Packing of Rigid Spanning Subgraphs and Spanning Trees

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Abstract

We prove that every \((6k + 2\ell, 2k)\)-connected simple graph contains \(k\) rigid and \(\ell\) connected edge-disjoint spanning subgraphs. This implies a theorem of Jackson and Jordán [6] providing a sufficient condition for the rigidity of a graph and a theorem of Jordán [8] on the packing of rigid spanning subgraphs. Both these results generalize the classic result of Lovász and Yemini [10] saying that every 6-connected graph is rigid. Our approach provides a transparent proof for this theorem.

Our result also gives two improved upper bounds on the connectivity of graphs that have interesting properties: (1) in every 8-connected graph there exists a packing of a spanning tree and a 2-connected spanning subgraph; (2) every 14-connected graph has a 2-connected orientation.

1 Introduction

In this paper, we consider sufficient conditions for the existence of a packing of spanning subgraphs in a given undirected graph \(G = (V, E)\), where by a packing we mean a set of pairwise edge-disjoint subgraphs of \(G\). Let us present a few examples in this area.

A first example is the existence of a packing of \(\ell\) spanning trees in every \(2\ell\)-edge-connected graph. This result is an easy consequence of the classic theorem of Tutte [12] and Nash-Williams [11] that characterizes the existence of such a packing. It is well known that this characterization can be derived from matroid theory as follows. The spanning trees of \(G\) correspond to the bases of the graphic matroid \(\mathcal{C}(G)\) of \(G\). Hence, by matroid union [4], the packings of \(\ell\) spanning trees of \(G\) correspond to the bases of the matroid \(\mathcal{N}_{0,\ell}\) defined as the union of \(\ell\) copies of \(\mathcal{C}(G)\). Thus, the existence of the required packing is characterized by the rank of \(E\) in \(\mathcal{N}_{0,\ell}\). Finally, using the formula of Edmonds [4] for the rank function of \(\mathcal{N}_{0,\ell}\) gives the theorem of Tutte and Nash-Williams.

A more recent example, due to Jordán [8], is the existence of a packing of \(k\) rigid spanning subgraphs in every \(6k\)-connected graph. The definition of rigidity is postponed to the next section but we mention here that the minimally rigid spanning subgraphs of \(G\) correspond to the bases of a matroid, namely the
rigidity matroid $\mathcal{R}(G)$ of $G$. So, as in the previous argument, the existence of a packing of $k$ rigid spanning subgraphs is characterized by the rank of $E$ in the matroid $\mathcal{N}_{k,0}$ defined as the union of $k$ copies of $\mathcal{R}(G)$. Jordán [8] used the formula of Edmonds [4] for the rank function of $\mathcal{N}_{k,0}$ to prove that $6k$-connectivity implies the desired lower bound on the rank of $E$.

Our main contribution is to provide a new example that gives a sufficient connectivity condition for the existence of a packing of $k$ rigid spanning subgraphs and $\ell$ spanning trees. To prove this result, we naturally introduce the matroid $\mathcal{N}_{k,\ell}$ defined as the union of $k$ copies of the rigidity matroid $\mathcal{R}(G)$ and $\ell$ copies of the graphic matroid $\mathcal{C}(G)$.

As a packing of rigid spanning subgraphs turns out to be a packing of spanning 2-connected subgraphs, the packing result of Jordán [8] allowed him to settle the base case of a conjecture of Kriesell (see in [8]) on removable spanning trees and that of a conjecture of Thomassen [13] on orientation of graphs. Our result on the packing of rigid spanning subgraphs and spanning trees enables us to improve the results of Jordán on these conjectures.

2 Definitions

Let $G = (V,E)$ be a graph. For $X \subseteq V$, denote by $d_G(X)$ the degree of $X$, that is, the number of edges of $G$ with one end vertex in $X$ and the other one in $V \setminus X$. We say that $G$ is Eulerian if each vertex of $G$ is of even degree.

A graph $G' = (V', E')$ is a subgraph of $G$ if $V' \subseteq V$ and $E' \subseteq E$. The subgraph $G'$ is called spanning if $V' = V$. A set of pairwise edge-disjoint subgraphs of $G$ is called a packing.

