Augmenting a hypergraph to have a matroid-based (f, g)-bounded (α, β)-limited packing of rooted hypertrees

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Joint work with

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Prior work

Packing trees

1. Nash-Williams (1961), Tutte (1961),
2. Peng, Chen, Koh (1991),
3. Frank, T. Király, Kriesell (2003),
4. Kaiser (2012),
   Katoh, Tanigawa (2013),
5. Hoppenot, Martin, Szigeti (2025),
   Hoppenot, Szigeti (2026)

Packing arborescences

1. Edmonds (1973), Lovász (1976), Frank (1978),
2. Frank, T. Király, Z. Király (2003),
   Bérczi, Frank (2008), Frank (2009),
   Kamiyama, Katoh, Takizawa (2009),
3. Bérczi, Frank (2010, 2018),
   Durand de Gevigney, Nguyen, Szigeti (2013),
   Cs. Király (2016),
   Fortier, Cs. Király, Léonard, Szigeti, Talon (2018),
   Matsuoka, Szigeti (2019),
4. Fortier, Cs. Király, Szigeti, Tanigawa (2020),
   Cs. Király, Szigeti, Tanigawa (2020),
   Gao, Yang (3 in 2021),
   Hörsch, Szigeti (2021, 2022),
   Szigeti (2023, 2024),
   Hörsch, Peyrille, Szigeti (2026)
   Hoppenot, Szigeti (2026)

Plan

Previous results

1. Packing spanning trees (Nash-Williams, Tutte),
2. Augmentation for packing spanning trees (Frank),
3. Packing spanning hypertrees (Frank, Király, Kriesell),
4. Complete matroid-based packing of rooted trees (Katoh, Tanigawa),

New result

1. Matroid-based (f, g)-bounded packing of k rooted trees ($\alpha \leq k \leq \beta$),
2. Generalization to hypergraphs, Augmentation version.

Tools

1. Generalized partition/Katoh-Tanigawa matroids, Matroid intersection,
2. Trimming a hypergraph covering two supermodular functions on partitions,
3. Covering two supermodular functions on partitions.

Packing spanning trees

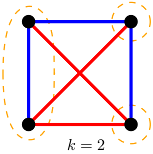
Theorem (Nash-Williams 1961, Tutte 1961)

Let $G = (V, E)$ be a graph and $k \in \mathbb{Z}_+$.
There exists a packing of k spanning trees in G $\Longleftrightarrow$

$$e_E(\mathcal{P}) \geq k(|\mathcal{P}| - 1) \quad \text{for every partition } \mathcal{P} \text{ of } V.$$

Definition

$e_E(\mathcal{P})$: number of edges between the members of a partition $\mathcal{P}$ of V.



Theorem (Frank 2006)

Let $G = (V, E)$ be a graph and $k, \gamma \in \mathbb{Z}_+$.
We can add γ edges to G to have a packing of k spanning trees $\Longleftrightarrow$

$$\gamma + e_E(\mathcal{P}) \geq k(|\mathcal{P}| - 1) \quad \text{for every partition } \mathcal{P} \text{ of } V.$$

Packing of k spanning hypertrees

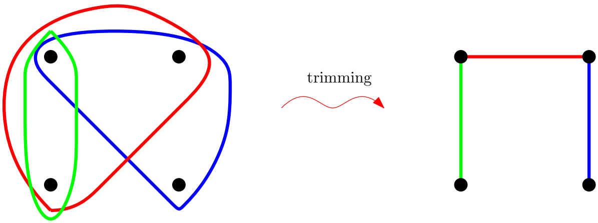
Theorem (Frank, Király, Kriesell 2003)

Let $\mathcal{G} = (V, \mathcal{E})$ *be a hypergraph and* $k \in \mathbb{Z}_+$.
There exists a packing of k *spanning hypertrees in* $\mathcal{G}$ $\Longleftrightarrow$

$$e_{\mathcal{E}}(\mathcal{P}) \quad \geq \quad k(|\mathcal{P}| - 1) \qquad \text{for every partition } \mathcal{P} \text{ of } V.$$

Definition

hypertree : a hypergraph that can be trimmed to a tree.



