Augmenting a hypergraph to have a matroid-based (f, g)-bounded (α, β)-limited packing of rooted hypertrees

Pierre Hoppenot, Zoltán Szigeti
University Grenoble Alpes, INP, G-SCOP

September 5, 2025

Abstract

The aim of this paper is to further develop the theory of packing trees in a graph. We first prove the classic result of Nash-Williams [13] and Tutte [14] on packing spanning trees by adapting Lovász' proof [12] of the seminal result of Edmonds [2] on packing spanning arborescences in a digraph. Our main result on graphs extends the theorem of Katoh and Tanigawa [11] on matroid-based packing of rooted trees by characterizing the existence of such a packing satisfying the following further conditions: for every vertex v, there are given a lower bound $f(v)$ and an upper bound $g(v)$ on the number of trees rooted at v and there are given a lower bound α and an upper bound β on the total number of roots. We also answer the hypergraphic version of the problem. Furthermore, we are able to solve the augmentation version of the latter problem, where the goal is to add a minimum number of edges to have such a packing. The methods developed in this paper to solve these problems may have other applications in the future.

1 Introduction

The first major result on packing spanning trees is due to Nash-Williams [13] and Tutte [14]. They independently characterized graphs having a packing of k spanning trees; in other words k pairwise edge-disjoint spanning trees. As a first contribution of this paper we provide a new proof of their result. We believe that the proof is new but we know that the approach is old. Actually, Lovász [12] provided an elegant and simple proof of Edmonds' result on packing spanning arborescences and here we work out how the same idea can be applied in the undirected case.

Since then, the result of Nash-Williams [13] and Tutte [14] has been extended in several ways. Notably, Katoh and Tanigawa [11] characterized graphs admitting a complete matroid-based packing of rooted trees, see Theorem 4. Here the rooted trees are not necessarily spanning. However, a matroid is given on the root set and the packing must satisfy a matroid constraint, informally meaning that every vertex is reachable from a basis of the matroid in the rooted trees of the packing. Katoh and Tanigawa explain in [11] an interesting application of this theorem in rigidity theory.

Our goal is to extend the result of Katoh and Tanigawa [11] on matroid-based packing of rooted trees. To do so, we develop useful tools mainly based on our improved knowledge of the uncrossing of two partitions of a set. First, we show the submodularity of some functions with two variables. We also give a tool which shows that it is possible to simultaneously cover, with an edge set, two supermodular functions on partitions of a vertex set, see Theorem 1. This has been proved in a special case in [8]. Likewise, we develop a tool which shows when it is possible to trim a hypergraph that covers two supermodular functions on partitions to a graph that covers the same functions, see Theorem 2.

The discovery of the submodularity of the above mentioned functions on partitions allows us to give the rank function of a new matroid which was inspired by the work of Katoh and Tanigawa [11]. More precisely, for a graph $G = (V, E)$, a multiset S of vertices in V, and a matroid $\mathsf{M} = (S, r_{\mathsf{M}})$, we give a matroid whose independent sets of size $r_{\mathsf{M}}(S)|V|$ are exactly the sets $F \cup R$, where F is the edge set and R is the root set of an M-based packing of rooted trees in G, see Theorem 6.

This matroid along with another matroid, the bounded direct sum of matroids (see Theorem 7), play a crucial role in the solution of the following problem. Given a graph $G = (V, E)$, a multiset S of vertices in V, $k \in \mathbb{Z}_+$, functions $f, g : V \to \mathbb{Z}_+$, and a matroid $\mathsf{M} = (S, r_{\mathsf{M}})$, characterize the existence of an M-based (f, g)-bounded packing of k rooted trees in G, where (f, g)-bounded means that every vertex $v \in V$ is the root of at least $f(v)$ and at most $g(v)$ rooted trees, see Theorem 9. We extend this result to (α, β)-limited packings, meaning that the given value of the number of roots is relaxed to an interval $[\alpha, \beta]$, see Theorem 10. Using the newly found submodularity of some functions on partitions and the previously mentioned result on trimming, we are then able to generalize the previous result to get a characterization of hypergraphs having an M-based (f, g)-bounded (α, β)-limited packing of rooted hypertrees, see Theorem 12. This extends an earlier theorem of Frank, Király and Kriesell [6] that generalized Nash-Williams and Tutte's theorem to hypergraphs. For further new results on packing hypertrees, see [7].

We are also able to formulate and prove the conditions under which a hypergraph can be augmented (in term of minimum number of edges) to contain an M-based (f, g)-bounded (α, β)-limited packing of rooted hypertrees, see Theorem 14. The readers interested in similar augmentation problems are invited to see [8].