Let $F \subseteq E$. We denote by $G_F$ the spanning subgraph of $G$ with edge set $F$, that is, $G_F = (V,F)$. Let us denote by $c(F)$ the number of connected components of $G_F$ and by $K_F$ the set of connected components of $G_F$ of size 1.

Let $T \subseteq V$. We denote by $F(T)$ the set of edges of $G_F$ induced by $T$. We say that $F$ is a T-join if the set of odd degree vertices of $G_F$ coincides with $T$. It is well known that if $G_F$ is a connected graph and $T$ is of even cardinality then $G_F$ contains a $T$-join.

For a collection $\mathcal{G}$ of subsets of $V$, we say that $(V,\mathcal{G})$ is a hypergraph. We denote by $V(\mathcal{G})$ the set of vertices that belong to at least one element of $\mathcal{G}$. We will use the following well-known fact:

the sum of the sizes of the elements of $\mathcal{G}$ is equal to the sum, for each vertex, of the number of elements of $\mathcal{G}$ containing it. \hfill (1)

A set $X$ of vertices is called connected in $(V,\mathcal{G})$ if, for any partition of $X$ into two non-empty parts, there exists an element of $\mathcal{G}$ intersecting both parts. In $(V,\mathcal{G})$ a connected component is a maximal connected vertex set. The number of connected components of this hypergraph is denoted by $c(\mathcal{G})$. Let $K_0$ be the set of connected components of $(V,\mathcal{G})$ of size 1.

For $X \in \mathcal{G}$, we define the border $X_B$ of $X$ as the set of vertices of $X$ that belong to another element of $\mathcal{G}$, that is, $X_B = X \cap (\cup_{Y \in \mathcal{G}(X) \setminus \{X\}} Y)$. We also define the inner part $X_I$ of $X$ as the set of vertices of $X$ that belong to no other
element of $G$, that is, $X_I = X \setminus X_B$. Let $I_G$ be the set of elements of $G$ whose inner part is not empty, that is, $I_G = \{X \in G : X_I \neq \emptyset\}$. Since every vertex of $V(G)$ is contained in at least two elements of $G \cup \{X_I : X \in I_G\}$, we have, by (1),
\[
\sum_{X \in G} |X| + \sum_{X \in I_G} |X_I| \geq 2|V(G)|. \tag{2}
\]

A graph $G = (V, E)$ is called rigid if \(\sum_{X \in G} (2|X| - 3) \geq 2|V| - 3\) for every collection $G$ of sets of $V$ such that \(\{E(X), X \in G\}\) partitions $E$. More details about rigid graphs will be given in Section 4.

We will use the following connectivity concepts. The graph $G$ is called $p$-edge-connected if $d_G(X) \geq p$ for every non-empty proper subset $X$ of $V$. We say that $G$ is $p$-connected if $|V| > p$ and $G - X$ is connected for all $X \subset V$ with $|X| \leq p - 1$. As in [1], for a pair of positive integers $(p, q)$, $G$ is called $(p, q)$-connected if $|V| > \frac{p}{q}$ and $G - X$ is $(p - q|X|)$-edge-connected for all $X \subset V$, that is, if for every pair of disjoint subsets $X$ and $Y$ of $V$ such that $Y \neq \emptyset$ and $X \cup Y \neq V$, we have
\[
d_{G - X}(Y) \geq p - q|X|. \tag{3}
\]

For a better understanding we mention that $G$ is $(6, 2)$-connected if $G$ is 6-edge-connected, $G - v$ is 4-edge-connected for all $v \in V$ and $G - \{u, v\}$ is 2-edge-connected for all $u, v \in V$. It follows from the definitions that $p$-edge-connectivity is equivalent to $(p, p)$-connectivity. Moreover, since loops and parallel edges do not play any role in vertex connectivity, by the definition of $(p, q)$-connectivity, we have the following remark.

