Second talk in Bonn

Orientations with given in-degree :

Remark : The in-degree vector characterizes the arc-connectivity properties.

1. If m is the **in-degree vector** of $\vec{G}$ ($m(v)=\rho_{\vec{G}}(v)$ $\forall$ $v \in V$), then $m(X)-i_{\vec{G}}(X)=\rho_{\vec{G}}(X)$.
2. The in-degree vector characterizes the in-degree function.
3. The in-degree function characterizes the connectivity properties.

Theorem of Hakimi

Given an undirected graph $G=(V,E)$ and a vector $m: V \to Z_+$,

there exists an orientation $\vec{G}$ of G with in-degree vector m $\Leftrightarrow$

1. $m(X) \geq i_G(X)$ $\forall X \subseteq V$,
2. $m(V) = |E|$.

Proof :

0. Take an arbitrary orientation $\vec{G}$ of G.
1. If $\rho_{\vec{G}}(v) \leq m(v)$ $\forall$ v, then it is an m-orientation, Stop.
   (Indeed, $|A|=\sum_{v \in V}\rho_{\vec{G}}(v) \leq \sum_{v \in V} m(v)=m(V)=|E|=|A|$.)
2. Otherwise, take a **big** vertex v : $\rho_{\vec{G}}(v)>m(v)$.
3. Let X be the set of vertices u from which there exists a path P_u to v.
4. Take a **small** vertex $u \in X$: $\rho_{\vec{G}}(u)<m(u)$.
   (It exists : $\sum_{x \in X} m(x)=m(X) \geq i_G(X)=i_G(X)+\rho_{\vec{G}}(X)=\sum_{x \in X}\rho_{\vec{G}}(x)$.)
5. Let $\vec{G}'$ be obtained from $\vec{G}$ by reorienting P_u. Go to 1.
   (It is better : $\sum_{w \in V} |\rho_{\vec{G}'}(w)-m(w)| = \sum_{w \in V} |\rho_{\vec{G}}(w)-m(w)|-2$.)
6. This algorithm finds an m-orientation in polynomial time.
   ($0 \leq \sum_{w \in V} |\rho_{\vec{G}}(w) - m(w)| \leq 2|E|$.)

Exercice : Prove Hakimi's theorem by uncrossing technique.

Applications :

1. Eulerian orientation of an undirected graph : [$m(v)= d_G(v)/2$ $\forall$ $v \in V$],
2. Eulerian orientation of a mixed graph : [$m(v)=(d_E(v)+\delta_A(v)+\rho_A(v))/2-\rho_A(v)$ $\forall$ $v \in V$],
3. Perfect matching in a bipartite graph : [$m(u)=1$ $\forall u \in U$, $m(w)=d(w)-1$ $\forall u \in W$],
4. f-factor in a bipartite graph : [$m(u)=f(u)$ $\forall u \in U$, $m(w)=d(w)-f(w)$ $\forall u \in W$].

Exercice : Derive from Hakimi's theorem the corresponding theorems.

1. **Theorem** (Euler) : There exists an *Eulerian orientation* of G $\Leftrightarrow$ $d_G(v)$ is even $\forall$ $v \in V$.
2. **Theorem** (Ford-Fulkerson) : There exists an *Eulerian orientation* of a *mixed* graph $(V, E \cup A)$ $\Leftrightarrow$
   1. $d_E(v)+\delta_A(v)+\rho_A(v)$ is even $\forall$ $v \in V$,

   2. $\rho_A(X)-\delta_A(X)\leq d_E(X)\ \forall\ X\subseteq V.$

3. **Theorem** (Hall, Frobenius) : There exists a *perfect matching* in a *bipartite* graph (U,W;E) ⇔
   1. $\Gamma(X)\geq|X|\ \forall X\subseteq W,$
   2. $|U|=|W|.$
4. **Theorem** (Ore) : There exists an *f-factor* in a *bipartite* graph (U,W;E) ⇔

   $i_E(X)\geq f(X)-f(U\cup W)/2\ \forall\ X\subseteq U\cup W.$

Theorem (Frank) :

Given an undirected graph G=(V,E) and a vector $m: V \to Z+$ with $m(V)=|E|$,

there exists an orientation $\vec{G}$ of G with in-degree vector m that is

1. *root-connected* ($\rho_{\vec{G}}(X) \geq 1\ \forall X\subset V-s$, s fixed, $\rho_{\vec{G}}(V)=0$) ⇔ $m(X)-i_G(X) \geq 1\ \forall\ X\subset V-s.$
2. *k-root-connected* ($\rho_{\vec{G}}(X) \geq k\ \forall X\subset V-s$, s fixed, $\rho_{\vec{G}}(V)=0$) ⇔ $m(X)-i_G(X) \geq k\ \forall\ X\subset V-s.$
3. *strongly connected* ($\rho_{\vec{G}}(X) \geq 1\ \forall X\subset V$, $\rho_{\vec{G}}(V)=0$) ⇔ $m(X)-i_G(X) \geq 1\ \forall\ X\subset V.$
4. *k-arc-connected* ($\rho_{\vec{G}}(X) \geq k\ \forall X\subset V$, $\rho_{\vec{G}}(V)=0$) ⇔ $m(X)-i_G(X) \geq k\ \forall\ X\subset V.$