The organization of this paper is as follows. In Section 2 we provide the necessary definitions. In Section 3 we first introduce some submodular and some supermodular functions on partitions. Then we prove our tools on covering and trimming about supermodular functions on partitions. Section 4 contains the above mentioned results on packing trees and their proofs.

2 Definitions

We denote by $\boldsymbol{\mathbb{Z}}$ the set of integers and $\boldsymbol{\mathbb{Z}_+}$ the set of non-negative integers. Let $\boldsymbol{V}$ be a finite set. For a function $m : V \to \mathbb{Z}$ and a subset X of V, we define $\boldsymbol{m(X)} = \sum_{x \in X} m(x)$. For $X \subseteq V$, we denote by $\overline{\boldsymbol{X}}$ its *complement*, that is $V \setminus X$. We say that X *separates* two distinct elements of V if X contains one of them and $\overline{X}$ contains the other one. A *multiset* of V is a set of elements of V allowing multiplicities. For a multiset S of V and $X \subseteq V$, $\boldsymbol{S_X}$ denotes the multiset of V consisting of the restriction of S on X. A set $\mathcal{S}$ of subsets of V is called a *family* if the subsets of V are taken with multiplicities in $\mathcal{S}$. For a family $\mathcal{S}$ of subsets of V and a subset X of V, we denote by $\boldsymbol{\mathcal{S}_X}$ the family containing the sets in $\mathcal{S}$ that intersect X. Two subsets X and Y of V are called *properly intersecting* if none of $X \cap Y$, $X \setminus Y$, and $Y \setminus X$ is empty. The operation that replaces two properly intersecting sets by their intersection and their union is called *uncrossing*. For a family $\mathcal{F}$ of subsets of V, the *uncrossing method* consists in applying repetitively the uncrossing operation as long as properly intersecting sets exist in the family.

A set of pairwise disjoint subsets of V such that their union is V is called a *partition* of V. We say that a subset X of V *crosses* a partition $\mathcal{P}$ of V if X intersects at least two members of $\mathcal{P}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of V and $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where the union is taken with multiplicities. We use the uncrossing method on the family $\mathcal{P}$ to obtain a new family $\mathcal{P}'$ which contains no properly intersecting sets. Taking respectively the minimal and maximal sets in $\mathcal{P}'$, we obtain two partitions $\mathcal{P}'_1$ and $\mathcal{P}'_2$ of V. We call $\mathcal{P}'_1$ the *intersection* of $\mathcal{P}_1$ and $\mathcal{P}_2$, and we denote it by $\boldsymbol{\mathcal{P}_1 \sqcap \mathcal{P}_2}$; we call $\mathcal{P}'_2$ the *union* of $\mathcal{P}_1$ and $\mathcal{P}_2$ and we denote it by $\boldsymbol{\mathcal{P}_1 \sqcup \mathcal{P}_2}$. We mention that while $\mathcal{P}_1 \sqcap \mathcal{P}_2$ depends on the choices during execution of the uncrossing method, $\mathcal{P}_1 \sqcup \mathcal{P}_2$ is uniquely defined.

Let S be a finite ground set. A set function b on S is called *non-decreasing* if $b(X) \leq b(Y)$ for all $X \subseteq Y \subseteq V$ and *subcardinal* if $b(X) \leq |X|$ for every $X \subseteq V$. We say that b is *submodular* (resp. *intersecting submodular*) if $b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y)$ for every sets (resp. properly intersecting sets) $X, Y \subseteq V$. A set function p on S is called *supermodular* if $-p$ is submodular. A set function m on S is called *modular* if it is submodular and supermodular.

Let V be a set, and b a function on the partitions of V. We say that b is *submodular on the partitions of* V if $b(\mathcal{P}_1) + b(\mathcal{P}_2) \geq b(\mathcal{P}_1 \sqcup \mathcal{P}_2) + b(\mathcal{P}_1 \sqcap \mathcal{P}_2)$ for every choice of $\mathcal{P}_1 \sqcap \mathcal{P}_2$, for every partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of V. A function p on the partitions of V is called *supermodular on the partitions of* V if $-p$

is submodular on the partitions of V.