**Remark 1.** Every $p$-connected graph contains a $(p, 1)$-connected simple spanning subgraph and $(p, 1)$-connectivity implies $(p, q)$-connectivity for all $q \geq 1$.

Let $D = (V, A)$ be a directed graph. We say that $D$ is strongly connected if for every ordered pair $(u, v) \in V \times V$ of vertices there is a directed path from $u$ to $v$ in $D$. The digraph $D$ is called $p$-arc-connected if $D - F$ is strongly connected for all $F \subset A$ with $|F| \leq p - 1$. We say that $D$ is $p$-connected if $|V| > p$ and $D - X$ is strongly connected for all $X \subset V$ with $|X| \leq p - 1$.

### 3 Results

Lovász and Yemini proved the following sufficient condition for a graph to be rigid.

**Theorem 1** (Lovász and Yemini [10]). Every 6-connected graph is rigid.

The following result of Jackson and Jordán is, by Remark 1, a sharpening of Theorem 1.

**Theorem 2** (Jackson and Jordán [6]). Every $(6, 2)$-connected simple graph is rigid.
Figure 1: A non-rigid \((6,3)\)-connected simple graph \(G = (V,E)\). The collection \(G\) of the four grey vertex-sets provides a partition of \(E\). Hence, since 
\[
\sum_{X \in G}(2|X| - 3) = 4(2 \times 8 - 3) = 52 < 53 = 2 \times 28 - 3 = 2|V| - 3,
\] 
\(G\) is not rigid. The reader can easily check that \(G\) is \((6,3)\)-connected.

Note that in Theorem 2 the connectivity condition is the best possible since there exist non-rigid \((5,2)\)-connected simple graphs (see [10]) and non-rigid \((6,3)\)-connected simple graphs, for an example see Figure 1.

Jordán generalized Theorem 1 by giving the following sufficient condition for the existence of a packing of rigid spanning subgraphs.

**Theorem 3 (Jordán [8]).** Let \(k \geq 1\) be an integer. In every \(6k\)-connected graph there exists a packing of \(k\) rigid spanning subgraphs.

The main result of this paper (Theorem 4) contains a common generalization of Theorems 2 and 3. It provides a sufficient condition to have a packing of rigid spanning subgraphs and spanning trees. The proof of Theorem 4 will be given in Section 5.

**Theorem 4.** Let \(k \geq 1\) and \(\ell \geq 0\) be integers. In every \((6k + 2\ell, 2k)\)-connected simple graph there exists a packing of \(k\) rigid spanning subgraphs and \(\ell\) spanning trees.

Note that Theorem 4 applied for \(k = 1\) and \(\ell = 0\) provides Theorem 2. By Remark 1, every \(6k\)-connected graph contains a \((6k, 2k)\)-connected simple spanning subgraph, thus Theorem 4 also implies Theorem 3. Let us see some corollaries of the previous results.

One can easily prove that rigid graphs with at least 3 vertices are 2-connected (see Lemma 2.6 in [7]) and so connected. Thus, Theorem 4 gives the following corollary.

**Corollary 1.** Let \(k \geq 1\) and \(\ell \geq 0\) be integers. In every \((6k + 2\ell, 2k)\)-connected simple graph there exists a packing of \(k\) 2-connected and \(\ell\) connected spanning subgraphs.

Corollary 1 allows us to improve two results of Jordán [8]. The first one deals with the following conjecture of Kriesell, see in [8].
Conjecture 1 (Kriesell). For every positive integer $p$, there exists a (smallest) integer $f(p)$ such that every $f(p)$-connected graph $G$ contains a spanning tree $T$ for which $G - E(T)$ is $p$-connected.

As Jordán [8] pointed out, Theorem 3 answers this conjecture for $p = 2$ by showing that $f(2) \leq 12$. Corollary 1 applied for $k = 1$ and $\ell = 1$ directly implies that $f(2) \leq 8$.

Corollary 2. Every 8-connected graph $G$ contains a spanning tree $T$ such that $G - E(T)$ is 2-connected.

The other improvement deals with the following conjecture of Thomassen.