(f,g)-bounded packing of spanning arborescences

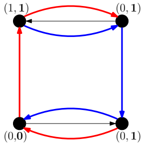
Theorem (Frank 1978, Cai 1983)

Let $D = (V, A)$ *be a digraph,* $f, g : V \to \mathbb{Z}_+$, $k \in \mathbb{Z}_+$.
There exists an (f,g)*-bounded packing of* k *spanning arborescences* $\iff$

1. $g(v) \geq f(v)$ *for every* $v \in V$,
2. $e_A(\mathcal{P}) \geq k|\mathcal{P}| - \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\}$ *for every subpart.* $\mathcal{P}$ *of* V.

Definition

(f,g)-bounded packing of arborescences :
number of v-arborescences in the packing is
at least $f(v)$ and at most $g(v)$ for every $v \in V$.



Matroid-based packing of rooted trees

Theorem (Katoh, Tanigawa 2013)

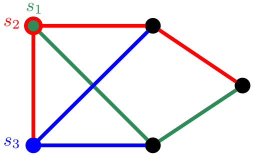
Let $G = (V, E)$ be a graph, S a multiset of V, $\mathrm{M} = (S, r_{\mathrm{M}})$ a matroid.

1. *There exists a complete M-based packing of rooted trees in G* $\Longleftrightarrow$
   - S_v *is independent in* M *for every* $v \in V$,
   - $e_E(\mathcal{P}) \geq \sum_{X \in \mathcal{P}}(r_{\mathrm{M}}(S) - r_{\mathrm{M}}(S_X))$ *for every partition* $\mathcal{P}$ *of* V.
2. $r_{KT}(F) = r_{\mathrm{M}}(S)|V| - |S| + \min_{\mathcal{P} \text{ part. of } V}\{e_F(\mathcal{P}) - \sum_{X \in \mathcal{P}}(r_{\mathrm{M}}(S) - r_{\mathrm{M}}(S_X))\}$

   is the rank function of a matroid M_{KT}.
3. F *is the edge set of a complete* M*-based packing of rooted trees* $\Longleftrightarrow$
   - F *is independent in* M_{KT},
   - $|F| = r_{\mathrm{M}}(S)|V| - |S|$.

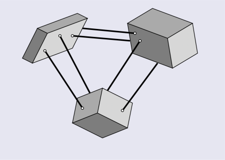
Definition

Matroid-based packing of rooted trees :
the root set of the rooted trees containing v
forms a basis of the matroid M for every $v \in V$.



Application : Rigidity

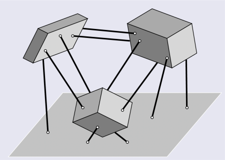
Body-Bar Framework



Theorem (Tay 1984)

"Rigidity" of a Body-Bar Framework can be characterized by the existence of a spanning tree decomposition.

Body-Bar Framework with Bar-Boundary



Theorem (Katoh, Tanigawa 2013)

"Rigidity" of a Body-Bar Framework with Bar-Boundary can be characterized by the existence of a complete matroid-based rooted tree decomposition.

Augmentation for M-based (f,g)-bounded (α,β)-limited packing of rooted hypertrees

Theorem (Hoppenot, Szigeti 2026)

Let $\mathcal{G}=(V,\mathcal{E})$ be a hypergraph, S a multiset of V, $\mathrm{M}=(S,r_{\mathrm{M}})$ a matroid, $\alpha,\beta,\gamma\in\mathbb{Z}_+$, $f,g:V\to\mathbb{Z}_+$. We can add γ edges to $\mathcal{G}$ to have M-based (f,g)-bounded (α,β)-limited packing of rooted hypertrees $\Longleftrightarrow$

$$
\begin{array}{rcll}
f(v) & \leq & \min\{r_{\mathrm{M}}(S_v),g(v)\} & \text{for every } v\in V,\\
\alpha & \leq & \min\{\beta,\sum_{v\in V}\min\{r_{\mathrm{M}}(S_v),g(v)\}\}, & \\
r_{\mathrm{M}}(S)-r_{\mathrm{M}}(S_Y) & \leq & \min\{\beta-f(Y),g(\overline{Y})\} & \text{for every } Y\subseteq V,\\
\gamma+g(V)+e_{\mathcal{E}}(\mathcal{P}) & \geq & \sum_{X\in\mathcal{P}}\max\{r_{\mathrm{M}}(S)-r_{\mathrm{M}}(S_Y)+g(Y):Y\subseteq X\}, & \\
\gamma+\beta+e_{\mathcal{E}}(\mathcal{P}) & \geq & \sum_{X\in\mathcal{P}}\max\{r_{\mathrm{M}}(S)-r_{\mathrm{M}}(S_Y)+f(Y):Y\subseteq X\} & \\
 & & & \text{for every partition } \mathcal{P} \text{ of } V.
\end{array}
$$

Definition

(α,β)-limited : number of roots is at least α and at most β.