Let r be a non-negative, integer-valued, non-decreasing, subcardinal and submodular set function on S. Then $\mathsf{M} = (S, r)$ is called a *matroid* and r is called the *rank function* of M. A subset X of S is called an *independent set* of M if $r(X) = |X|$. The set of independent sets of M is denoted by $\boldsymbol{\mathcal{I}_{\mathsf{M}}}$. A maximal independent set of M is called a *basis* of M. If S is a basis of M, then M is called *free matroid*. If the bases are all the subsets of S of size k, then M is called a *uniform matroid* of rank k. For $T \subseteq S$, the matroid $\boldsymbol{\mathsf{M}|_T} = (T, r)$, obtained from M by deleting the elements $S \setminus T$, is called *restricted* matroid on T. For an independent set X in M, the matroid $\boldsymbol{\mathsf{M}/X} = (S \setminus X, r_{/X})$, whose set of independent sets is $\{Y \subseteq S \setminus X : X \cup Y \in \mathcal{I}_{\mathsf{M}}\}$ and whose rank function is $\boldsymbol{r_{/X}}(Z) = r(X \cup Z) - |X|$ for every $Z \subseteq S \setminus X$, is called *contracted matroid*.

Let $\boldsymbol{G} = (V, E)$ be a graph with *vertex set* $\boldsymbol{V}$ and *edge set* $\boldsymbol{E}$. For any graph G, $\boldsymbol{V(G)}$ denotes its vertex set and $\boldsymbol{E(G)}$ its edge set. A graph G' is a *subgraph* of G if it is obtained from G by deleting some vertices and some edges. Furthermore, if $V(G') = V(G)$, then G' is called a *spanning* subgraph of G. Let X be a subset of V. We denote by $\boldsymbol{i_E(X)}$ the number of edges of E in X. It is known that i_E is supermodular. We denote by $\boldsymbol{G[X]}$ the subgraph of G after deleting $\overline{X}$. We denote by $\boldsymbol{G/X}$ the graph obtained from G by contracting X, that is by replacing X by a new vertex $\boldsymbol{v_X}$, by deleting all the edges in X, and replacing every edge uv in E such that $v \in X$ and $u \in \overline{X}$ by an edge $v_X u$. For disjoint $X, Y \subseteq V$, $\boldsymbol{d_E(X, Y)}$ denotes the number of edges xy in E with $x \in X$ and $y \in Y$. A *tree* of G is a connected subgraph of G that contains no cycle. For a tree T of G and $s \in V(T)$, the couple (s, T) is a called a *rooted tree* of G and s is called the *root* of the rooted tree. By a *packing* of rooted trees in G, we mean a set $\mathcal{B}$ of rooted trees of G that are edge disjoint. For two functions $f, g : V \to \mathbb{Z}_+$, we say that the packing $\mathcal{B}$ is (f, g)-*bounded* if for every vertex v of G, v is the root in at least $f(v)$ and at most $g(v)$ rooted trees in $\mathcal{B}$. For two non-negative integers α and β, we say that the packing $\mathcal{B}$ is (α, β)-*limited* if the number of trees in $\mathcal{B}$ is at least α and at most β. Given a multiset S of V and a matroid M on S, a packing of rooted trees is called *(complete)* M-*based* if the multiset of roots of the rooted trees in the packing is (the set S) a subset of S and the set of roots of the rooted trees containing v in the packing is a basis of M for every vertex v of G.

Let $\boldsymbol{\mathcal{G}} = (V, \mathcal{E})$ be a hypergraph with vertex set V and *hyperedge set* $\boldsymbol{\mathcal{E}}$. A *hyperedge* is a subset of V of size at least two. For any hypergraph $\mathcal{G}$, $\boldsymbol{V(\mathcal{G})}$ denotes its vertex set and $\boldsymbol{E(\mathcal{G})}$ its hyperedge set. A hypergraph $\mathcal{G}'$ is a *subhypergraph* of $\mathcal{G}$ if it is obtained from $\mathcal{G}$ by deleting some vertices and some hyperedges. For a partition $\mathcal{P}$ of V and a hyperedge set $\mathcal{H}$, we denote by $\boldsymbol{e_{\mathcal{H}}(\mathcal{P})}$ the number of hyperedges in $\mathcal{H}$ that are not contained in a member of $\mathcal{P}$. If $\mathcal{H}$ is an edge set H, then $\boldsymbol{e_H(\mathcal{P})}$ is the number of edges in H that are between different members of $\mathcal{P}$. The operation that consists in replacing a hyperedge Z by an edge whose end-vertices belong to Z is called *trimming*. The *trimming of a hypergraph* consists in the trimming of all its hyperedges, resulting in a graph. For a subhypergraph $\mathcal{T}$ of $\mathcal{G}$ and $s \in V(\mathcal{T})$, the couple $(s, \mathcal{T})$ is called a *rooted hypertree* of $\mathcal{G}$ if $\mathcal{T}$ can be trimmed to a graph T such that (s, T') is a rooted tree where T' is obtained from T by deleting the isolated vertices different from s. A set $\mathcal{B}$ of rooted hypertrees is called a *packing* if $\mathcal{B}$ can be trimmed to a packing $\mathcal{B}'$ of rooted trees. Furthermore, $\mathcal{B}$ is said to be (f, g)-*bounded*, (α, β)-*limited* and M-*based* if $\mathcal{B}'$ is (f, g)-bounded, (α, β)-limited and M-based.