Conjecture 2 (Thomassen [13]). For every positive integer $p$, there exists a (smallest) integer $g(p)$ such that every $g(p)$-connected graph $G$ has a $p$-connected orientation.

By applying Theorem 3 and an orientation result of Berg and Jordán [2], Jordán [8] proved the conjecture for $p = 2$ by showing that $g(2) \leq 18$. Applying the same approach, that is, using a packing theorem (Corollary 1) and an orientation theorem (Theorem 5), we can prove a more general result (Corollary 3) that, in turn, implies $g(2) \leq 14$.

Theorem 5 (Király and Szigeti [9]). An Eulerian graph $G = (V, E)$ has an orientation $D$ such that $D - v$ is $p$-arc-connected for all $v \in V$ if and only if $G - v$ is $2p$-edge-connected for all $v \in V$.

Corollary 1 and Theorem 5 imply the following corollary which, specialized for $p = 1$, gives, by Remark 1, the claimed upper bound for $g(2)$.

Corollary 3. Every simple $(12p + 2, 4p)$-connected graph $G$ has an orientation $D$ such that $D - v$ is $p$-arc-connected for all $v \in V$.

Proof. Let $G = (V, E)$ be a simple $(12p + 2, 4p)$-connected graph. By Theorem 5 it suffices to prove that $G$ contains an Eulerian spanning subgraph $H$ such that $H - v$ is $2p$-edge-connected for all $v \in V$. By Corollary 1, in $G$ there exists a packing of $2p$ 2-connected spanning subgraphs $H_i = (V, E_i) (i = 1, \ldots, 2p)$ and a spanning tree $F$. Define $H' = (V, \bigvee_{i=1}^{2p} E_i)$. For all $i = 1, \ldots, 2p$, since $H_i$ is 2-connected, $H_i - v$ is connected; hence $H' - v$ is $2p$-edge-connected for all $v \in V$. Let $T$ be the set of vertices of odd degree in $H'$ and $F'$ a $T$-join in the tree $F$. Now, adding the edges of this $T$-join $F'$ to $H'$ provides the required spanning subgraph of $G$.

Finally, we mention the following conjecture of Frank that would imply $g(2) = 4$.

Conjecture 3 (Frank [5]). A graph has a 2-connected orientation if and only if it is (4, 2)-connected.

4 Preliminaries

Let $G = (V, E)$ be a graph. In this section we present some simple facts about the graphic matroid $\mathcal{C}(G)$, the rigidity matroid $\mathcal{R}(G)$ and the matroid $N_{k,\ell}(G)$.
introduced in the Introduction.

We will denote by \( C(G) \) the \textbf{graphic matroid} of \( G \) on ground-set \( E \), that is an edge set \( F \) of \( G \) is independent in \( C(G) \) if and only if \( G_F \) is a forest. Let \( n = |V| \) be the number of vertices in \( G \). It is well known that the rank function \( r_C \) of \( C(G) \) satisfies the following:

\[
 r_C(F) = n - c(F). \tag{4}
\]

We will denote by \( R(G) \) the \textbf{rigidity matroid} of \( G \) on ground-set \( E \) with rank function \( r_R \) (for a definition we refer the reader to \cite{10}). For \( F \subseteq E \), by a theorem of Lovász and Yemini \cite{10}, we have

\[
 r_R(F) = \min \sum_{X \in \mathcal{G}} (2|X| - 3), \tag{5}
\]

where the minimum is taken over all collections \( \mathcal{G} \) of subsets of \( V \) such that \( \{F(X), X \in \mathcal{G}\} \) partitions \( F \). Note that

\[
 r_R(E) \leq 2|V| - 3 \tag{6}
\]

and equality holds if and only if \( G \) is rigid.

For a subset \( F \) of \( E \), let \( \mathcal{G} \) be a collection of subsets of \( V \) such that \( \{F(X), X \in \mathcal{G}\} \) partitions \( F \) that minimizes the right hand side of \( (5) \). It is well known that each element of \( \mathcal{G} \) induces a rigid subgraph of \( G_F \). (For example, see the proof of Lemma 2.4 in \cite{7}.) Note also that, if \( G \) is simple, then every element of \( \mathcal{G} \) of size 2 induces at most one (in fact exactly one) edge and contributes exactly one to the sum. So we have the following simple but very useful observation.