Proof of necessity

Proof

Let $\mathcal{B}$ be such a packing with root set T after adding the edge set F.

$$
\begin{array}{rcccll}
f(v) & \leq & |T_v| & \leq & \min\{r_M(S_v), g(v)\} & \text{for every } v \in V, \\
\alpha & \leq & |T| & \leq & \min\{\beta, \sum_{v\in V} \min\{r_M(S_v), g(v)\}\}, & \\
r_M(S) - r_M(S_Y) & \leq & |T_{\overline{Y}}| & \leq & \min\{\beta - f(Y), g(\overline{Y})\} & \text{for every } Y \subseteq V,
\end{array}
$$

$$
\begin{array}{rcl}
\gamma + e_{\mathcal{E}}(\mathcal{P}) & \geq & e_{\mathcal{E}\cup F}(\mathcal{P}) \geq \sum_{X\in\mathcal{P}} (r_M(S) - r_M(T_X)) \\
& \geq & \sum_{X\in\mathcal{P}} \max\{r_M(S) - r_M(T_Y) - |T_{X\setminus Y}| : Y \subseteq X\} \\
\text{(by } g(v) \geq |T_v|\text{)} & \geq & \sum_{X\in\mathcal{P}} \max\{r_M(S) - r_M(S_Y) + g(Y) : Y \subseteq X\} - g(V), \\
\text{(by } |T_v| \geq f(v), \text{ and } \beta \geq |T|\text{)} & \geq & \sum_{X\in\mathcal{P}} \max\{r_M(S) - r_M(S_Y) + f(Y) : Y \subseteq X\} - \beta.
\end{array}
$$

Generalized partition / Katoh-Tanigawa matroids

Theorem

Let S be a multiset of V, $k \in \mathbb{Z}_+, f, g : V \to \mathbb{Z}_+$. Let

$$\mathcal{B} = \{T \subseteq S : f(v) \le |T_v| \le g(v) \text{ for every } v \in V, |T| = k\}.$$

Then $\mathcal{B}$ forms the set of bases of a matroid $\Longleftrightarrow$

$$f(v) \le \min\{|S_v|, g(v)\} \quad \text{for every } v \in V,$$
$$f(V) \le k \le \sum_{v \in V} \min\{|S_v|, g(v)\}.$$

Theorem (Hoppenot, Szigeti 2026)

Let $G = (V, E)$ be a graph, S a multiset of V, $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid.

1. $r'_{KT}(F \cup T) = r_{\mathsf{M}}(S)|V| + \min_{\mathcal{P} \text{ part. of } V} \{e_F(\mathcal{P}) - \sum_{X \in \mathcal{P}} (r_{\mathsf{M}}(S) - r_{\mathsf{M}}(T_X))\}$ *is the rank function of a matroid* M'_{KT}.
2. M*-based packing of rooted trees exists with edge set* F*, root set* T $\Longleftrightarrow$
   - $F \cup T$ *is independent in* M'_{KT},
   - $|F \cup T| = r_{\mathsf{M}}(S)|V|$.

Intersection and union of partitions

Definitions

- Uncrossing : For a family $\mathcal{F}$ of subsets of V, while $X, Y \in \mathcal{F}$, $X \cap Y, X \setminus Y, Y \setminus X \neq \emptyset$ replace X and Y by $X \cap Y$ and $X \cup Y$.
- For $\mathcal{P}_1, \mathcal{P}_2$ partitions of V, let $\mathcal{F} := \mathcal{P}_1 \cup \mathcal{P}_2$ (with multiplicities).
- By uncrossing $\mathcal{F}$ we obtain a laminar family $\mathcal{F}'$.
- The minimal and maximal sets in $\mathcal{F}'$ form partitions $\mathcal{P}'_1$ and $\mathcal{P}'_2$ of V.
- $\mathcal{P}_1 \sqcap \mathcal{P}_2 := \mathcal{P}'_1$ (depends on execution of uncrossing).
- $\mathcal{P}_1 \sqcup \mathcal{P}_2 := \mathcal{P}'_2$ (uniquely defined).

