

Note

A generalization of Petersen's theorem

Michel X. Goemans

Department of Mathematics, Room 2-372, Massachusetts Institute of Technology, Cambridge, MA 02139, USA.

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Abstract

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Petersen's theorem asserts that any cubic graph with at most 2 cut edges has a perfect matching. We generalize this classical result by showing that any cubic graph $G=(V, E)$ with at most 1 cut edge has a T -join of cardinality less than or equal to $|V|/2$ for every even subset T of vertices. Our result is based on the Edmonds–Johnson min–max relation for T -joins.

In his 1891 landmark paper on the factorization of regular graphs, Petersen [6] (see Biggs et al. [1]) proved the following famous result.

Theorem 1 (Petersen [6]). *If G is a cubic graph with at most 2 cut edges then G has a perfect matching.*

A graph is *cubic* if every vertex has degree 3. The vertex set of any cubic graph is of even cardinality. Let T be an even subset of vertices of a graph $G=(V, E)$. A T -join is a set A of edges such that T is exactly the set of vertices of odd degree in the graph (V, A) . When $T=\{u, v\}$, the minimal T -joins are the paths from u to v . When $T=V$ (assuming $|V|$ is even), every perfect matching of G is a T -join and these are the only T -joins of cardinality $n/2$, where n denotes the number of vertices of G . Let $\tau(G, T)$

Correspondence to: Michel X. Goemans, Dept. of Mathematics, M.I.T., Room 2-372, Cambridge, MA 02139, USA.

denote the minimum size of a T -join. Using T -joins, Petersen's theorem can be expressed as follows.

Theorem 2. *If G is a cubic graph with at most 2 cut edges then $\tau(G, V) \leq n/2$.*

In this paper, we generalize this result by showing the following.

Theorem 3. *Let $G=(V, E)$ be a connected cubic graph with at most one cut edge. Let T be a subset of V of even cardinality. Then $\tau(G, T) \leq n/2$.*

Before proving this result, we need some definitions and results related to T -joins. A cut C is a set of edges of the form $\delta(S)$ with $S \subset V$ and $S \neq \emptyset$, where $\delta(S)$ denotes the set of edges between S and $V \setminus S$. The cut $C = \delta(S)$ is said to be induced by the set S of vertices. Cubic graphs have the property that S and $\delta(S)$ have the same parity for any subset of vertices. A family $\mathcal{F} = \{\delta(S_1), \dots, \delta(S_k)\}$ of cuts is called *laminar* if $S_i \subseteq S_j$ or $S_j \subseteq S_i$ or $S_i \cap S_j = \emptyset$ for all i, j . A cut edge e is an edge for which $\{e\}$ is a cut.

A T -cut is a cut of the form $\delta(S)$ with $|S \cap T|$ odd. A *2-packing* of T -cuts is a family of T -cuts such that every edge of G is contained in at most 2 of them. The following theorem due to Edmonds and Johnson [2] (see [3] for some related min-max relations) gives a good characterization for the problem of finding a T -join of minimum cardinality.

Theorem 4 (Edmonds and Johnson [2]). *The minimum size $\tau(G, T)$ of a T -join is equal to half the maximum number of T -cuts in a 2-packing of G .*

Proof of Theorem 3. Let $\mathcal{F} = \{\delta(S_1), \dots, \delta(S_k)\}$ be a 2-packing of T -cuts of maximum cardinality. By Theorem 4, $k = 2\tau(G, T)$. Without loss of generality, $\mathcal{F}$ can be assumed to be laminar. Indeed, if $S_i \not\subseteq S_j$, $S_j \not\subseteq S_i$ and $S_i \cap S_j \neq \emptyset$ for some i and j , we can perform the classical uncrossing procedure, namely, we replace the cuts $\delta(S_i)$ by $\delta(S_j)$ by $\delta(S_i \setminus S_j)$ and $\delta(S_i \cap S_j)$ if $|S_i \cap S_j \cap T|$ is even and by $\delta(S_i \cup S_j)$ and $\delta(S_i \cap S_j)$ otherwise. This operation does not modify the cardinality of $\mathcal{F}$ and does not increase the number of cuts in which any edge is contained. We can also assume that if G has a cut edge e then, for any cut C of $\mathcal{F}$, either $e \notin C$ or $C = \{e\}$.

