.
   - Similarly, $\max\{\overline{s}_2, \overline{t}_1\} = k$.
   - $\overline{s}_1 \overline{t}_2 + \overline{s}_2 \overline{t}_1 = k(\min\{\overline{s}_1, \overline{t}_2\} + \min\{\overline{s}_2, \overline{t}_1\}) = k(k+1)$.





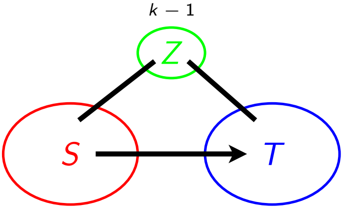
Remark

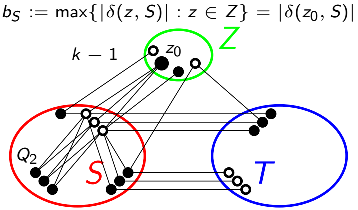
The incidence graph of a projective plane is a bipartite graph $(V_1, V_2; E)$ such that any two vertices of V_i have exactly one common neighbour.

Theorem

The incidence graph of a projective plane of order $2k - 1$ is good.

Incidence graphs of projective planes





Claim

There exists a vertex in S (T) that has no neighbour outside of S (T).

Proof

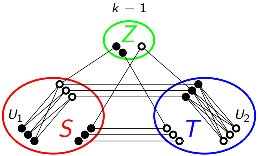
$|Q_1| \leq |\delta(S)| \leq 2|\delta(S,Z)| = 2\sum_{z\in Z}|\delta(z,S)| \leq 2|Z|b_S = 2(k-1)b_S.$
$|Q_2| = |Q| - |Q_1| \geq b_S(2k-1) - 2b_S(k-1) = b_S > 1.$

Incidence graphs of projective planes

Proof

$$\begin{aligned} |V_1| &= (2k-1)^2 + (2k-1) + 1 \\ |\delta(Z)| &\geq 2|\delta(S,T)| \\ 4k(k-1) &\geq 2|\delta(Z)| \\ |\delta(Z) \cup \delta(S,T)| &\geq |V_1| - |U_1| \\ |\delta(S,T)| &\geq |N(U_1)| \\ |N(U_1)| &\geq |U_1| \end{aligned}$$

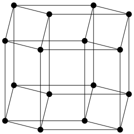
$\hookleftarrow$



$U_i := V_i \setminus V(\delta(Z) \cup \delta(S,T))$.

$U_1 \cup N(U_1) \subseteq S \quad U_2 \cup N(U_2) \subseteq T$

Hypercube



Theorem (Levit, Chandran, Cheriyan)

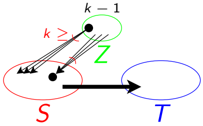
A $2k$-dimensional hypercube Q_{2k} is good.

Lemma (Levit, Chandran, Cheriyan)

$|N_{2k}(S)| \geq k \min\{k, |S|+1\}$ for all $S \subseteq V(Q_{2k})$ with $1 \leq |S| \leq 2^{2k-1}$.

Proof of theorem

$k \min\{|Z|, |S|\} \geq d^-(S) = d^+(S) \geq |N_{2k}(S)| - |Z| \geq$
$k \min\{k, |S|+1\} - k + 1 = k \min\{|Z|, |S|\} + 1.$



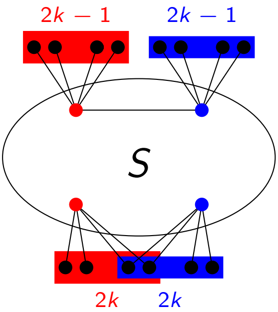
Hypercube : Proof of the lemma (Szigeti)

Lemma (a)

$|N_{2k}(S)| \geq k(|S|+1)$ if $1 \leq |S| \leq k$.

