Packing mixed hyperarborescences

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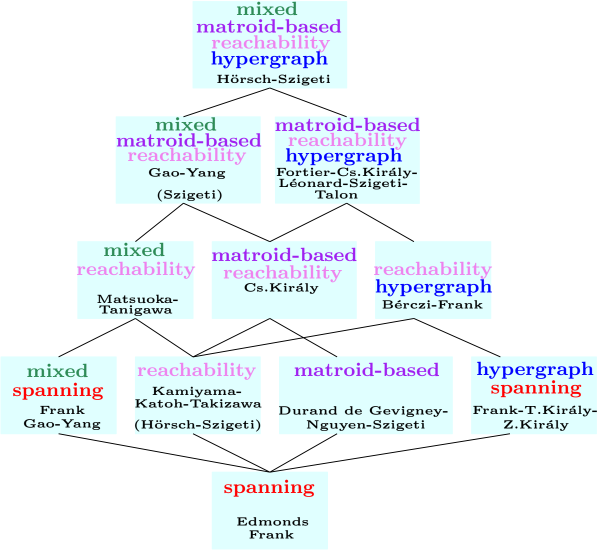
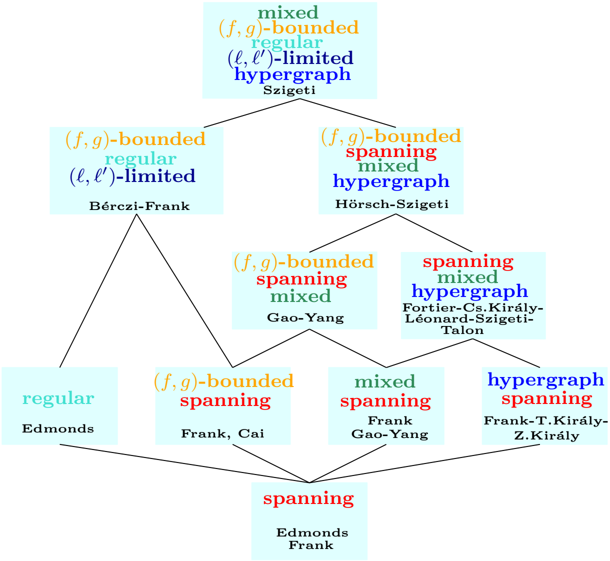
Results 1


mixed
matroid-based
reachability
hypergraph
Hörsch-Szigeti
mixed
matroid-based
reachability
Gao-Yang
(Szigeti)
matroid-based
reachability
hypergraph
Fortier-Cs.Király-Léonard-Szigeti-Talon
mixed
reachability
Matsuoka-Tanigawa
matroid-based
reachability
Cs.Király
reachability
hypergraph
Bérczi-Frank
mixed
spanning
Frank
Gao-Yang
reachability
Kamiyama-Katoh-Takizawa
(Hörsch-Szigeti)
matroid-based
Durand de Gevigney-Nguyen-Szigeti
hypergraph
spanning
Frank-T.Király-Z.Király
spanning
Edmonds
Frank

Results 2


mixed
(f,g)-bounded
regular
(ℓ,ℓ')-limited
hypergraph
Szigeti
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regular
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Bérczi-Frank
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Frank

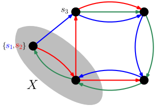
Packing spanning arborescences : fixed roots

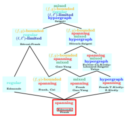
Theorem 1 (Edmonds 1973)

Let $D = (V, A)$ be a digraph and S a multiset of V.
There exists a packing of spanning s-arborescences $(s \in S)$ $\iff$
$|S_X| + d_A^-(X) \geq |S|$ *for all* $\emptyset \neq X \subseteq V$.

Definitions

1. s-arborescence: unique (s, v)-dipath for all $v \in V$.
2. spanning subgraph: contains all the vertices.
3. packing subgraphs: pairwise arc-disjoint subgraphs.
4. S-branching: unique (S, v)-dipath for all $v \in V$.
5. multiset S of V: vertex set with multiplicities.
6. S_X for $X \subseteq V$: restriction of S in X.

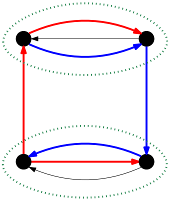




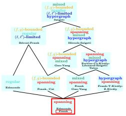
Packing spanning arborescences : flexible roots

Theorem 2 (Frank 1978)

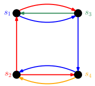
Let $D = (V, A)$ be a digraph and $k \in \mathbb{Z}_+$.
There exists a packing of k spanning arborescences $\Longleftrightarrow$
$k \geq \sum_{X \in \mathcal{P}} (k - d_A^-(X))$ *for every subpartition $\mathcal{P}$ of V.*



Theorem 2 implies Theorem 1.