\textbf{Remark 2.} If \( G \) is simple, then

\[
 r_R(F) = \min \sum_{X \in \mathcal{H}} (2|X| - 3) + |F \setminus H|, \tag{7}
\]

where the minimum is taken over all subsets \( H \subseteq F \) and all collections \( \mathcal{H} \) of subsets of \( V \) such that \( \{F(X), X \in \mathcal{H}\} \) partitions \( H \) and each element of \( \mathcal{H} \) induces a rigid subgraph of \( G_H \) of size at least 3.

The following claim provides insight into the structure of the minimizers of \( (7) \).

\textbf{Claim 1.} Let \( G = (V,E) \) be a simple graph and \( F \subseteq E \). Let \( H \subseteq F \) and \( \mathcal{H} \) be a collection of subsets of \( V \) that minimize the right hand side of \( (7) \).

(i) For every \( \mathcal{H}^* \subseteq \mathcal{H} \), \( r_R(\cup_{X \in \mathcal{H}^*} F(X)) = \sum_{X \in \mathcal{H}^*} (2|X| - 3) \).

(ii) For every non-empty \( \mathcal{H}^* \subseteq \mathcal{H} \), there exists a vertex in \( V(\mathcal{H}^*) \) that is contained in a single element of \( \mathcal{H}^* \).

(iii) \( |\mathcal{I}_H| + |\mathcal{K}_H| \geq c(\mathcal{H}) \).

(iv) The connected components of \( (V,\mathcal{H}) \) and those of \( G_H \) coincide.
Proof. (i) Since \{F(X), X \in \mathcal{H}\} partitions \(H\), we have, by (7) and subadditivity of \(r_R\),

\[
\sum_{X \in \mathcal{H}} (2|X| - 3) + |F \setminus H| = r_R(F) 
\leq r_R(\cup_{X \in \mathcal{H}^*} F(X)) + r_R(\cup_{X \in \mathcal{H} \setminus \mathcal{H}^*} F(X)) + r_R(F \setminus H) 
\leq \sum_{X \in \mathcal{H}^*} r_R(F(X)) + \sum_{X \in \mathcal{H} \setminus \mathcal{H}^*} r_R(F(X)) + |F \setminus H| 
\leq \sum_{X \in \mathcal{H}^*} (2|X| - 3) + \sum_{X \in \mathcal{H} \setminus \mathcal{H}^*} (2|X| - 3) + |F \setminus H|.
\]

So equality holds everywhere and (i) follows.

(ii) By contradiction, suppose that every vertex of \(V(\mathcal{H}^*)\) is contained in at least two elements of \(\mathcal{H}^*\). Hence, by (5), (i), since the size of each element of \(\mathcal{H}^*\) is at least 3 and by (1), we have \(2|V(\mathcal{H}^*)| - 3 \geq r_R(\cup_{X \in \mathcal{H}^*} F(X)) = \sum_{X \in \mathcal{H}^*} (2|X| - 3) = \sum_{X \in \mathcal{H}^*} |X| + \sum_{X \in \mathcal{H} \setminus \mathcal{H}^*} (|X| - 3) \geq 2|V(\mathcal{H}^*)| + 0\), a contradiction.

(iii) Let \(C\) be a connected component of \((V, \mathcal{H})\) that is not in \(\mathcal{K}_H \cup \mathcal{H}^*\) the elements of \(\mathcal{H}\) contained in \(C\). By (ii), there exists in \(C\) a vertex \(v\) contained in a single element \(X\) of \(\mathcal{H}^*\). Hence, by definition of \(\mathcal{H}^*, v \in X_I\) and so \(X \in \mathcal{I}_H\). Thus we proved that \(C\) contains an element of \(\mathcal{I}_H\). Since the connected components of \((V, \mathcal{H})\) are disjoint, (iii) follows.

