Packing of arborescences versus matroid intersection

Zoltán Szigeti

Laboratoire G-SCOP
INP Grenoble, France

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Joint work with :
Csaba Király (EGRES Group, Eötvös University, Budapest)

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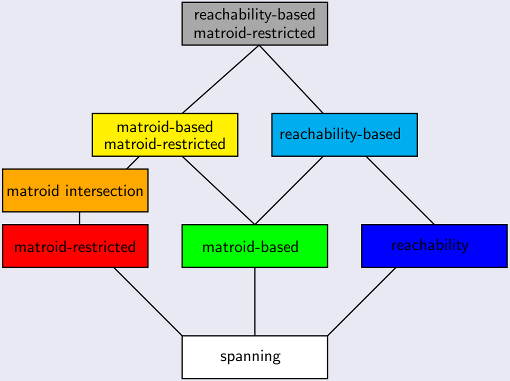


Matroids

Definition

For $\mathcal{I} \subseteq 2^E$, $\mathcal{M} = (E, \mathcal{I})$ is a matroid if

1. $\mathcal{I} \neq \emptyset$,
2. If $X \subseteq Y \in \mathcal{I}$ then $X \in \mathcal{I}$,
3. If $X, Y \in \mathcal{I}$ with $|X| < |Y|$ then $\exists\ y \in Y \setminus X$ such that $X \cup y \in \mathcal{I}$.

Examples

1. Linear matroid : Sets of linearly independent vectors in a vector space,
2. Graphic matroid : Edge-sets of forests of a graph,
3. Uniform matroid $U_{n,k}$: $\{X \subseteq E : |X| \leq k\}$ where $|E| = n$,
4. Free matroid : $U_{n,n}$.

Matroid intersection

Notion

1. independent : sets in $\mathcal{I}$,
2. base : maximal independent set,
3. bridge : an element contained in all bases,
4. rank function : $r(X) = \max\{|Y| : Y \in \mathcal{I}, Y \subseteq X\}$,
   1. submodular ($r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)\ \forall X, Y \subseteq E$),
   2. $X \in \mathcal{I}$ if and only if $r(X) = |X|$.

Theorem (Edmonds 1970)

Two matroids $\mathcal{M}_1 = (E, r_1)$ and $\mathcal{M}_2 = (E, r_2)$ have a common independent set of size k $\iff r_1(X) + r_2(E - X) \geq k\ \forall\ X \subseteq E$.

Matroid Operations

Definition

$\mathcal{M} = (E, \mathcal{I})$ matroid, $e \in E$, $\mathcal{M}' = (E', \mathcal{I}')$ matroid with $E \cap E' = \emptyset$.

1. deletion of e : $\mathcal{M} - e = (E - e, \{I \subseteq E - e : I \in \mathcal{I}\})$,
2. contraction of e : $\mathcal{M}/e = (E - e, \{I \subseteq E - e : I \cup e \in \mathcal{I}\})$,
3. k-sum : $k\mathcal{M} = (E, \{\cup_1^k I_i : I_i \in \mathcal{I}\})$,
4. direct sum : $\mathcal{M} \oplus \mathcal{M}' = (E \cup E', \{I \cup I' : I \in \mathcal{I}, I' \in \mathcal{I}'\})$.

Example

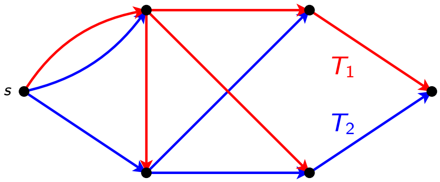
Graphic matroids of $G = (V, E)$, $G' = (V', E')$ with $V \cap V' = \emptyset$, $e \in E$.

1. Graphic matroid of $G - e$,
2. Graphic matroid of G/e,
3. Unions of edge sets of k edge-disjoint forests,
4. Graphic matroid of $(V \cup V', E \cup E')$.