A laminar family $\mathcal{F} = \{\delta(S_1), \dots, \delta(S_k)\}$ of cuts can be represented as a tree on $k+1$ nodes. The root of the tree corresponds to V while the other nodes correspond to the subsets S_i . The nodes of the tree are indexed $0, 1, \dots, k$, node 0 representing the root. For ease of notation, let $S_0 = V$. The children of node i consist of a collection of disjoint maximal sets of $\{S_1, \dots, S_k\}$ included in S_i . By convention, if $S_i = S_j$ ($i \neq j$), then i and j are not siblings but either i is the son of j or vice versa. Let $p(i)$ denote the parent of node i and let C_i be the

set of children of node i . This tree representation induces a partition of V into $\{V_0, V_1, \dots, V_k\}$, where

$$V_i = S_i \setminus \left(\bigcup_{j \in C_i} S_j \right).$$

An example of this representation is depicted in Fig. 1. A family $\mathcal{F}$ can have different tree representations. Indeed, replacing S_i by $V \setminus S_i$ for some maximal subset of $\{S_1, \dots, S_k\}$ is equivalent to performing a rotation of the tree as described in Fig. 2. By repeating this operation, we can assume without loss of generality that if G has a cut edge (u, v) then no set $S_i, i = 1, \dots, k$, contains both u and v and if (u, v) is contained in

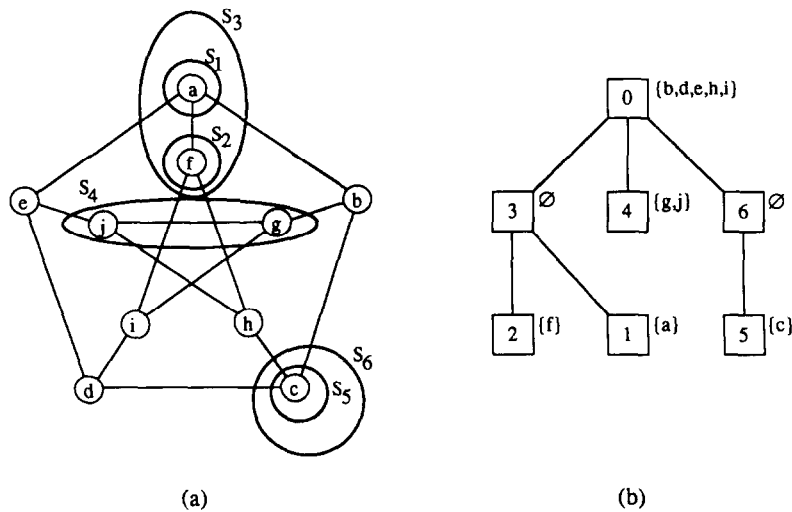


Fig. 1.

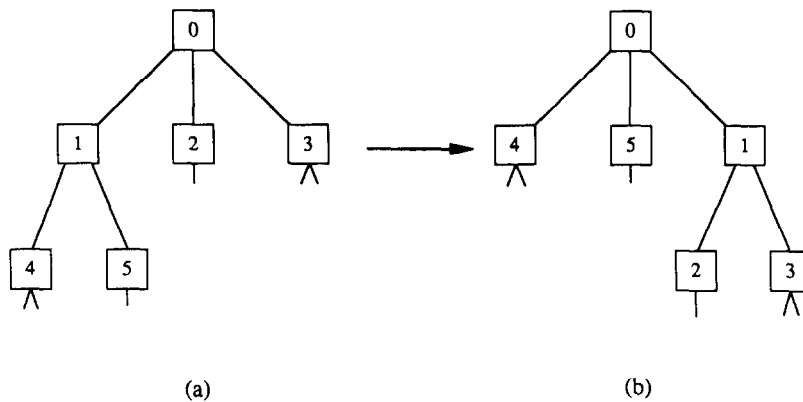


Fig. 2.

two cuts of $\mathcal{F}$, say $\delta(S_p)$ and $\delta(S_q)$, then $S_p = V \setminus S_q$. As a consequence, if i is an internal node of the tree representation (i.e. $i \neq 0$) with $V_i = \emptyset$ then $|\delta(S_i)| \geq 2$.