(iv) Let \(U\) be a connected component of \(G_H\) and \(\emptyset \neq W \subseteq U\). Then, there exists an edge of \(H\) with one end in \(W\) and the other end in \(U \setminus W\). Since \{\(F(X), X \in \mathcal{H}\}\} partitions \(H\), this edge is contained in an element of \(\mathcal{H}\) that intersects both \(W\) and \(U \setminus W\). So \(U\) is connected in \((V, \mathcal{H})\).

Let \(U\) be a connected component of \((V, \mathcal{H})\) and \(W \subseteq U\). Then, there exists an element \(X\) of \(\mathcal{H}\) intersecting both \(W\) and \(U \setminus W\). Since \(X \subseteq U\) and \(X\) induces a rigid, and so connected, subgraph of \(G_H\), there exists an edge of \(H\) with one end in \(X \cap W \subseteq W\) and the other in \(X \setminus W \subseteq U \setminus W\). So \(U\) is connected in \(G_H\). This ends the proof of (iv).

As we mentioned in the Introduction, to have a packing of \(k\) rigid spanning subgraphs and \(\ell\) spanning trees in \(G\), we must find \(k\) bases in the rigidity matroid \(R(G)\) and \(\ell\) bases in the graphic matroid \(C(G)\) all pairwise disjoint. To do that we will need the following matroid. For \(k \geq 0\) and \(\ell \geq 0\), define \(\mathcal{N}_{k,\ell}(G)\) as the matroid on ground-set \(E\), obtained by taking the matroid union of \(k\) copies of the rigidity matroid \(R(G)\) and \(\ell\) copies of the graphic matroid \(C(G)\). Let \(r_{k,\ell}\) be the rank function of \(\mathcal{N}_{k,\ell}(G)\). By a theorem of Edmonds [4], for the rank of matroid unions,

\[
r_{k,\ell}(E) = \min_{F \subseteq E} k r_R(F) + \ell r_C(F) + |E \setminus F|. 
\]

Observe that

\[
r_{k,\ell}(E) \leq k r_R(E) + \ell r_C(E) \leq k(2n - 3) + \ell(n - 1). 
\]
Jordán [8] used the matroid $\mathcal{N}_{k,0}(G)$ to prove Theorem 3 and pointed out that using $\mathcal{N}_{k,\ell}(G)$ one could prove a theorem on the packing of rigid spanning subgraphs and spanning trees. We tried to fulfill this gap by following the proof of [8] but we failed. To achieve this aim we had to find a new proof technique.

5 Proofs

In this section we provide the proofs of our results. Let us first demonstrate our proof technique by giving a transparent proof for Theorems 1 and 2. We emphasize that in the first two proofs we use only Remark 2 from the Preliminaries.

Proof of Theorem 1. By Remark 1, we may assume that $G$ is simple. Then, by (7), there exist a subset $H \subseteq E$ and a collection $\mathcal{H}$ of subsets of $V$ of sizes at least 3 such that $\{E(X), X \in \mathcal{H}\}$ partitions $H$ and $r_R(E) = \sum_{X \in \mathcal{H}} (2|X| - 3) + |E \setminus H|$. If $V \in \mathcal{H}$, then $r_R(E) \geq 2|V| - 3$, hence, by (6), $G$ is rigid. So in the following we assume that $V \notin \mathcal{H}$ and find a contradiction.

Recall that, for $X \in \mathcal{H}$, $X_B = X \cap (\cup_{Y \in \mathcal{H} - X} Y)$, $X_I = X \setminus X_B$ and $I_H = \{X \in \mathcal{H} : X_I \neq \emptyset\}$.