Packing of spanning s-arborescences

Definition

1. s-arborescence : directed tree, indegree of every vertex except s is 1,
2. spanning subgraph of D : subgraph that contains all the vertices of D,
3. packing of arborescences : arc-disjoint arborescences,



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4. $\partial(Z, X)$: set of arcs from Z to X, for $Z \subseteq V(D) - X$,
5. $|\partial(X)|$: indegree of X.

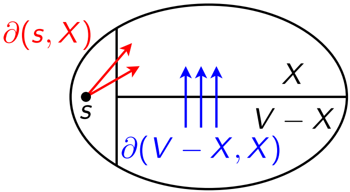
Theorem (Edmonds 1973)

Let $D = (V + s, A)$, $k \in \mathbb{Z}_+$.

- *D has a packing of k spanning s-arborescences*

$\Longleftrightarrow$

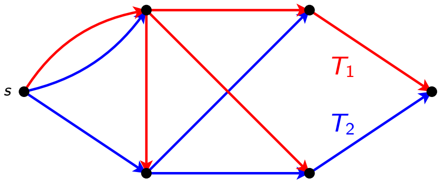
- $|\partial(X)| \geq k \quad \forall\, \emptyset \neq X \subseteq V.$



Remark

Let $D = (V + s, A)$ and G be the underlying undirected graph of D.

1. $\vec{F} \subseteq A$ is a packing of k spanning s-arborescences of $D \Longleftrightarrow$
2. F is a packing of k spanning trees of G, $|\partial^{\vec{F}}(v)| = k \ \forall \ v \in V \Longleftrightarrow$
3. F is a common base of $\mathcal{M}_1 = k$-sum of the graphic matroid of G and $\mathcal{M}_2 = \oplus_{v \in V} U_{|\partial(v)|,k}$.



Matroid-restricted packing of spanning s-arborescences

Definition

Given a digraph $D = (V + s, A)$ and a matroid $\mathcal{M} = (A, \mathcal{I})$, a packing of spanning s-arborescences $T_1, \ldots, T_k$ is matroid-restricted if $\cup_1^k A(T_i) \in \mathcal{I}$.

Theorem

Given a digraph $D = (V + s, A)$, $k \in \mathbb{Z}_+$ and a matroid $(\mathcal{M}, r)$ which is the direct sum of the matroids $\mathcal{M}_v = (\partial(v), r_v)$ $\forall v \in V$.

- *D has an $\mathcal{M}$-restricted packing of k spanning s-arborescences $\Longleftrightarrow$*
- *$r(\partial(X)) \geq k$ $\forall$ $\emptyset \neq X \subseteq V$.*

Remarks

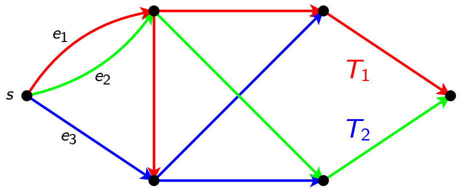
1. For free matroid, we are back to packing of k spanning s-arborescen.
2. This problem can also be formulated as matroid intersection.
3. For general matroid $\mathcal{M}$, the problem is NP-complete, even for $k = 1$.

Matroid-based packing of s-one-arborescences

Definition

Let $D = (V + s, A)$ be a digraph and $\mathcal{M}$ a matroid on $\partial(s, V)$.

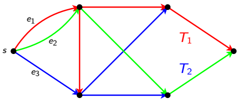
1. s-one-arborescence : s-arborescence containing one arc leaving s.
2. A packing of s-one-arborescences $\{T_1, \dots, T_t\}$ is matroid-based if $\{A(T_i) \cap \partial(V) : v \in V(T_i)\}$ is a base of $\mathcal{M}$ $\forall v \in V$.

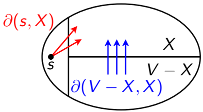


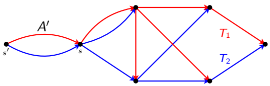
Theorem (Durand de Gevigney, Nguyen, Szigeti 2013)

Let $D = (V + s, A)$ be a digraph and $\mathcal{M} = (\partial(s, V), r)$ a matroid.

