# Third talk in Bonn

"There are two ways of being happy: We must either diminish our wants or **augment** our means - either may do - the result is the same and it is for each man to decide for himself and to do that which happens to be easier." Benjamin Franklin

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**Story for augmentation** :

A telecommunication network company possesses some transmission antennas that are connected by communication channels. Between two antennas there may be none, one or more channels, each channel being assigned a separate frequency bandwidth.

This telecommunication network is considered to be functional if any two antennas can communicate via the channels, eventually through other antennas of the network.

For technical reasons some channels may be out of service. The network should be functional after these failures. Unfortunately, today this is not the case, the company must hence augment the reliability of the network by assigning new frequencies, and consequently by creating new channels between antennas. Frequency assignment has no real cost, so the number of assignments is to be minimized. (1️⃣)

The antennas are not of the same importance, for different pairs of antennas the network should tolerate different numbers of failures. Can the company augment the reliability of the network optimally under these conditions? (2️⃣)

The company has recently understood that they can use satellites for communications between given number of antennas. How can they solve the above problems? (3️⃣, 4️⃣)

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**Problems in the story** :

adding edges :

1️⃣ Global edge-connectivity augmentation,

2️⃣ Local edge-connectivity augmentation,

adding stars :

3️⃣ Global edge-connectivity augmentation,

4️⃣ Local edge-connectivity augmentation.

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**Global edge-connectivity augmentation :**

0️⃣ **Problem** : Given a graph G=(V,E) and k∊Z_+, what is the minimum `number` **OPT** of new edges whose addition results in a k-edge-connected graph?

**Remark** : Min cost k-edge-connected augmentation is NP-complete :

c(e)=1 for all non edges of a graph G and 0 for all edges,

min cost augmentation of empty graph is 0 iff G has a hamiltonian cycle.

**Remark** : min cost subgraph problem solvable ⇒ augmentation problem trivial

**Example** : Minimum rooted k-arc-connectivity augmentation is polynomial, because min cost rooted k-arc-connected subgraph (min cost matroid intersection) is polynomial.

1. **Lower bound** : **deficiency** of a subpartition is the sum of the deficiencies $k-d(X_i)$ of the $X_i$'s,

   $Opt \geq \lceil$ half of the maximum deficiency of a subpartition of $V \rceil$ .

2. Theorem of Watanabe-Nakamura :

   $Opt = \lceil$ half of the maximum deficiency of a subpartition of $V \rceil$ .

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"Every **extension** of knowledge arises from making the conscious the unconscious." Friedrich Nietzsche

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3. **Algorithm** of Frank :
   1. **Minimal extension** :
      1. Add a new vertex s to G and connect it to each vertex of G by k edges.
         The resulting graph is k-edge-connected in V.
      2. Delete as many new edges as possible preserving k-edge-connect. in V to get $G'=(V,E\cup F')$.
      3. If $d_{G'}(s)$ is odd, then add an arbitrary new edge incident to s obtaining $G''=(V+s,E\cup F'')$
         that is k-edge-connected in V with $d_{G''}(s)$=even.
   2. **Complete splitting off** :
      Execute a complete splitting of to obtain a k-edge-connected graph $G^*=(V,E\cup F)$ .

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4. **Optimality** :
   1. Each edge $e\in F'$ enters a maximal tight set, that is, $d_G(X_e)+d_{F'}(X_e)=k$
      (by minimality of the extension).
   2. These sets provide a subpartition $\{X_1, \dots, X_l\}$ of V if $\sum|X_i|$ minimal (by uncrossing lemma).
      We also need : $d(X)+d(Y)=d(X-Y)+d(Y-X)+2d(X\cap Y,V-(X\cup Y))$.
   3. $Opt \leq |F| = \lceil |F'|/2 \rceil = \lceil 1/2 \sum d_{F'}(X_i) \rceil = \lceil 1/2\sum(k-d_G(X_i)) \rceil \leq \alpha \leq Opt$.

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**Minimal extension** :

Theorem of Frank :

1. True for global arc-connectivity. Proof
2. True for symmetric skew-supermodular functions in graphs. Proof

   **Exercise** (Z.Király) : Prove that covering symmetric skew-supermodular functions by edges is NP-complete (by 3D matching).

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**Splitting off theorems**:

1. Lovász : Global undirected : d(s) even, $k\geq 2$.

   Bang-Jensen, Gabow, Jordán, Szigeti : Global undirected :
   d(s)/2(-1) admissible pairs for `every` edge, $k\geq 2$.

2. Mader : Local undirected : $d(s)\neq 3$, no cut edge incident to s.

   Szigeti : Local undirected : d(s)/3 admissible pairs for `some` edge.

