Packing forests

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Prior work

Packing trees

1. Nash-Williams (1961),Tutte (1961),
2. Peng, Chen, Koh (1991),
3. Frank, T. Király, Kriesell (2003),
4. Katoh, Tanigawa (2013),
5. Hoppenot, Martin, Szigeti (2023+)

Packing arborescences

1. Edmonds (1973), Lovász (1976), Frank (1978),
2. Frank, T. Király, Z. Király (2003), Bérczi, Frank (2008), Frank (2009), Kamiyama, Katoh, Takizawa (2009),
3. Bérczi, Frank (2010, 2018), Durand de Gevigney, Nguyen, Szigeti (2013), Cs. Király (2016), Fortier, Cs. Király, Léonard, Szigeti, Talon (2018), Matsuoka, Szigeti (2019),
4. Fortier, Cs. Király, Szigeti, Tanigawa (2020), Cs. Király, Szigeti, Tanigawa (2020), Gao, Yang (2021, 2021), Hörsch, Szigeti (2021, 2022)

Packing spanning trees

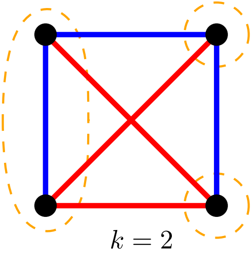
Theorem 1 (Nash-Williams (1961), Tutte (1961))

There exists in a graph $G = (V, E)$ a packing of k spanning trees $\iff$

$$e_E(\mathcal{P}) \geq k(|\mathcal{P}| - 1) \quad \forall \text{ partition } \mathcal{P} \text{ of } V.$$

Definition

$e_E(\mathcal{P})$: number of edges between the members of a partition $\mathcal{P}$ of V.



Packing spanning forests of given size

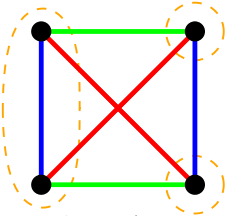
Theorem 2 (Peng, Chen, Koh (1991))

There exists in a graph $G = (V, E)$ a packing of k spanning forests each with ℓ connected components $\iff$

$$\begin{aligned} |V| &\geq \ell, \\ e_E(\mathcal{P}) &\geq k(|\mathcal{P}| - \ell) \quad \forall \text{ partition } \mathcal{P} \text{ of } V. \end{aligned}$$

Remark

For $\ell = 1$, Theorem 2 reduces to Theorem 1.



Packing spanning forests of given sizes

Theorem 3

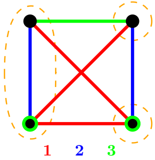
There exists in a graph $G = (V, E)$ a packing of k spanning forests with $\ell(1), \dots, \ell(k)$ connected components $\iff$

$$\begin{aligned} |V| &\geq \ell(i) && \forall\, i, \\ \sum_{1 \leq i \leq k} \min\{\ell(i), |\mathcal{P}|\} + e_E(\mathcal{P}) &\geq k|\mathcal{P}| && \forall \text{ partition } \mathcal{P} \text{ of } V, \end{aligned}$$

or equivalently $$e_E(\mathcal{P}) \geq \sum_{1 \leq i \leq k} \max\{0, |\mathcal{P}| - \ell(i)\}.$$

Remarks

1. For $\ell(i) = \ell\ \forall i$, Theorem 3 reduces to Theorem 2.
2. Theorem 3 can easily be proved using matroid theory.
3. We did not find Theorem 3 in the literature.



Packing spanning branchings of given sizes

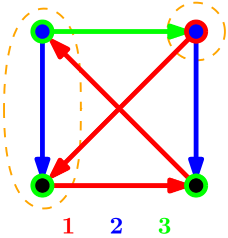
Theorem 4 (Bérczi, Frank (2018))

There exists in a digraph $D = (V, A)$ a packing of k spanning branchings with $\ell(1), \dots, \ell(k)$ connected components $\iff$

$$\begin{aligned} |V| &\geq \ell(i) && \forall\, i, \\ \sum_{1 \leq i \leq k} \min\{\ell(i), |\mathcal{P}|\} + e_A(\mathcal{P}) &\geq k|\mathcal{P}| && \forall \text{ subpartition } \mathcal{P} \text{ of } V. \end{aligned}$$

Definitions

1. Branching : each connected component is an arborescence,
2. $e_A(\mathcal{P})$: number of arcs entering a member of a subpartition $\mathcal{P}$ of V.