3 Results on partitions

In this section we present and demonstrate the necessary results on functions on partitions. We hope that these results on supermodular functions on partitions will have interesting applications later on as well. Actually, the results of this section will allow us to prove the new results on packing trees in Section 4.

3.1 Submodularity on partitions

We here introduce two submodular and two supermodular functions on partitions of a set V.

We start with the following observation about the uncrossing method on partitions which comes from [7].

Claim 1 (Hoppenot, Szigeti [7]). *For all partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of a set V and $X \subseteq V$, we have*

(a) If X crosses $\mathcal{P}_1 \sqcup \mathcal{P}_2$, then it crosses both $\mathcal{P}_1$ and $\mathcal{P}_2$.

(b) If X crosses $\mathcal{P}_1 \sqcap \mathcal{P}_2$, then it crosses $\mathcal{P}_1$ or $\mathcal{P}_2$.

(c) $|\mathcal{P}_1| + |\mathcal{P}_2| = |\mathcal{P}_1 \sqcap \mathcal{P}_2| + |\mathcal{P}_1 \sqcup \mathcal{P}_2|$.

In the proof of Theorem 9.5.1 in [4], Frank proved that for a graph $G = (V, E)$, e_E is submodular on the partitions of V. We generalized this in [8] by showing that for a hypergraph $\mathcal{G} = (V, \mathcal{E})$, $e_{\mathcal{E}}$ is submodular on the partitions of V. Here we propose the following further extension that we will need to be able to introduce a new matroid in Subsection 4.2.

Lemma 1. *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph, $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{E}$ and $\mathcal{P}_1, \mathcal{P}_2$ partitions of V. Then*

$$e_{\mathcal{E}_1}(\mathcal{P}_1) + e_{\mathcal{E}_2}(\mathcal{P}_2) \geq e_{\mathcal{E}_1 \cap \mathcal{E}_2}(\mathcal{P}_1 \sqcap \mathcal{P}_2) + e_{\mathcal{E}_1 \cup \mathcal{E}_2}(\mathcal{P}_1 \sqcup \mathcal{P}_2). \tag{1}$$

Proof. Let $X \in (\mathcal{E}_1 \cup \mathcal{E}_2) \setminus (\mathcal{E}_1 \cap \mathcal{E}_2)$. Then X does not contribute to $e_{\mathcal{E}_1 \cap \mathcal{E}_2}(\mathcal{P}_1 \sqcap \mathcal{P}_2)$. If X contributes to the right hand side of (1), then $e_{\{X\}}(\mathcal{P}_1 \sqcup \mathcal{P}_2) = 1$, hence, by Claim 1(a), X contributes at least one to the left hand side of (1) and exactly one to the right hand side of (1). Let $X \in \mathcal{E}_1 \cap \mathcal{E}_2$. If $e_{\{X\}}(\mathcal{P}_1 \sqcup \mathcal{P}_2) = 1$, then, by Claim 1(a), X contributes two to the left hand side of (1) and at most two to the right hand side of (1). Suppose now that $e_{\{X\}}(\mathcal{P}_1 \sqcup \mathcal{P}_2) = 0$. If X contributes to the right hand side of (1), then $e_{\{X\}}(\mathcal{P}_1 \sqcap \mathcal{P}_2) = 1$, hence, by Claim 1(b), X contributes at least one to the left hand side of (1) and exactly one to the right hand side of (1). It follows that (1) holds. $\square$

If we are given an intersecting submodular function b on a set V and we define the value of a partition $\mathcal{P}$ of V as the sum of the b-values of the members of $\mathcal{P}$, then we obtain a submodular function on the partitions of V. We will need the following extension of this observation in the proof of the submodularity of the rank function of the above mentioned matroid.

