First talk in Bonn

"I was thinking that we all learn by experience, but some of us have to go to **summer school.**"
Peter De Vries

Theme :

Submodular Functions

Connectivity Problems

Framework :

1. Graphs : Undirected, Directed
2. Hypergraphs : Undirected, Directed
3. Set functions : Submodular, Supermodular (intersecting, crossing, skew)

Topics :

1. **Touching** : 1 Orientation 2 Augmentation
2. **little bit touching** : 1 vertex-connectivity, 2 weighted versions (usually NP-complete)
3. **Not touching** : 1 Edge-disjoint paths 2 spanning subgraph 3 packing problems,

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Reminders :

Exemple :

Undirected graph **$G=(V,E)$,**

Directed graph **$\vec{G}=(V,A)$.**

Notation :

Undirected cut : **$\delta_G(X)$** = # of edges leaving X in G,

Directed cut : **$\delta_{\vec{G}}(X)$** = # of arcs leaving X in $\vec{G}$, **$\rho_{\vec{G}}(X)$** = # of arcs entering X in $\vec{G}$.

Definition :

Undirected local edge-connectivity : **$\lambda_G(u,v)$** = maximum # of edge-disjoint paths from u to v in G,

Directed local arc-connectivity : **$\lambda_{\vec{G}}(u,v)$** = maximum # of arc-disjoint paths from u to v in $\vec{G}$.

Results :

Undirected Menger : $\lambda_G(u,v) \geq k \Leftrightarrow$ if $\delta_G(X) \geq k$ for all $v \in X \subseteq V-u$.

Directed Menger : $\lambda_{\vec{G}}(u,v) \geq k \Leftrightarrow \rho_{\vec{G}}(X) \geq k$ for all $v \in X \subseteq V-u$.

Remark :

Undirected version follows from directed version (by replacing the edges by opposite arcs)

Directed version follows easily from flows ($c \equiv 1$: k paths $\Leftrightarrow$ flow of value k)

Definition :

Undirected global edge-connectivity : G is k-edge-connected $\Leftrightarrow$

1 if $\lambda_G(u,v) \geq k$ $\forall u,v \in V$, 2 $\delta_G(X) \geq k$ $\forall \emptyset \neq X \subset V$, 3 G-F is connected $\forall F \subseteq E$, $|F|<k$.

Directed global arc-connectivity : $\vec{G}$ is k-arc-connected $\Leftrightarrow$

1 if $\lambda_{\vec{G}}(u,v) \geq k$ $\forall (u,v) \in V \times V$, 2 $\rho_{\vec{G}}(X) \geq k$ $\forall \emptyset \neq X \subset V$, 3 $\vec{G}$-F is strongly-connected $\forall F \subseteq A$, $|F|<k$.

Definition :

Undirected vertex-connectivity : G-X is connected for all $X \subseteq V$, $|F|<k$.

Directed vertex-connectivity : $\vec{G}$-X is strongly-connected for all $X \subseteq V$, $|F|<k$.

"**Method** is much, technique is much, but inspiration is even more." Benjamin Cardozo

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Methods : 1 **Uncrossing,** 2 **Splitting off.**

Definition : Function $b: 2^V \to Z$ ($p: 2^V \to Z$) is

1 *Submodular* : if $b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y)$ for all $X,Y \subseteq V$,

Exemple : 1 degree functions : δ_G, $\delta_{\vec{G}}$, $\rho_{\vec{G}}$, 2 rank function of a matroid.