Packing spanning branchings with bounded sizes

Theorem 5 (Bérczi, Frank (2018))

There exists in a digraph $D = (V, A)$ an (ℓ, ℓ')-bordered (α, β)-limited packing of k spanning branchings $\iff$

$$\min\{\sum_{1\leq i\leq k} \ell'(i), \beta\} \geq \alpha, \qquad \min\{|V|, \ell'(i)\} \geq \ell(i) \quad \forall\ i,$$

$$\beta + \sum_{1\leq i\leq k} \min\{0, |\mathcal{P}| - \ell(i)\} + e_A(\mathcal{P}) \quad \geq \quad k|\mathcal{P}| \quad \forall \text{ subpartition } \mathcal{P} \text{ of } V,$$

$$\sum_{1\leq i\leq k} \min\{\ell'(i), |\mathcal{P}|\} + e_A(\mathcal{P}) \quad \geq \quad k|\mathcal{P}| \quad \forall \text{ subpartition } \mathcal{P} \text{ of } V.$$

Definitions

1. (ℓ, ℓ')-bordered : $\ell(i) \leq \text{conn.comp}(B_i) \leq \ell'(i)$ $\forall$ branching B_i,
2. (α, β)-limited : $\alpha \leq \sum_{i=1}^{k} \text{conn.comp}(B_i) \leq \beta$.

Packing spanning branchings with bounded sizes

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Remark

For $\ell(i) = \ell'(i)\ \forall i$, $\alpha = \beta = \sum_{i=1}^{k} \ell(i)$, Theorem 5 reduces to Theorem 4.

Packing spanning forests with bounded sizes

Theorem 6 (Hoppenot, Martin, Szigeti (2023+))

There exists in a graph $G = (V, E)$ an (ℓ, ℓ')-bordered (α, β)-limited packing of k spanning forests $\iff$

$$\min\{\sum_{1\leq i\leq k} \ell'(i), \beta\} \geq \alpha, \qquad \min\{|V|, \ell'(i)\} \geq \ell(i) \quad \forall\ i,$$

$$\beta + \sum_{1\leq i\leq k} \min\{0, |\mathcal{P}| - \ell(i)\} + e_E(\mathcal{P}) \geq k|\mathcal{P}| \quad \forall \text{ partition } \mathcal{P} \text{ of } V,$$

$$\sum_{1\leq i\leq k} \min\{\ell'(i), |\mathcal{P}|\} + e_E(\mathcal{P}) \geq k|\mathcal{P}| \quad \forall \text{ partition } \mathcal{P} \text{ of } V.$$

Remarks

1. For $\ell(i) = \ell'(i)\ \forall i$, $\alpha = \beta = \sum_{i=1}^{k} \ell(i)$, Theorem 6 reduces to Thm 3.
2. Theorem 6 can be proved using Theorem 3.

Regular packing of trees

Theorem 7 (Katoh, Tanigawa (2013))

There exists in a graph $G = (V, E)$ an h-regular packing of k trees $\Longleftrightarrow$

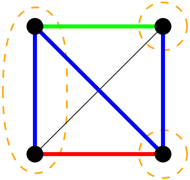
$$\begin{aligned} h|V| &\geq k, \\ e_E(\mathcal{P}) &\geq h|\mathcal{P}| - k \quad \forall \text{ partition } \mathcal{P} \text{ of } V. \end{aligned}$$

Definition

h-regular packing of trees : every vertex belongs to h of them.

Remarks

1. $e_E(\mathcal{P}) \geq \sum_{T \in \mathcal{T}} (|\mathcal{P}_T| - 1) = \sum_{X \in \mathcal{P}} |\mathcal{T}_X| - k \geq h|\mathcal{P}| - k.$
2. For $h = k$, Theorem 7 reduces to Theorem 1.
3. Thm 7 is a special case of Thm 6 : $\exists$ an h-regular packing of k trees $\Longleftrightarrow$ $\exists$ a packing of h spanning forests with k total number of conn. comp.