Definitions

Function p on partitions of V is supermodular if $\forall$ partitions $\mathcal{P}_1, \mathcal{P}_2$ of V,

$$p(\mathcal{P}_1) + p(\mathcal{P}_2) \quad \leq \quad p(\mathcal{P}_1 \sqcap \mathcal{P}_2) + p(\mathcal{P}_1 \sqcup \mathcal{P}_2).$$

Example : $p(\mathcal{P}) = \sum_{X \in \mathcal{P}} p^*(X)$, p^* intersecting supermod. set function.

Trimming to cover 2 supermodular functions on partitions

Theorem (Hoppenot, Szigeti 2026)

Let p_1, p_2 be supermodular functions on the partitions of V, $\mathcal{G} = (V, \mathcal{E})$ a hypergraph. Then $\mathcal{G}$ can be trimmed to a graph $G = (V, E)$ such that

$$e_E(\mathcal{P}) \geq \max\{p_1(\mathcal{P}), p_2(\mathcal{P})\} \qquad \text{for every partition } \mathcal{P} \text{ of } V \iff$$

$$e_{\mathcal{E}}(\mathcal{P}) \geq \max\{p_1(\mathcal{P}), p_2(\mathcal{P})\} \qquad \text{for every partition } \mathcal{P} \text{ of } V.$$

Remark

Let S be a multiset of V, $\beta \in \mathbb{Z}_+$, $f, g : V \to \mathbb{Z}_+$, $\mathrm{M} = (S, r_{\mathrm{M}})$ a matroid. Then $\hat{p}_1$ and $\hat{p}_2$ are supermodular on partitions of V.

$$\hat{p}_1(\mathcal{P}) = -g(V) + \sum_{X \in \mathcal{P}} \max\{r_{\mathrm{M}}(S) + g(Y) - r_{\mathrm{M}}(S_Y) : Y \subseteq X\},$$
$$\hat{p}_2(\mathcal{P}) = -\beta + \sum_{X \in \mathcal{P}} \max\{r_{\mathrm{M}}(S) + f(Y) - r_{\mathrm{M}}(S_Y) : Y \subseteq X\}.$$

Covering 2 supermodular functions on partitions

Theorem (Hoppenot, Szigeti 2026)

Let p_1, p_2 be supermodular functions on the partitions of a set V, $\gamma \in \mathbb{Z}_+$. There exists an edge set F on V of size γ such that

$$\begin{aligned} e_F(\mathcal{P}) &\geq \max\{p_1(\mathcal{P}), p_2(\mathcal{P})\} && \text{for every partition } \mathcal{P} \text{ of } V \iff \\ \gamma &\geq \max\{p_1(\mathcal{P}), p_2(\mathcal{P})\} && \text{for every partition } \mathcal{P} \text{ of } V, \\ 0 &\geq \max\{p_1(\{V\}), p_2(\{V\})\}. \end{aligned}$$

Remark

Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph, S multiset of V, $\beta \in \mathbb{Z}_+$, $f, g : V \to \mathbb{Z}_+$, $\mathrm{M} = (S, r_\mathrm{M})$ a matroid. Then p_1, p_2 are supermodular on partitions of V.

$$\begin{aligned} p_1(\mathcal{P}) &= \hat{p}_1(\mathcal{P}) - e_\mathcal{E}(\mathcal{P}), \\ p_2(\mathcal{P}) &= \hat{p}_2(\mathcal{P}) - e_\mathcal{E}(\mathcal{P}). \end{aligned}$$

Conclusion

Summary

1. Augmentation for matroid-based (f, g)-bounded (α, β)-limited packing of rooted hypertrees.
2. Submodular technique works not only for set functions and biset functions, but also for functions on partitions.

Remarks on directed counterparts

1. Complete matroid-based packing of arborescences (Durand de Gevigney, Nguyen, Szigeti 2013),
2. Augmentation for matroid-based (f, g)-bounded packing of hyperarborescences (Szigeti 2024+),
3. Matroid-based (α, β)-limited packing of hyperarborescences, (Hörsch, Peyrille, Szigeti 2026).
4. Open problem: Matroid-based (f, g)-bounded (α, β)-limited packing of arborescences.