Remark : Useful inequalities about degree functions :

1 $d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X-Y,Y-X)$

2 $d(X) + d(Y) = d(X-Y) + d(Y-X) + 2d(X \cap Y, V-(X \cup Y))$

3 $d(A)+d(B)+d(C) \geq d(A \cap B \cap C)+d(A-(B \cup C))+d(B-(A \cup C))+d(C-(A \cup B))$
$+2d(A \cap B \cap C, V-(A \cup B \cup C))$

4 $\rho(X) + \rho(Y) = \rho(X \cap Y) + \rho(X \cup Y) + d(X-Y,Y-X)$

5 $\rho(X)+\rho(Y)=\rho(X-Y)+\rho(Y-X)+d(V-(X \cup Y),X \cap Y) +\rho(X \cap Y)-\delta(X \cap Y)$

6 $\sum_{v \in X}(\rho(v)-\delta(v)) = \rho(X)-\delta(X)$

2 *Supermodular* : if $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ for all $X,Y \subseteq V$,

Exemple : 1 **$i_G(X)$** = number of edges in X.

3 *Intersecting supermodular* : Supermodular on intersecting sets ($X \cap Y \neq \emptyset$),

4 *Crossing supermodular* : Supermodular on crossing sets ($X \cap Y$, $X-Y$, $Y-X$, $V-(X \cup Y) \neq \emptyset$).

"**Uncrossing** can handle just about any situation you will come across in your Magical life."

Definition :

1 G *covers* a function $p: 2^V \to Z$ if $d_G(X) \geq p(X)$ for all $X \subseteq V$.

2 X is *tight* if $d_G(X) = p(X)$.

1 **Uncrossing Lemma** : If G covers a crossing supermodular function p then intersection and union of crossing tight sets are tight.

Proof : $p(X) + p(Y) = d(X) + d(Y) \geq d(X \cap Y) + d(X \cup Y) \geq p(X \cap Y) + p(X \cup Y) \geq p(X) + p(Y)$. □

Application (Theorem of Mader) : A minimally k-edge-connected graph has a vertex of degree k.

Proof : 0 G covers $p(X)=\{k$ if $\emptyset \neq X \subset V$, 0 otherwise$\}$; p is crossing supermodular.

1 Each edge belongs to a tight cut (by minimality of G).

2 Let $\delta_G(S)$ be a minimal tight cut. Suppose that S is not a vertex.

3 There exists an edge e in S (by minimality of S).

4 Let $\delta\_G(T)$ be a tight cut containing e.

5 Note that S and T are crossing (by minimality of S).

6 Then $S \cap T$ is a smaller tight cut, ↯. □

Exercise (Theorem of Mader) : A minimally k-arc-connected digraph contains a vertex of in- and out-degree k.

Proof by uncrossing but much more complicated.

OPEN problem : Minimally k-vertex-connected digraph contains a vertex of in- and out-degree k. (Mader) (for k=2 it is proved by Mader)

"When I **split** an infinitive, God damn it, I split it so it will stay split." Raymond Chandler

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Definition :

Splitting off at s : $G=(V+s,E)$, su, sv∈E. **G_uv**:=(V+s, E-su-sv+uv).

Complete Splitting off at s : Executing a sequence of splitting off and deleting the vertex s when its degree becomes 0.

2 **Splitting Theorem** (Lovász) : If G=(V+s,E) is k-edge-connected (k≥2) and d(s) is even, then ∃ a complete splitting off at s that preserves k-edge-connectivity.

Proof : 1 We show that for every su there exists sv so that G_uv is k-edge-connected in V.

2 If not, there exists a *dangerous* set ($d(X) \le k+1$) containing u and v.

3 Let M be a minimal set of such dangerous sets containing all the neighbors of s.

4 Any set X of M contains at most d(s)/2 neighbors of s.

($k+1 \ge d(X)=d(V-X)-d(s,V-X)+d(s,X) \ge k-d(s)+2d(s,X)$.)

5 For A,B,C in M, $3(k+1) \ge d(A)+d(B)+d(C) \ge d(A \cap B \cap C)+d(A-(B \cup C))+d(B-(A \cup C))$ $+d(C-(A \cup B))+2d(A \cap B \cap C, V+s-(A \cup B \cup C)) \ge k+k+k+k+2$, ↯.□

Application :

1 Theorem of Lovász : constructive characterization of 2k-e-c graphs, starting from K_2^2k, by 1 adding edges, 2 pinching k edges.