3. Mader : Global directed : $\rho_{\vec{G}}(s)=\delta_{\vec{G}}(s)$, $k\geq 1$.
4. Local directed : no result is known.

"He who would be serene and pure needs but one thing, **detachment**." Meister Eckhart

**Detachment theorems** :

5 B. Fleiner : Detachment global undirected ⇔

$\sum d_i = d_G(s)$, $k \geq 2$, $d_i \geq 2$, G-s is k-$\sum \lfloor d_i/2 \rfloor$ -edge-connected.

6 Jordán-Szigeti : Detachment local undirected ⇔

$\sum d_i = d_G(s)$, $r \geq 2$, $d_i \geq 2$, $\lambda_{G-s}(u,v) \geq r(u,v) - \sum \lfloor d_i/2 \rfloor$ $\forall u,v \in V$.

7 Nash-Williams : Detachment global undirected in whole graph, $k \geq 2$ ⇔

$\sum d^v_i = d_G(v)$, $d^v_i \geq k$ $\forall v \in V$,

except : 1 k=odd, cut vertex v, $d^v_1 = d^v_2 = k$, no loop on v,

2 k=odd, $|V|=2$, $d^v_1 = d^v_2 = k$, no loop on v $\forall v \in V$.

8 Berg-Jackson-Jordán : Detachment global directed in whole digraph ⇔

$\sum f^v_i = \rho_{\vec{G}}(v)$, $\sum g^v_i = \delta_{\vec{G}}(v)$, $f^v_i$, $g^v_i \geq k$ $\forall v \in V$.

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**Augmentation results** :

1 Watanabe-Nakamura : OPT = ⌈half of the maximum k-deficiency of a subpartition of V⌉ ($k \geq 2$).

2 Frank : OPT = ⌈half of the maximum R-deficiency of a subpartition of V⌉ ($r \geq 2$).

3 Frank : OPT = maximum of in-k-deficiency, out-k-deficiency of a subpartition of V ($k \geq 1$).

4 **Exercise** : Prove that the local arc-connectivity augmentation is NP-complete (by set covering) :

T subset of V-s, r(s,v)=1 if v in T and 0 otherwise.

5 B. Fleiner : ($k \geq 2$, $d_i \geq 2$) ⇔

1 $\sum d_i \geq$ maximum k-deficiency of a subpartition of V,

2 G is k-$\sum \lfloor d_i/2 \rfloor$ -edge-connected.

6 Jordán-Szigeti : ($r \geq 2$, $d_i \geq 2$) ⇔

1 $\sum d_i \geq$ maximum R-deficiency of a subpartition of V,

2 $\lambda_G(u,v) \geq r(u,v) - \sum \lfloor d_i/2 \rfloor$ $\forall u,v \in V$.

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**Vertex-connectivity augmentation** :

1 **directed** graphs :

1 Frank-Jordán : min-max theorem,

2 Frank-Végh : polynomial by one.

2 **undirected** graphs :

1 Jackson-Jordán : polynomial for fixed k,

2 Végh : polynomial by one.

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**Augmentation in Hypergraphs** :

1 Bang-Jensen, Jackson : Global undirected

OPT = maximum of ⌈half of the k-deficiency of a subpartition of V⌉ ,

number of connected components -1 after deleting at most k-1 hyperedges.

T. Király : Global undirected, adding hyperdges of size t

2. Cosh-Jackson-Z. Király : Local undirected, adding edges (NP-complete)
3. Szigeti : Local undirected, adding hyperedges of minimum total size

   OPT = maximum R-deficiency of a subpartition of V.

4. Berg-Jackson-Jordán : Global directed, (adding dyperarcs of size most t)

   OPT = maximum of in-k-deficiency, ⌈out-k-deficiency(/(t-1))⌉ of a subpartition of V.

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**Augmentation with partition constraints** :

1. Bang-Jensen, Gabow, Jordán, Szigeti : Graph

   OPT = maximum of ⌈half of the k-deficiency of a subpartition of V⌉ and

   k-deficiency of a subpartition of $P_i \in \mathcal{P}$ if k=even

   this lower bound or plus 1 (can be characterized) if k=odd. (example : $C_4$ and $C_6$, k=3).

2. Gabow-Jordán : Bipartite Digraphs

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**Augmentation in Hypergaphs with partition constraints** :

1. Cosh : Bipartite Hypergraph
2. Bernáth-Grappe-Szigeti : Hypergraph

   3 lower bounds, 2 configurations

**Remark** : Bernáth-Grappe-Szigeti : symmetric crossing supermodular function

   3 lower bounds, 3 configurations

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**OPEN problems** :

1. Local edge-connectitivity augmentation in graphs with partition constraints

   = splitting preserving Local edge-connectivity with partition constraints

   (Frank : that would imply Detachment preserving local edge-connectivity)