Having defined the tree representation of $\mathcal{F}$, we divide the set of nodes of the tree into the following 4 categories:

$$\begin{aligned} i \in N_o & \text{ if } V_i = \emptyset \text{ and } |S_i| \text{ is odd,} \\ i \in N_e & \text{ if } V_i = \emptyset \text{ and } |S_i| \text{ is even,} \\ i \in N_s & \text{ if } V_i \neq \emptyset, p(i) \in N_e \text{ and } |S_i| \text{ is even,} \\ i \in N_f & \text{ otherwise.} \end{aligned}$$

Moreover, let $N_{oe} = N_o \cup N_e$ and $N_{sf} = N_s \cup N_f$. In other words, $i \in N_{oe}$ if $V_i = \emptyset$ and $i \in N_{sf}$ if $V_i \neq \emptyset$.

We now derive two basic properties of this representation.

Property 1. *If $j \in C_i$ then either $j \in N_{sf}$ or $i \in N_{sf}$.*

Otherwise, consider any edge e in $\delta(S_j)$. Such an edge must exist since G is connected. Edge e links two vertices from different members of the partition, say V_k and V_l , with $V_k \subseteq S_j$, $V_l \cap S_j = \emptyset$ and $k \neq j \neq l$. This contradicts the assumptions since e is contained in more than 2 cuts, namely, $\delta(S_k)$, $\delta(S_l)$ and $\delta(S_j)$ if $S_l \subseteq S_i$ and $\delta(S_k)$, $\delta(S_i)$ and $\delta(S_j)$ otherwise.

Property 2. *If $i \in N_e$ then there exists $j \in (C_i \cap N_s)$.*

Indeed, since $V_i = \emptyset$, we have

$$\sum_{j \in C_i} |S_j \cap T| = |S_i \cap T|.$$

By definition of a T -cut, $|S_k \cap T|$ is odd and, therefore, we must have $|C_i|$ odd. Combined with $|S_i|$ even and $\sum_{j \in C_i} |S_j| = |S_i|$, this implies that there exists a node $j \in C_i$ with $|S_j|$ even. By Property 1, $V_j \neq \emptyset$, which proves that $j \in N_s$.

In order to prove that $k \leq n$, we show that every node of the tree, except the root node, can be assigned to a *different* vertex of V . When $i \in N_{sf}$, V_i is nonempty and, therefore, we can simply assign i to a vertex of V_i . For $i \in N_o$, we assign it to a vertex of $V_{p(i)}$. Note that $V_{p(i)}$ is nonempty by Property 1. A node $i \in N_e$ is assigned to a vertex of the set V_j corresponding to one of its children j in N_s . Such a child exists by Property 2.

In order to complete the proof, we need to show that the cardinality of V_i is large enough to accommodate all the assignments. Consider first a node $i \in N_f$. Let $B_i = C_i \cap N_o$ and let $k \in B_i$. Since every edge is contained in at most 2 cuts, all edges of $\delta(S_k)$ link S_k to V_i . Therefore, the graph $G - V_i$ has at least b_i connected components,

where $b_i = |B_i|$. Moreover, $|\delta(S_k)| \geq 3$ since $|\delta(S_k)|$ is odd by definition of N_o and not equal to 1 by construction of the tree. Hence, we have at least $3b_i$ edges between V_i and M , where