Proof : G must be reduced to K_2^2k via 2k-e-c graphs by the inverse operations :

1 deleting an edge and 2 splitting of at a vertex of degree 2k.

This can be done by Mader's and by Lovász' theorems. □

Exercice : Prove Mader's directed splitting off theorem.

Exercice : Theorem of Mader : constructive characterization of k-arc-connected digraphs, starting from K_2^(k,k) by 1 adding arcs, 2 pinching k arcs.

OPEN problem : Construction of 2-vertex-connected digraphs.

2 Weak Orientation theorem of Nash-Williams : G has a k-arc-connected orientation $\vec{G}$ if and only if G is 2k-edge-connected.

Proof : Necessity : For all $\emptyset \ne X \subset V$, $d\_G(X)=\rho\_\vec{G}(X)+\delta\_\vec{G}(X) \ge k+k$.

Sufficiency : By constructive characterization.

0 K_2^2k has trivially a k-arc-connected orientation.

1️⃣ If an edge is added then it can be oriented in any direction.

2️⃣ If a pinching is executed then the natural orientation is OK.□

3️⃣ Augmentation (see later).

"Every **orientation** presupposes a disorientation." Hans Magnus Enzensberger

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Story for orientation :

Mr. Orient, the Mayor of the city called ”The Edges”,
having wanted to make the main street a one way street,
unfortunately made a mistake by ordering the ONE WAY sign
and received 100 signs, as many as the number of streets in the city.
To be justified, he decides to use all the signs, i.e. to make all the streets of the city one way (0️⃣).

Having finished his plan, he realizes that it does not enable him to go home.
He thus goes back to work while keeping in mind that he must be able,
from the City Hall, to reach any point of the city (1️⃣).
After one moment of reflexion, he realizes that he must be able,
from any point of the city, to reach all the others (3️⃣).

Being proud of himself, he presents his project to his assistant, a well-balanced man, who reminds him that during summer,
some streets of the city may be blocked by floods, they thus try to conceive a plan where
blocking any street does not make a district inaccessible (4️⃣).

But they are still not satisfied ; examining their plan, they see that
there are far too many paths from the downtown to the shopping center
and not enough in the other direction.
They try an ultimate improvement : to place the ”one way” signs so that
the orientation of the streets be well-balanced (6️⃣).

Since then, the city was renamed ”The Arcs”.

Problems in the story :

0️⃣ Orientation, 1️⃣ rooted-connected, (2️⃣ k-rooted-connected,) 3️⃣ strongly-connected,
4️⃣ 2-arc-connected, (5️⃣ k-arc-connected,) 6️⃣ well-balanced.

1️⃣ **Rooted-connected orientation** :

Given an undirected graph G and a vertex s of G,

there exists a *root-connected* orientation of G at s ⇔
there exists an orientation of G containing an s-arborescence ⇔
there exists a spanning tree of G ⇔
G is *connected*.

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2 **k-rooted-connected orientation** : (Frank)

Given an undirected graph G, a vertex s of G and an integer $k \geq 1$,

there exists $\vec{G}$ of G that is *k-root-connected* at s ⇔ (Menger)

there exists $\vec{G}$ of G with $\rho_{\vec{G}}(X) \geq k \ \forall X \subseteq V-s$ ⇔ (Edmonds)

there exists $\vec{G}$ of G containing k arc-disjoint s-arborescences ⇔

there exist k edge-disjoint spanning trees of G ⇔ (Nash-Williams)

G is ***k-partition-connected*** for every partition $\mathcal{P}$ of V, $|E(\mathcal{P})| \geq k(|\mathcal{P}| - 1)$.

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3 **strongly-connected orientation** (Robbins)

Given an undirected graph G,

there exists a *strongly-connected* orientation of G ⇔

there is an orientation of G having a directed ear-decomposition ⇔

there exists an ear-decomposition of G ⇔

G is *2-edge-connected*.

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4 **2-arc-connected orientation**, 5 **k-arc-connected orientation**, already seen

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6 **well-balanced orientation**, see later.