Lemma 2. *Let S be a multiset of a set V and b an intersecting submodular function on S. For all $S^1, S^2 \subseteq S$ and $\mathcal{P}_1, \mathcal{P}_2$ partitions of V, we have*

$$\sum_{X \in \mathcal{P}_1} b(S^1_X) + \sum_{Y \in \mathcal{P}_2} b(S^2_Y) \geq \sum_{Z \in \mathcal{P}_1 \sqcap \mathcal{P}_2} b((S^1 \cap S^2)_Z) + \sum_{W \in \mathcal{P}_1 \sqcup \mathcal{P}_2} b((S^1 \cup S^2)_W). \tag{2}$$

Proof. Let S^1, S^2 be subsets of S and $\mathcal{P}_1, \mathcal{P}_2$ partitions of V. Let $\boldsymbol{\mathcal{Q}} = \{(X, S^1_X) : X \in \mathcal{P}_1\} \cup \{(Y, S^2_Y) : Y \in \mathcal{P}_2\}$. For $U \subseteq V$ and $R \subseteq S_U$, let $\text{val}(U, R) = b(R)$. We define $\text{val}(\mathcal{Q}) = \sum_{(U,R) \in \mathcal{Q}} \text{val}(U, R) = \sum_{X \in \mathcal{P}_1} b(S^1_X) + \sum_{Y \in \mathcal{P}_2} b(S^2_Y)$. We say that two couples (U_1, R_1) and (U_2, R_2) in $\mathcal{Q}$ are *properly intersecting* if U_1 and U_2 are properly intersecting sets. The uncrossing of such two couples is the operation that replaces them by $(U_1 \cap U_2, R_1 \cap R_2)$ and $(U_1 \cup U_2, R_1 \cup R_2)$. We apply the uncrossing operation on $\mathcal{Q}$ to obtain a new family $\mathcal{Q}''$ which contains no properly intersecting couples. Actually, the uncrossing of $\mathcal{Q}$ will mimic the uncrossing of $\mathcal{P}_1$ and $\mathcal{P}_2$. Note that in each step $Z = U_1 \cap U_2$ will be a member of $\mathcal{P}_1 \sqcap \mathcal{P}_2$ and that $R_1 \cap R_2 = S^1_Z \cap S^2_Z$. Note also that when $W = U_1 \cup U_2$ becomes a member of $\mathcal{P}_1 \sqcup \mathcal{P}_2$, then $R_1 \cup R_2 = S^1_W \cup S^2_W$. It follows that the value of $\mathcal{Q}''$ is $\sum_{Z \in \mathcal{P}_1 \sqcap \mathcal{P}_2} b((S^1 \cap S^2)_Z) + \sum_{W \in \mathcal{P}_1 \sqcup \mathcal{P}_2} b((S^1 \cup S^2)_W)$. If $\mathcal{Q}_{i+1}$ is obtained from $\mathcal{Q}_i$ by uncrossing two couples, then, by the intersecting submodularity of b, we have $\text{val}(\mathcal{Q}_i) \geq \text{val}(\mathcal{Q}_{i+1})$. Hence the lemma follows. $\square$

We introduce two supermodular functions on partitions that we will need later.

Claim 2. *Let S be a multiset of a set V, $\beta \in \mathbb{Z}_+$, $f, g : V \to \mathbb{Z}_+$ functions, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. Let the functions p_1 and p_2 be defined as follows. For every partition $\mathcal{P}$ of V,*

$$\boldsymbol{p_1}(\mathcal{P}) = -g(V) + \sum_{X \in \mathcal{P}} \max\{r_{\mathsf{M}}(S) + g(Y) - r_{\mathsf{M}}(S_Y) : Y \subseteq X\}, \tag{3}$$

$$\boldsymbol{p_2}(\mathcal{P}) = -\beta + \sum_{X \in \mathcal{P}} \max\{r_{\mathsf{M}}(S) + f(Y) - r_{\mathsf{M}}(S_Y) : Y \subseteq X\}. \tag{4}$$

The functions p_1 and p_2 are supermodular on partitions of V.

Proof. Since $r_{\mathsf{M}}(S)$ is constant, g and f are modular and r_{M} is submodular, $r_{\mathsf{M}}(S)+g(\cdot)-r_{\mathsf{M}}(S_\cdot)$ and $r_{\mathsf{M}}(S)+f(\cdot)-r_{\mathsf{M}}(S_\cdot)$ are supermodular. Then, by [4, Theorem 14.3.1], the functions p_g and p_f, defined by $p_g(X) := \max\{r_{\mathsf{M}}(S)+g(Y)-r_{\mathsf{M}}(S_Y) : Y \subseteq X\}$ and $p_f(X) := \max\{r_{\mathsf{M}}(S)+f(Y)-r_{\mathsf{M}}(S_Y) : Y \subseteq X\}$, are supermodular. It follows, by Lemma 2 applied for $S = S^1 = S^2 = V$ to $-p_g$ and $-p_f$, that $\sum_{X\in\mathcal{P}} p_g(X)$ and $\sum_{X\in\mathcal{P}} p_f(X)$ are supermodular on partitions of V. Then, since $g(V)$ and β are constant, we may conclude that the functions p_1 and p_2 are supermodular on partitions of V. $\square$

3.2 Covering two supermodular functions on the partitions

In edge-connectivity augmentation problems the aim is to cover a function on the subsets of vertices by a set of edges. The directed version was considered in Corollary 2.48 of [5]. Here we have to cover a function on partitions of a vertex set by an edge set. We can even cover two such functions simultaneously.