Each edge of $H$ being induced by an element of $\mathcal{H}$, it contributes neither to $d_G - X_B(X_I)$ for $X \in I_H$ nor to $d_G(v)$ for $v \in V \setminus V(\mathcal{H})$. Thus, since for $X \in I_H$, $\emptyset \neq X_I \neq V \setminus X_B$, we have, by 6-connectivity of $G$,

$$|E \setminus H| \geq \frac{1}{2} \left( \sum_{X \in I_H} d_{G - X_B}(X_I) + \sum_{v \in V \setminus V(\mathcal{H})} d_G(v) \right)$$

$$\geq \frac{1}{2} \left( \sum_{X \in I_H} (6 - |X_B|) + \sum_{v \in V \setminus V(\mathcal{H})} 6 \right) \quad (*)$$

$$\geq \sum_{X \in I_H} (3 - |X_B|) + 2(|V| - |V(\mathcal{H})|). \quad (10)$$

By $|X| \geq 3$ for $X \in \mathcal{H} \setminus I_H$, (10) and (2), we have

$$r_R(E) = \sum_{X \in \mathcal{H}} (2|X| - 3) + |E \setminus H|$$

$$\geq \left( \sum_{X \in \mathcal{H}} |X| + \sum_{X \in I_H} (|X| - 3) \right) + \left( \sum_{X \in I_H} (3 - |X_B|) + 2(|V| - |V(\mathcal{H})|) \right)$$

$$\geq \sum_{X \in \mathcal{H}} |X| + \sum_{X \in I_H} |X_I| + 2(|V| - |V(\mathcal{H})|)$$

$$\geq 2|V|.$$

Hence, by (6), we have $2|V| - 3 \geq r_R(E) \geq 2|V|$, a contradiction. ■

Proof of Theorem 2. The proof of Theorem 2 is obtained from the proof of Theorem 1 by replacing $d_{G - X_B}(X_I) \geq 6 - |X_B|$ by $d_{G - X_B}(X_I) \geq 6 - 2|X_B|$ in the inequality $(*)$. This means that in the proof of Theorem 1 we used $(6,2)$-connectivity instead of 6-connectivity. ■
Here comes the proof of the main result.

**Proof of Theorem 4.** Let \( k \geq 1 \) and \( \ell \geq 0 \) be integers and \( G = (V, E) \) a \((6k + 2\ell, 2k)\)-connected simple graph. To prove the theorem we use the matroid \( \mathcal{N}_{k, \ell} \) defined in Section 4 and show that

\[
r_{k, \ell}(E) = k(2n - 3) + \ell(n - 1).
\]  

(11)

Choose \( F \) a smallest-size set of edges that gives the rank of \( E \) in \( \mathcal{N}_{k, \ell} \), that is, which minimizes the right hand side of (8). By (7), there exist a subset \( H \subseteq F \) and a collection \( \mathcal{H} \) of subsets of \( V \) of sizes at least 3 such that \( \{F(X), X \in \mathcal{H}\} \) partitions \( H \) and

\[
r_\mathcal{H}(F) = \sum_{X \in \mathcal{H}} (2|X| - 3) + |F \setminus H|.
\]  

(12)

**Claim 2.** \( H = F \).

**Proof.** Since \( \mathcal{H} \) is a collection of subsets of \( V \) of sizes at least 3 such that \( \{H(X), X \in \mathcal{H}\} \) partitions \( H \), we have, by (12), \( r_\mathcal{H}(H) \leq \sum_{X \in \mathcal{H}} (2|X| - 3) = r_\mathcal{H}(F) - |F \setminus H| \). Hence, since the rank function \( r_\mathcal{H} \) is non-decreasing and \( k \geq 1 \), we have

\[
kr_\mathcal{H}(H) + \ell r_\mathcal{H}(H) + |E \setminus H| \leq kr_\mathcal{H}(F) + \ell r_\mathcal{H}(F) + |E \setminus H| - k|F \setminus H|
\]

\[
\leq kr_\mathcal{H}(F) + \ell r_\mathcal{H}(F) + |E \setminus F|.
\]

Thus \( H \) also minimizes the right hand side of (8) and, by \( H \subseteq F \) and the minimality of \( F, H = F \). \( \blacksquare \)

If \( V \in \mathcal{H} \), then, by (12), \( r_\mathcal{H}(F) \geq \sum_{X \in \mathcal{H}} (2|X| - 3) \geq 2n - 3 \) and, by Claim 2 and Remark 2, \( G_F \) is connected, that is, \( r_C(F) = n - 1 \). Hence, by (9), we have (11) and the theorem is proved. From now on, we assume that \( V \notin \mathcal{H} \) and we will show a contradiction.