$k = 3, h = 2$

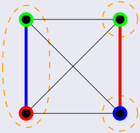
Regular packing of forests of given size

Theorem 8 (Szigeti (2023))

There exists in a graph $G = (V, E)$ an h-regular packing of k forests each with ℓ connected components $\iff$

$$k \geq h, \qquad h|V| \geq k\ell,$$

$$e_E(\mathcal{P}) \geq h|\mathcal{P}| - k\ell \quad \forall \text{ partition } \mathcal{P} \text{ of } V.$$



$k = 3, h = 2, \ell = 2$

Remarks

1. $e_E(\mathcal{P}) \geq \sum_{F \in \mathcal{F}} (|\mathcal{P}_F| - \ell) = \sum_{X \in \mathcal{P}} |\mathcal{F}_X| - k\ell \geq h|\mathcal{P}| - k\ell.$
2. For $h = k$, Theorem 8 reduces to Theorem 2.
3. For $\ell = 1$, Theorem 8 reduces to Theorem 7.
4. The decision problem whether there exists in G an h-regular packing of k forests each with ℓ edges is NP-complete.

Regular packing of forests of given sizes

Theorem 9 (Hoppenot, Martin, Szigeti (2023+))

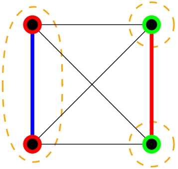
There exists in a graph $G = (V, E)$ an h-regular packing of k forests with $\ell(1), \dots, \ell(k)$ connected components $\iff$

$$|V| \geq \ell(i) \quad \forall\, i, \qquad h|V| \geq \sum_{1 \leq i \leq k} \ell(i),$$

$$\sum_{1 \leq i \leq k} \min\{\ell(i), |\mathcal{P}|\} + e_E(\mathcal{P}) \;\geq\; h|\mathcal{P}| \qquad \forall \text{ partition } \mathcal{P} \text{ of } V.$$

Remark

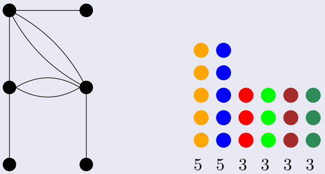
1. For $h = k$, Theorem 9 reduces to Theorem 3.
2. For $\ell(i) = \ell \ \forall i$, Theorem 9 reduces to Theorem 8.



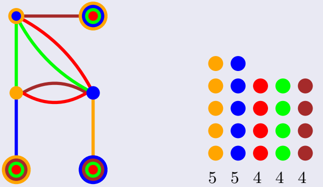
$k = 3, h = 2,$ 1 2 3

Sketch of the proof

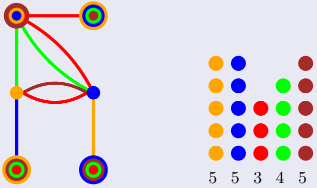
1. Instance : $h = 5$, $k = 6$,



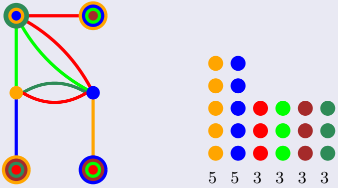
2. Packing of 5 spanning forests of $5, 5, 4, 4, 4$ conn. comp. :



3. Uncrossing :



4. 5-regular packing of 6 forests of $5, 5, 3, 3, 3, 3$ conn. comp. :



Regular packing of forests with bounded sizes

Theorem 10 (Hoppenot, Martin, Szigeti (2023+))

There exists an h-regular (ℓ, ℓ')-bordered (α, β)-limited packing of k forests in a graph $G = (V, E)$ $\iff$

$$\min\{\sum_{1\leq i\leq k} \ell'(i), \beta, h|V|\} \geq \alpha, \qquad \min\{|V|, \ell'(i)\} \geq \ell(i) \quad \forall\ i,$$

$$\beta + \sum_{1\leq i\leq k} \min\{0, |\mathcal{P}| - \ell(i)\} + e_E(\mathcal{P}) \geq h|\mathcal{P}| \quad \forall \text{ partition } \mathcal{P} \text{ of } V,$$

$$\sum_{1\leq i\leq k} \min\{\ell'(i), |\mathcal{P}|\} + e_E(\mathcal{P}) \geq h|\mathcal{P}| \quad \forall \text{ partition } \mathcal{P} \text{ of } V.$$

Remarks

1. For $h = k$, Theorem 10 reduces to Theorem 6.
2. For $\ell(i) = \ell'(i)\ \forall i$, $\alpha = \beta = \sum_{i=1}^{k} \ell(i)$, Thm 10 reduces to Thm 9.
3. Theorem 10 can be proved using Theorem 9.