Regular packing of arborescences



Definition

k-regular packing of arborescences:
each vertex belongs to k arborescences in the packing.

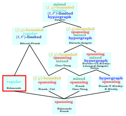
Theorem 3 (Edmonds 1973)

Let $D = (V, A)$ be a digraph, S a multiset of V and $k \in \mathbb{Z}_+$.
There exists a k-regular packing of s-arborescences $(s \in S' \subseteq S)$ $\iff$
$|S_X| + |d_A^-(X)| \geq k$ *for all* $\emptyset \neq X \subseteq V$.

Remark

There exists a k-regular packing of s-arborescences $(s \in S)$ $\iff$
there exists a packing of k S_i-branchings with $\bigcup S_i = S$.

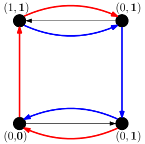
Theorem 3 implies Theorem 1.



(f,g)-bounded packing of spanning arborescences

Definition

(f,g)-bounded packing of arborescences:
number of v-arborescences in the packing is at least $f(v)$ and at most $g(v)$ $\forall v \in V$.

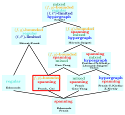


Theorem 4 (Frank 1978, Cai 1983)

Let $D=(V,A)$ be a digraph, $f,g: V \to \mathbb{Z}_+$ functions and $k \in \mathbb{Z}_+$. There exists an (f,g)-bounded packing of k spanning arborescences $\iff$

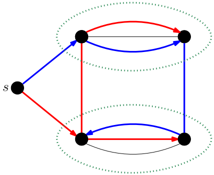
1. $g(v) \geq f(v)$ *for every* $v \in V$,
2. $\min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \geq \sum_{X\in\mathcal{P}}(k - d_A^-(X))$ *for every subpartition* $\mathcal{P}$ *of* V.

Theorem 4 implies Theorem 2.



Definition

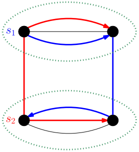
1. mixed s-arborescence:
   it can be oriented to obtain an s-arborescence.
2. $e_E(\mathcal{P})$: number of edges entering at least one member of a subpartition $\mathcal{P}$ of V.

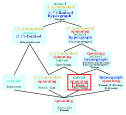


Theorem 5 (Frank 1978)

Let $F = (V, E \cup A)$ be a mixed graph and S a multiset of V.
There exists a packing of spanning mixed s-arborescences $(s \in S)$ $\Longleftrightarrow$
$e_E(\mathcal{P}) \geq \sum_{X \in \mathcal{P}}(|S| - |S_X| - d_A^-(X))$ *for every subpartition $\mathcal{P}$ of V.*

Theorem 5 implies Theorem 1.





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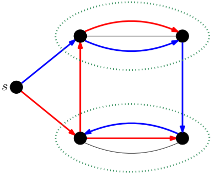

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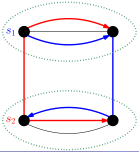
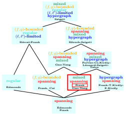
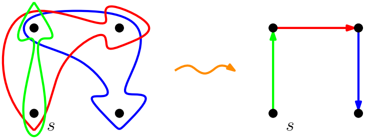




Packing spanning hyperarborescences

Definition

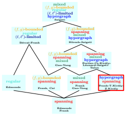
(spanning) (s-)hyperarborescence:
dypergraph that can be trimmed to a (spanning) (s-)arborescence.



Theorem 6 (Frank, T. Király, Z. Király 2003)

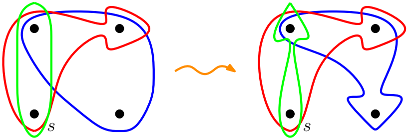
Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph and S a multiset of V.
There exists a packing of spanning s-hyperarborescence ($s \in S$) $\iff$
$|S_X| + d^-_{\mathcal{A}}(X) \geq |S|$ for all $\emptyset \neq X \subseteq V$.

Theorem 6 implies Theorem 1.



Definition

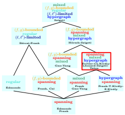
(spanning) mixed (s-)hyperarborescence: mixed hypergraph that can be oriented to a (spanning) (s-)hyperarborescence.



Theorem 7 (Fortier, Cs. Király, Léonard, Szigeti, Talon 2018)

Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, S a multiset of V.
There exists a packing of spanning mixed s-hyperarborescence $(s \in S)$ $\Longleftrightarrow$
$e_{\mathcal{E}}(\mathcal{P}) \geq \sum_{X \in \mathcal{P}}(|S| - |S_X| - d^-_{\mathcal{A}}(X))$ for every subpartition $\mathcal{P}$ of V.

Theorem 7 implies Theorems 5 and 6.