Recall the definitions of the border \( X_B = X \cap (\cup_{Y \in \mathcal{H} \setminus X} Y) \), the inner part \( X_I = X \setminus X_B \) for \( X \in \mathcal{H} \), \( \mathcal{I}_\mathcal{H} = \{X \in \mathcal{H} : X_I \neq \emptyset\} \) and the sets \( \mathcal{K}_F \) and \( \mathcal{K}_H \) of connected components of \( G_F \) and \( (V, \mathcal{H}) \) of size 1. By Claim 1 (iv), \( \mathcal{K}_F = \mathcal{K}_H \).

Let us use the connectivity condition on \( G \) to show a lower bound on \( |E \setminus F| \).

**Claim 3.** \( |E \setminus F| \geq k \left( \sum_{X \in \mathcal{I}_H} (3 - |X_B|) + 3|\mathcal{K}_F| \right) + \ell \left( |\mathcal{I}_H| + |\mathcal{K}_F| \right) \).

**Proof.** By \( V \notin \mathcal{H} \), for \( X \in \mathcal{I}_H \), \( \emptyset \neq X_I \neq V \setminus X_B \). Then, for \( X \in \mathcal{I}_H \) and for \( v \in \mathcal{K}_F \), we have, by \( (6k + 2\ell, 2k) \)-connectivity of \( G \),

\[
d_{G - X_B}(X_I) \geq (6k + 2\ell) - 2k|X_B|.
\]  

(13)

\[
d_G(v) \geq 6k + 2\ell.
\]  

(14)

Since, by Claim 2, every edge of \( F \) is induced by an element of \( \mathcal{H} \) and by definition of \( X_I \), for \( X \in \mathcal{I}_H \), no edge of \( F \) contributes to \( d_{G - X_B}(X_I) \). Each \( v \in \mathcal{K}_F \) is a connected component of the graph \( G_F \), thus no edge of \( F \) contributes
to $d_G(v)$. Hence, by (13), (14) and $\ell \geq 0$, we obtain the required lower bound on $|E \setminus F|$,

$$|E \setminus F| \geq \frac{1}{2} \left( \sum_{X \in I_H} d_{G-H}(X) + \sum_{v \in K_F} d_G(v) \right) \geq \frac{1}{2} \left( (6k + 2\ell)|I_H| - 2k \sum_{X \in I_H} |X_B| + (6k + 2\ell)|K_F| \right) \geq k \left( \sum_{X \in I_H} (3 - |X_B|) + 3|K_F| \right) + \ell \left( |I_H| + |K_F| \right).$$

Thus, by (12), Claims 2, 3, $|X| \geq 3$ ($X \in H \setminus I_H$), Claim 1 (iv), (iii) and (2), we get

$$r_{k,\ell}(E) = k \sum_{X \in H} (2|X| - 3) + |E \setminus F| + \ell(n - c(F)) \geq k \left( \sum_{X \in H} |X| + \sum_{X \in I_H} (|X| - 3) \right) + k \left( \sum_{X \in I_H} (3 - |X_B|) + 3|K_F| \right) + \ell \left( |I_H| + |K_F| \right) + \ell(n - c(F)) \geq k \left( \sum_{X \in H} |X| + \sum_{X \in I_H} |X_I| + 2|K_H| \right) + \ell \left( c(H) + n - c(F) \right) \geq 2kn + \ell n.$$  

By $k \geq 1$ and $\ell \geq 0$, this contradicts (9).

Remark that the proof actually shows that if $G$ is simple and $(6k + 2\ell, 2k)$-connected and if $F \subseteq E$ is such that $|F| \leq 3k + \ell$, then in $G' = (V, E \setminus F)$ there exists a packing of $k$ rigid spanning subgraphs and $\ell$ spanning trees.

We mention that Theorem 4 was slightly generalized by Durand de Gevigney and Nguyen [3] for finding bases of a particular count matroid and spanning trees pairwise edge-disjoint. Their proof applies the discharging method.

References