Theorem 1. *Let p_1 and p_2 be supermodular functions on the partitions of a set V and $\gamma \in \mathbb{Z}_+$. There exists an edge set F on V of size γ such that*

$$e_F(\mathcal{P}) \geq \max\{p_1(\mathcal{P}), p_2(\mathcal{P})\} \qquad \text{for every partition } \mathcal{P} \text{ of } V \tag{5}$$

if and only if

$$0 \geq \max\{p_1(\{V\}), p_2(\{V\})\}, \tag{6}$$

$$\gamma \geq \max\{p_1(\mathcal{P}), p_2(\mathcal{P})\} \qquad \text{for every partition } \mathcal{P} \text{ of } V. \tag{7}$$

Proof. Since the necessity is immediate, we only prove the sufficiency. It is enough to prove the theorem for $\boldsymbol{\gamma} = \max\{p_1(\mathcal{P}), p_2(\mathcal{P}) : \mathcal{P} \text{ partition of } V\}$. The proof is by induction on γ. If $\gamma = 0$, then there is nothing to prove. Suppose that the theorem is true for $\gamma - 1 \geq 0$. Let $\boldsymbol{p_1}$ and $\boldsymbol{p_2}$ be two supermodular functions on the partitions of a set V such that p_1, p_2 and γ satisfy (6) and (7). Let $\boldsymbol{\mathcal{Q}_1} := \{\mathcal{P} \text{ partition of } V : p_1(\mathcal{P}) = \gamma\}$ and $\boldsymbol{\mathcal{Q}_2} := \{\mathcal{P} \text{ partition of } V : p_2(\mathcal{P}) = \gamma\}$. Note that at least one of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is not empty.

Claim 3. *If $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{Q}_i$, then $\mathcal{P}_1 \sqcap \mathcal{P}_2, \mathcal{P}_1 \sqcup \mathcal{P}_2 \in \mathcal{Q}_i$ for $i = 1, 2$.*

Proof. By $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{Q}_i$, p_i is supermodular on the partitions of V, and (7), we have $\gamma + \gamma = p_i(\mathcal{P}_1) + p_i(\mathcal{P}_2) \leq p_i(\mathcal{P}_1 \sqcap \mathcal{P}_2) + p_i(\mathcal{P}_1 \sqcup \mathcal{P}_2) \leq \gamma + \gamma$, so equality holds everywhere and the claim follows. $\square$

If $\mathcal{Q}_i \neq \emptyset$, then let $\boldsymbol{X_i}$ be a maximal set among the members of the partitions in $\mathcal{Q}_i$ and $\boldsymbol{\mathcal{P}_i^1} \in \mathcal{Q}_i$ such that $X_i \in \mathcal{P}_i^1$. Since, by $\mathcal{P}_i^1 \in \mathcal{Q}_i$, $\gamma > 0$, and (6), we have $p_i(\mathcal{P}_i^1) = \gamma > 0 \geq p_i(\{V\})$, we get that $\emptyset \neq X_i \neq V$. Thus there exists $u_i \in X_i$ and $v_i \in \overline{X}_i$.

Therefore, if $\mathcal{Q}_1 \neq \emptyset \neq \mathcal{Q}_2$, then there exists $u, v \in V$ such that both X_1 and X_2 separate u and v. If only one of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is non-empty, say $\mathcal{Q}_i$, then let $u = u_i$ and $v = v_i$.