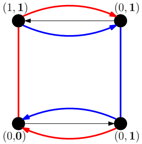


(f, g)-bounded packing spanning mixed arborescences

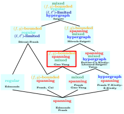
Theorem 8 (Gao, Yang 2021)

Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \to \mathbb{Z}$ functions and $k \in \mathbb{Z}_+$. An (f, g)-bounded packing of k spanning mixed arborescences exists $\iff$

1. $g(v) \geq f(v)$ *for every* $v \in V$,
2. $e_E(\mathcal{P}) + \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \geq \sum_{X \in \mathcal{P}}(k - d_A^-(X))$ *for every subpartition* $\mathcal{P}$ *of* V.



Theorem 8 implies Theorems 4 and 5.

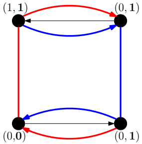


(f,g)-bounded packing spanning mixed hyperarborescences

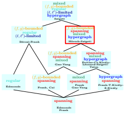
Theorem 9 (Hörsch-Szigeti 2021)

Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}$ functions, $k \in \mathbb{Z}_+$. An (f,g)-bounded packing of k spanning mixed hyperarborescences exists $\iff$

1. $g(v) \geq f(v)$ *for every* $v \in V$,
2. $e_{\mathcal{E}}(\mathcal{P}) + \min\{k - f(\overline{\cup \mathcal{P}}), g(\cup \mathcal{P})\} \geq \sum_{X \in \mathcal{P}} (k - d^-_{\mathcal{A}}(X))$ *for every subpartition* $\mathcal{P}$ *of* V.



Theorem 9 implies Theorems 8 and 7.



(f, g)-bounded regular (ℓ, ℓ')-limited packing arborescences

Definition

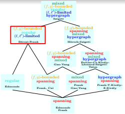
(ℓ, ℓ')-limited packing of arborescences:
packing of at least ℓ and at most ℓ' arborescences.

Theorem 10 (Bérczi, Frank 2018)

Let $D = (V, A)$ be a digraph, $f, g : V \to \mathbb{Z}_+$ functions and $k, \ell, \ell' \in \mathbb{Z}_+$.
An (f, g)-bounded k-regular (ℓ, ℓ')-limited packing of arborescences exists $\Longleftrightarrow$

1. $g_k(v) := \min\{g(v), k\} \geq f(v)$ *for every* $v \subseteq V$,
2. $\min\{g_k(V), \ell'\} \geq \ell$,
3. $\min\{\ell' - f(\overline{\cup \mathcal{P}}), g_k(\cup \mathcal{P})\} \geq \sum_{X \in \mathcal{P}}(k - d_A^-(X))$
   for every subpartition $\mathcal{P}$ of V.

Theorem 10 implies Theorems 3 and 4.



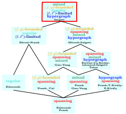
(f, g)-bounded regular (ℓ, ℓ')-limited packing mixed hyperarborescences

Theorem 11 (Szigeti 2022+)

Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}$ functions, $k, \ell, \ell' \in \mathbb{Z}_+$. An (f, g)-bounded k-regular (ℓ, ℓ')-limited packing of mixed hyperarborescences exists $\Longleftrightarrow$

1. $g_k(v) \geq f(v)$ *for every* $v \in V$,
2. $\min\{g_k(V), \ell'\} \geq \ell$,
3. $e_{\mathcal{E}}(\mathcal{P}) + \min\{\ell' - f(\overline{\cup \mathcal{P}}), g_k(\cup \mathcal{P})\} \geq \sum_{X \in \mathcal{P}} (k - d^-_{\mathcal{A}}(X))$ *for every subpartition* $\mathcal{P}$ *of* V.

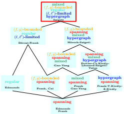
Theorem 11 implies Theorems 9 and 10.



(f,g)-bounded regular (ℓ,ℓ')-limited packing mixed hyperarborescences

Sketch of the proof

1. characteristic vectors of the dyperedge sets of the (f,g)-bounded k-regular (ℓ,ℓ')-limited packings of hyperarborescences in orientations of $\mathcal{F}$ = integer points of the intersection of the two generalized polymatroids
   - $\sum_{v\in V}(Q(0,r_v)\cap K(k-g_k(v),k-f(v)))$ and
   - $Q(0,r_{\mathsf{M}^k_{\mathcal{F}}})\cap K(k|V|-\ell',k|V|-\ell)$.
2. Frank's theorem on intersection of two generalized polymatroids provides the result.



Open problems

Open problems

1. Packing of mixed branchings with given root set sizes ℓ_i.
   1. For directed graphs: (Bérczi, Frank).
   2. For undirected graphs: (easy exercise).
   3. For each $\ell_i = \ell$: (Szigeti).
2. Packing of k mixed hyperbranchings each of root set size ℓ.
   1. For mixed graphs: (previous observation).
   2. For dypergraphs: (Bérczi, Frank).
   3. For hypergraphs: (Martin, Szigeti).

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Thanks for your attention!