Proof

$$\begin{aligned}
|N_{2k}(S)| &\geq \sum_{v\in S} d_{2k}(v) - 2\binom{|S|}{2} \\
&= |S|(2k+1-|S|) \\
&= k(|S|+1) + (k-|S|)(|S|-1) \\
&\geq k(|S|+1) \\
&\geq |S|(k+1).
\end{aligned}$$



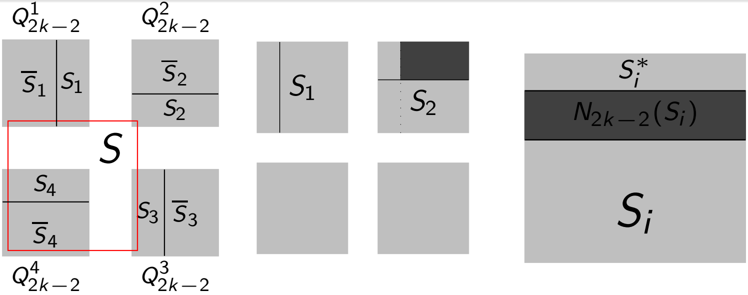
Hypercube : Proof of the lemma

Lemma (b)

$|N_{2k}(S)| \geq k^2$ if $k \leq |S| \leq 2^{2k-1}$.

Proof

1. Induction for k. For $k = 1$, the lemma is true since Q_2 is connected.
2. $|N_{2k}(S) \cap V(Q^i)| \geq \max\{|S_{i-1}| - |S_i|, |N_{2k-2}(S_i)|\}$.
3. $|N_{2k-2}(S_i)| \geq (k-1)^2$ if $k-1 \leq |S_i| \leq g(k-1) = 2^{2k-2} - ((k-1)^2 + k - 2)$.

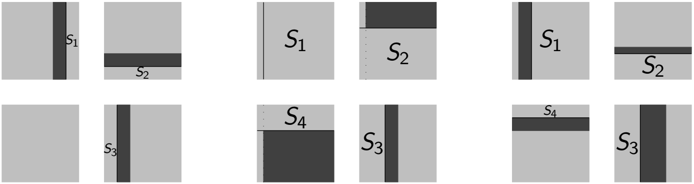


Hypercube : Proof of the lemma

Proof

Let $p := |\{i : |S_i| \geq 1\}|$ and $q := |\{i : 1 \leq |S_i| \leq k-1\}|$.

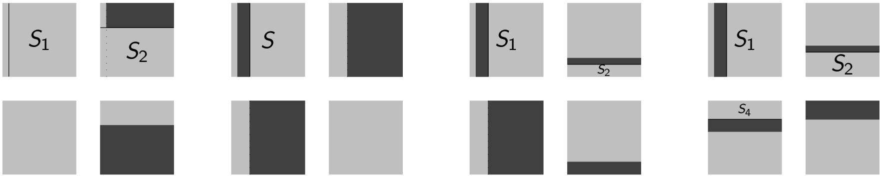
1. $p = q$: $|N_{2k}(S)| \geq \sum |N_{2k-2}(S_i)| \geq \sum |S_i|k = |S|k \geq k^2$.
2. $p = 4$:
   1. If $|S_1| \geq g(k\text{-}1)+1$: $|N_{2k}(S)| \geq (|S_1| - |S_2|) + (|S_1| - |S_4|) + |N_{2k-2}(S_3)| = 3|S_1| + |S_3| - |S| + |N_{2k-2}(S_3)| \geq 3(g(k-1)+1) + (2k-1) - 2^{2k-1} \geq k^2$.
   2. Otherwise, $|N_{2k}(S)| \geq qk + (4-q)(k-1)^2 = k^2 + ((3-q)(k-1)+1)(k-2) + q \geq k^2$.



Hypercube : Proof of the lemma

Proof ($q < p < 4$)

1. If $|S_1| \geq g(k-1)+1 : |N_{2k}(S)| \geq |S_1| \geq g(k-1)+1 \geq k^2$.
2. $p = 1 : |N_{2k}(S)| \geq (k-1)^2 + 2|S| = k^2 + 2(|S|-k) + 1 > k^2$.
3. $p = 2 : |N_{2k}(S)| \geq qk + (2-q)(k-1)^2 + |S|$
   $= k^2 + (1-q)(k-2)(k-1) + (|S|-k) \geq k^2$.
4. $p = 3 : |N_{2k}(S)| \geq qk + (3-q)(k-1)^2 + 1 = k^2 + (2-q)(k-2)(k-1) + q \geq k^2$.



Thank you for your attention !