- There exists an $\mathcal{M}$-based packing of s-one-arborescences in D $\Longleftrightarrow$
- $r(\partial(s, X)) + |\partial(V - X, X)| \geq r(\partial(s, V))\ \forall X \subseteq V$.







Remark

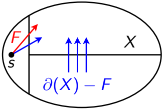
A packing of k spanning s-arborescences in $D = (V + s, A)$ can be obtained as an $\mathcal{M}$-based packing of s'-one-arborescences in $D' = (V + s + s', A \cup A')$, where $A' = \{k \times s's\}$ and free matroid $\mathcal{M}$ on A'.

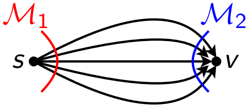
$\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-one-arborescences

Theorem (Cs. Király, Szigeti 2016-)

Let $D = (V + s, A)$, $\mathcal{M}_1 = (\partial(s, V), r_1)$, $\mathcal{M}_2 = (A, r_2) = \oplus_{v \in V} \mathcal{M}_v$.

- D has an $\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-one-arborescen. $\Longleftrightarrow$
- $r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial(s, V)) \;\; \forall X \subseteq V, F \subseteq \partial(s, X)$.





Remarks

1. It contains matroid-restricted packing of spanning s-arborescences, even matroid intersection. For matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on S, our problem on $(D = (\{s, v\}, \{|S| \times sv\}), \mathcal{M}_1, \mathcal{M}_2)$ reduces to it.
2. For free $\mathcal{M}_2$, we are back to $\mathcal{M}_1$-based packing of s-one-arborescen.

Algorithmic aspects

Remark : Our condition

$$r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial(s, V))\ \forall X \subseteq V, F \subseteq \partial(s, X) \iff$$

$$\min_{X \subseteq V} \left\{ \min_{F \subseteq \partial(s,X)} \{r_1(F) + r_2(\partial(X) - F)\} \right\} \geq r_1(\partial(s, V))$$

Remark : How to check it in polynomial time

1. $b_1(F) = r_1(F) + r_2(\partial(X) - F)$ for $F \subseteq \partial(s, X)$ is submodular.
2. $b_2(X) = \min\{b_1(F) : F \subseteq \partial(s, X)\}$ for $X \subseteq V$ is submodular.
3. By submodular function minimization (Iwata, Fleischer, Fujishige (2001)/Schrijver(2000)), we are done.

Algorithmic aspects

Algorithm

INPUT : $(D, \mathcal{M}_1, \mathcal{M}_2)$.
OUTPUT : Either the required packing or a pair violating our condition.

1. If $(D, \mathcal{M}_1, \mathcal{M}_2)$ doesn't satisfy our condition then stop with the pair violating our condition.
2. If $\mathcal{M}_2$ is the free matroid then use Durand de Gevigney, Nguyen, Szigeti's algorithm for $\mathcal{M}_1$-based packing of s-one-arborescences and stop with the packing.
3. Otherwise, let e be a non-bridge edge in $\mathcal{M}_2$.
4. If $(D - e, \mathcal{M}_1 - e, \mathcal{M}_2 - e)$ satisfies our condition and e is not a bridge in $\mathcal{M}_1$ then use recursively our algorithm for it and stop with the packing.
5. Otherwise, $(D, \mathcal{M}_1, \mathcal{M}_2' = (\mathcal{M}_2/e) \oplus e)$ satisfies our condition. Use recursively our algorithm for $(D, \mathcal{M}_1, \mathcal{M}_2')$ and stop with the packing.

Conclusion

Summary

- A theorem on matroid-based matroid-restricted packing of s-one-arborescences that generalizes
  - Durand de Gevigney, Nguyen, Szigeti's result on matroid-based packing of s-one-arborescences,
  - Edmonds' result on matroid intersection.
- A polynomial algorithm to solve our problem.
- The problem of reachability-based matroid-restricted packing of s-one-arborescences can also be solved.

Open problem

Algorithm for finding a matroid-based matroid-restricted packing of s-one-arborescences of minimum weight ?

Thank you for your attention !