Regular packing of hyperforests with bounded sizes

Theorem 11 (Hoppenot, Martin, Szigeti (2023+))

There exists an h-regular (ℓ, ℓ')-bordered (α, β)-limited packing of k hyperforests in a hypergraph $\mathcal{G} = (V, \mathcal{E})$ $\Longleftrightarrow$

$$\min\{\sum_{1\leq i\leq k} \ell'(i), \beta, h|V|\} \geq \alpha, \qquad \min\{|V|, \ell'(i)\} \geq \ell(i) \quad \forall\, i,$$

$$\beta + \sum_{1\leq i\leq k} \min\{0, |\mathcal{P}| - \ell(i)\} + e_{\mathcal{E}}(\mathcal{P}) \geq h|\mathcal{P}| \quad \forall \text{ partition } \mathcal{P} \text{ of } V,$$

$$\sum_{1\leq i\leq k} \min\{\ell'(i), |\mathcal{P}|\} + e_{\mathcal{E}}(\mathcal{P}) \geq h|\mathcal{P}| \quad \forall \text{ partition } \mathcal{P} \text{ of } V.$$

Definition

$e_{\mathcal{E}}(\mathcal{P})$: number of hyperedges between the members of a partition $\mathcal{P}$ of V.

Regular packing of hyperforests with bounded sizes

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$$\sum_{1 \le i \le k} \min\{\ell'(i), |\mathcal{P}|\} + e_{\mathcal{E}}(\mathcal{P}) \ \ge \ h|\mathcal{P}| \quad \forall \text{ partition } \mathcal{P} \text{ of } V.$$

Remarks

1. For $\mathcal{G}$ is a graph, Theorem 11 reduces to Theorem 10.
2. Theorem 11 can be proved by the trimming operation using Thm 10.

Regular packing of hyperbranchings with bounded sizes

Theorem 12 (Hoppenot, Martin, Szigeti (2023+))

There exists an h-regular (ℓ, ℓ')-bordered (α, β)-limited packing of k hyperbranchings in a dypergraph $\mathcal{D} = (V, \mathcal{A})$ $\iff$

$$\min\{\sum_{1 \le i \le k} \ell'(i), \beta, h|V|\} \ge \alpha, \qquad \min\{|V|, \ell'(i)\} \ge \ell(i) \quad \forall\, i,$$

$$\beta + \sum_{1 \le i \le k} \min\{0, |\mathcal{P}| - \ell(i)\} + e_{\mathcal{A}}(\mathcal{P}) \ \ge \ h|\mathcal{P}| \quad \forall \text{ subpartition } \mathcal{P} \text{ of } V,$$

$$\sum_{1 \le i \le k} \min\{\ell'(i), |\mathcal{P}|\} + e_{\mathcal{A}}(\mathcal{P}) \ \ge \ h|\mathcal{P}| \quad \forall \text{ subpartition } \mathcal{P} \text{ of } V.$$

Definition

$e_{\mathcal{A}}(\mathcal{P})$: number of hyperedges entering a member of subpartition $\mathcal{P}$ of V.

Regular packing of hyperbranchings with bounded sizes

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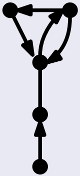
Remarks

1. For $h = k$ and $\mathcal{D}$ is a digraph, Theorem 12 reduces to Theorem 5.
2. The proof of Bérczi, Frank works for this general version as well.

Conclusion

Remarks

1. Directed hypergraphs : h-regular (ℓ, ℓ')-bordered (α, β)-limited packing of k hyperbranchings,
2. Hypergraphs : h-regular (ℓ, ℓ')-bordered (α, β)-limited packing of k hyperforests,
3. Mixed hypergraphs : natural generalization to
   1. h-regular packing of k mixed hyperbranchings each with ℓ connected components holds,
   2. packing of k spanning mixed branchings with $\ell(1), \dots, \ell(k)$ connected components does not hold. Counterexample :



$k = 2$

$\ell(1) = 1$

$\ell(2) = 3$

Conclusion

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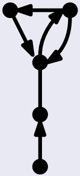

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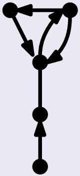

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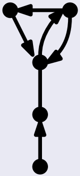

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Thank you for your attention !