“The univers is so **well-balanced** that the mere fact that you have a problem also serves as a sign that there is a solution.” Steve Maraboli

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Well-balanced orientation :

Exercise :

An orientation $\vec{G}$ of an eulerian graph G is *eulerian* ⇔ $\lambda_{\vec{G}}(u, v) = 1/2\lambda_G(u,v)\ \forall(u,v)\in V\times V.$ Proof

Definition :

Well-balanced orientation $\vec{G}$ of G : $\lambda_{\vec{G}}(u, v) \geq \lfloor 1/2\lambda_G(u, v)\rfloor\ \forall(u,v)\in V\times V.$

Exercise : $\vec{G}$ is well-balanced ⇔ $\rho_{\vec{G}}(X)-\delta_{\vec{G}}(X) \leq d_G(X)-2\lfloor R_G(X)/2\rfloor\ \forall X \subseteq V.$ Proof

1. Well-balanced orientation **Theorem** of Nash-Williams :

   Every graph G admits a well-balanced orientation.

   Remark : Strong orientation implies weak orientation.

   Proof : $\lambda_{\vec{G}}(u, v) \geq \lfloor 1/2\lambda_G(u, v)\rfloor \geq \lfloor 1/2\times 2k\rfloor = k\ \forall(u,v)\in V\times V.$
2. Smooth well-balanced orientation **Theorem** of Nash-Williams :

   There `exists` a **pairing** M of **T_G** and there `exists` an eulerian orientation $\vec{G}+\vec{M}$ of G+M such that $\vec{G}$ is well-balanced.
3. Strong Pairing **Theorem** of Nash-Williams :

   There `exists` a **pairing** M of **T_G** such that for `every` eulerian orientation $\vec{G}+\vec{M}$ of G+M such that $\vec{G}$ is well-balanced.
4. Feasible Pairing **Theorem** of Nash-Williams :

   There `exists` a **pairing** M of **T_G** such that
   1. for every orientation of M, there exists an eulerian orientation of G+M,
   2. for `every` eulerian orientation $\vec{G}+\vec{M}$ of G+M, $\vec{G}$ is well-balanced.

Exercise : Prove it is equivalent to $d_M(X) \le d_G(X) - 2\lfloor R_G(X)/2 \rfloor$ $\forall X \subseteq V$.

Applications of the pairing theorem :

5 **Theorem** of Király-Szigeti : For `every` **pairing** M of **T_G,**

there `exists` an eulerian orientation $\vec{G}$ +→M of G+M such that $\vec{G}$ is well-balanced.

6 **Theorem** of Király-Szigeti : Every Eulerian graph G has an Eulerian orientation $\vec{G}$

such that $\vec{G}-v$ is a well-balanced orientation of G−v for all v∈V.

Corollary : An Eulerian graph G has an Eulerian orientation $\vec{G}$ so that $\vec{G}-v$ is

k-arc-connected ∀v∈V ⇔ G−v is 2k-edge-connected ∀v∈V.

Corollary (Berg-Jordán) : An Eulerian graph G has a 2-vertex-connected

Eulerian orientation ⇔ G−v is 2-edge-connected ∀v∈V.

7 Subgraph **Theorem** of Nash-Williams :

For every subgraph H of G, ∃ $\vec{G}$: $\vec{G}$ and $\vec{G}(H)$ are well-balanced.

8 Edge-partition **Theorem** of Király-Szigeti :

For every partition $\{E_1, \ldots, E_k\}$ of E(G), there exists $\vec{G}$ of G :

$\vec{G}$ and $\vec{G}(E_i)$ ∀i are well-balanced orientations of the corresponding graphs.

9 Vertex-partition **Theorem** of Király-Szigeti :

For every partition $\{V_1, \ldots, V_k\}$ of V(G), there exists $\vec{G}$ of G :

$\vec{G}$ and $\vec{G}/(V-V_i)$ ∀i are well-balanced orientations of the corresponding graphs.

Exercices : Prove 4 implies 5, 6, 7, 8, 9.

Exercices : Prove : pairing theorem for global edge-connectivity is easy. (by Lovasz and Mader)

Weighted problems :

1 Theorem : Minimum weight in-degree-constrained orientation problem can be solved in polynomial time. (min-cost flow)

2 Theorem of Edmonds : Minimum weight k-rooted-connected orientation problem can be solved in polynomial time. (submodular flow)

3 Theorem of Frank : Minimum weight k-arc-connected orientation problem can be solved in polynomial time. (submodular flow)

4 Theorem of Bernáth, Iwata, T. Király, Z. Király, Szigeti :

Minimum weight well-balanced orientation problem is NP-complete.

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k-vertex-connected orientation :

1 Theorem of Thomassen : G has a 2-vertex-connected orientation ⇔

G is 4-edge-coneted and G−v is 2-edge-connected ∀v∈V.

2 Theorem of Durand de Gevigney : k-vertex-connected orientation problem is NP-complete for k≥3.