Claim 4. *$e_{uv}(\mathcal{P}) = 1$ for every $\mathcal{P} \in \mathcal{Q}_1 \cup \mathcal{Q}_2$.*

Proof. Suppose that there exists a partition in $\mathcal{Q}_1 \cup \mathcal{Q}_2$, say $\boldsymbol{\mathcal{P}_i^2} \in \mathcal{Q}_i$ such that $e_{uv}(\mathcal{P}_i^2) = 0$, that is a set $\boldsymbol{Y_i} \in \mathcal{P}_i^2$ contains both u and v. Recall that X_i contains exactly one of u and v, say u. By Claim 3, we have $\mathcal{P}_i^1 \sqcup \mathcal{P}_i^2 \in \mathcal{Q}_i$. By Claim 1, we get that an element of $\mathcal{P}_i^1 \sqcup \mathcal{P}_i^2$ contains X_i and an element of $\mathcal{P}_i^1 \sqcup \mathcal{P}_i^2$ contains Y_i. Since $u \in X_i \cap Y_i$, it follows that an element $\boldsymbol{Z_i}$ of $\mathcal{P}_i^1 \sqcup \mathcal{P}_i^2$ contains $X_i \cup Y_i$. The fact that $X_i \subset X_i \cup \{v\} \subseteq X_i \cup Y_i \subseteq Z_i$ contradicts the maximality of X_i. $\square$

Let $\boldsymbol{p_i'}(\mathcal{P}) = p_i(\mathcal{P}) - e_{uv}(\mathcal{P})$ for every partition $\mathcal{P}$ of V and for $i = 1, 2$. Since, by assumption and (1), p_i and $-e_{uv}$ are supermodular on the partitions of V, so is p_i'. Note that, by (6), we have $\max\{p_1'(\{V\}), p_2'(\{V\})\} = \max\{p_1(\{V\}), p_2(\{V\})\} \leq 0$. Let $\boldsymbol{\gamma'} = \max\{p_1'(\mathcal{P}), p_2'(\mathcal{P}) : \mathcal{P} \text{ partition of } V\}$. By Claim 4, we have $\gamma' = \gamma - 1$. Then, by induction, there exists an edge set $\boldsymbol{F'}$ on V of size γ' such that $e_{F'}(\mathcal{P}) \geq \max\{p_1'(\mathcal{P}), p_2'(\mathcal{P})\}$ for every partition $\mathcal{P}$ of V. Let $\boldsymbol{F} = F' \cup \{uv\}$. Then $|F| = |F'| + 1 = \gamma' + 1 = \gamma$ and $e_F(\mathcal{P}) = e_{F'}(\mathcal{P}) + e_{uv}(\mathcal{P}) \geq p_i'(\mathcal{P}) + e_{uv}(\mathcal{P}) = p_i(\mathcal{P})$ for every partition $\mathcal{P}$ of V and for $i = 1, 2$. Hence F is the desired edge set for p_1 and p_2. $\square$

3.3 Trimming on partitions

It was proved in [7] that a hypergraph covering two particular supermodular functions on partitions of a set can be trimmed to a graph covering the same functions. Here we provide the general form of it. We will use this to extend Theorem 10 to hypergraphs. We hope there will be other applications of Theorem 2 later on.

Theorem 2. *Let p_1 and p_2 be supermodular functions on the partitions of V and $\mathcal{G} = (V, \mathcal{E})$ a hypergraph. Then $\mathcal{G}$ can be trimmed to a graph $G = (V, E)$ such that*

$$e_E(\mathcal{P}) \quad \geq \quad \max\{p_1(\mathcal{P}), p_2(\mathcal{P})\} \quad \textit{for every partition } \mathcal{P} \textit{ of } V \tag{8}$$

if and only if

$$e_{\mathcal{E}}(\mathcal{P}) \quad \geq \quad \max\{p_1(\mathcal{P}), p_2(\mathcal{P})\} \quad \textit{for every partition } \mathcal{P} \textit{ of } V. \tag{9}$$

Proof. Since the necessity is immediate, we only prove the sufficiency. We prove the theorem by induction on $\sum_{X\in\mathcal{E}} |X|$. If for every $X \in \mathcal{E}$, $|X| = 2$, then $\mathcal{G}$ is a graph and, (9) coincides with (8). Otherwise, there exists a hyperedge $\boldsymbol{X} \in \mathcal{E}$ of size at least 3. We show that we can remove a vertex from X without violating (9); and then the induction hypothesis completes the proof. Suppose for a contradiction that for every $v \in X$, (9) is violated after the removal of v from X. By $|X| \geq 3$, there exist at least two vertices of X, say v_1 and v_2, such that the removal v_1 and the removal of v_2 violate the same p_i. We fix this index i for the rest of the proof. Since this condition is satisfied before the removal of the vertex, there exist partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of V, such that $p_i(\mathcal{P}_1) = e_{\mathcal{E}}(\mathcal{P}_1)$ and $p_i(\mathcal{P}_2) = e_{\mathcal{E}}(\mathcal{P}_2)$, and $e_{\mathcal{E}}(\mathcal{P}_j)$ decreases when we remove v_j from X for $j = 1, 2$. It follows that $X \setminus \{v_j\}$ is contained in a member $\boldsymbol{Y_j}$ of $\mathcal{P}_j$ for $j = 1, 2$; and hence, by $|X| \geq 3$, we have $Y_1 \cap Y_2 \supseteq X \setminus \{v_1, v_2\} \neq \emptyset$. By $p_i(\mathcal{P}_1) = e_{\mathcal{E}}(\mathcal{P}_1)$ and $p_i(\mathcal{P}_2) = e_{\mathcal{E}}(\mathcal{P}_2)$, Lemma 1, (9), and since p_i is supermodular on the partitions of V, we obtain that