$$M = \bigcup_{k \in B_i} S_k.$$

Therefore, since G is cubic, we have $|V_i| \geq b_i$. Moreover, if i is not the root node, we must also have some edges in $\delta(S_i)$, implying that $|V_i| \geq b_i + 1$. This means that we can assign node i (if i is not the root node) as well as all its children in N_o to V_i . Now, consider a node $i \in N_s$. By the same reasoning, we have at least $3b_i$ edges between V_i and M . By definition of N_s , i is not the root node and S_i is even. Therefore, $\delta(S_i)$ is nonempty and even, implying that $|\delta(S_i)| \geq 2$. Moreover, since $V_{p(i)} = \emptyset$, all edges in $\delta(S_i)$ have one endpoint in V_i . Therefore, the number of edges between V_i and $M \cup (V \setminus S_i)$ is at least $3b_i + 2$. If $|V_i| = b_i + 1$ then we would have exactly one edge with one endpoint in V_i and the other in $L = S_i \setminus (M \cup V_i)$. This leads to a contradiction since $|\delta(L)| \geq 2$ by construction of the tree representation and all edges in $\delta(L)$ have one endpoint in V_i . Therefore, $|V_i| \geq b_i + 2$. This means that we can assign to V_i node i , all its children in N_o and its parent $p(i)$. This proves that every node, except the root node, can be assigned to different vertices of V and, hence, completes the proof of the theorem. $\square$

There is a slight discrepancy between the assumptions of Petersen's theorem and Theorem 3. However, our theorem does not hold when G has 2 cut edges. In Fig. 3, we have a cubic graph with 14 vertices and 2 cut edges, but the length of the shortest path between u and v is 8, which is strictly greater than 7.

The bound given in Theorem 3 is tight in the following sense. For every even $n \geq 10$ and every even $k \leq n$, there exists a cubic graph $G = (V, E)$ with one cut edge and $|V| = n$ such that $\tau(G, T) = n/2$ for some subset of cardinality k . This worst-case behavior is attained for the graph G_n depicted in Fig. 4. We leave it to the reader to verify that $\tau(G_n, \{1, \dots, k\}) = n/2$ for every even $k \leq n$.

For weighted 2-connected cubic graphs G , Naddef and Pulleyblank [5] have proved an extension of Petersen's theorem by showing that there exists a matching whose weight is at most a third of the total weight of G . A similar result does not hold

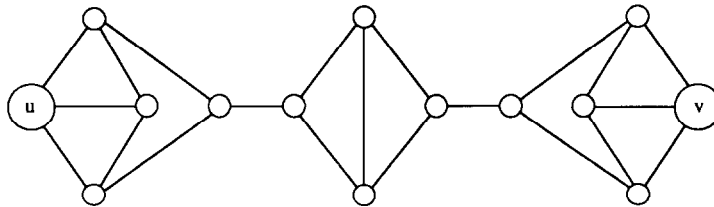


Fig. 3.

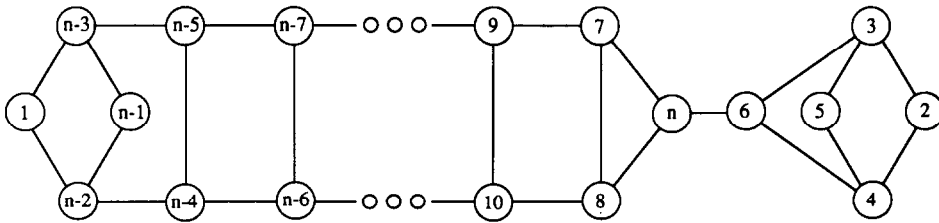


Fig. 4.

for T -joins. The minimum weight of a T -join can be half the total weight of the graph, and this bound is tight.

Recently, Frank [4] has obtained a very nice and deep min-max relation involving the minimum number of even ears in an ear decomposition of a graph and the maximum over all T of the minimum size of a T -join. Among other results, he proves the following one.

Theorem 5 (Frank [4, Corollary 2.6]). *Let G be a 2-edge-connected graph with a perfect matching. Then*

$$\max_T \tau(G, T) = \frac{n}{2}$$

iff there are no two disjoint nonempty subsets A and B of nodes so that $G - (A \cup B)$ contains $|A| + |B|$ odd components, among which $|A|$ components are linked only to A and the other $|B|$ components are linked only to B .

In particular, the latter condition is satisfied by 2-edge-connected cubic graphs and by graphs¹ obtained from a cubic graph with one cut edge e by duplicating e . This gives an alternate but indirect proof of Theorem 3.

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¹ This was pointed out by one of the referees.