$$\begin{aligned}
p_i(\mathcal{P}_1) + p_i(\mathcal{P}_2) \quad &= \quad e_{\mathcal{E}}(\mathcal{P}_1) + e_{\mathcal{E}}(\mathcal{P}_2) \quad = \quad e_{\mathcal{E}\setminus\{X\}}(\mathcal{P}_1) + e_{\mathcal{E}\setminus\{X\}}(\mathcal{P}_2) \quad + \quad e_X(\mathcal{P}_1) + e_X(\mathcal{P}_2) \\
&\geq \quad e_{\mathcal{E}\setminus\{X\}}(\mathcal{P}_1 \sqcap \mathcal{P}_2) + e_{\mathcal{E}\setminus\{X\}}(\mathcal{P}_1 \sqcup \mathcal{P}_2) \quad + \quad e_X(\mathcal{P}_1 \sqcap \mathcal{P}_2) + e_X(\mathcal{P}_1 \sqcup \mathcal{P}_2) \\
&= \quad e_{\mathcal{E}}(\mathcal{P}_1 \sqcap \mathcal{P}_2) + e_{\mathcal{E}}(\mathcal{P}_1 \sqcup \mathcal{P}_2) \quad \geq \quad p_i(\mathcal{P}_1 \sqcap \mathcal{P}_2) + p_i(\mathcal{P}_1 \sqcup \mathcal{P}_2) \\
&\geq \quad p_i(\mathcal{P}_1) + p_i(\mathcal{P}_2).
\end{aligned}$$

We hence have equality everywhere, in particular, $e_X(\mathcal{P}_1) + e_X(\mathcal{P}_2) = e_X(\mathcal{P}_1 \sqcap \mathcal{P}_2) + e_X(\mathcal{P}_1 \sqcup \mathcal{P}_2)$. Thus, since X crosses both $\mathcal{P}_1$ and $\mathcal{P}_2$, we get that X also crosses $\mathcal{P}_1 \sqcup \mathcal{P}_2$. However, by Claim 1, we get that a member of $\mathcal{P}_1 \sqcup \mathcal{P}_2$ contains Y_1 and a member of $\mathcal{P}_1 \sqcup \mathcal{P}_2$ contains Y_2. Since $Y_1 \cap Y_2 \neq \emptyset$, it follows that a member of $\mathcal{P}_1 \sqcup \mathcal{P}_2$ contains $Y_1 \cup Y_2 \supseteq X$, which contradicts the fact that X crosses $\mathcal{P}_1 \sqcup \mathcal{P}_2$. $\square$

4 Results on packings

In the previous section we proved all the necessary tools to be applied in this section. We now may present the results on packing trees and we are ready to prove them. This section contains seven subsections containing more and more general results, starting with the basic result on packing spanning trees, and finishing with a result on augmentation for matroid-based (f, g)-bounded (α, β)-limited packing of rooted hyperforests.

4.1 Packing of spanning trees

The classic result on packing spanning trees is due to Nash-Williams [13] and Tutte [14]. We provide a new proof of it that imitates the proof of Lovász, which he gave in [12] for Edmonds' theorem on packing spanning arborescences, and is inspired by Theorem 10.4.4 in [4]. This method of Lovász has been successfully applied in more general settings as well. We hope that our method will also be applied later. We think that it is natural that this method works for the undirected case as well and that this fact is worth being known.

Theorem 3 (Nash-Williams [13], Tutte [14]). *Let $G = (V, E)$ be a graph and $k \in \mathbb{Z}_+$. There exists a packing of k spanning trees in G if and only if*

$$e_E(\mathcal{P}) \quad \geq \quad k(|\mathcal{P}| - 1) \qquad \textit{for every partition } \mathcal{P} \textit{ of } V. \tag{10}$$

Proof. We prove only the difficult direction that is, the sufficiency. Let $(G = (V, E), k)$ be a smallest counter-example. Then $k \geq 1$. The following claim is well-known, see for example in [4].

Claim 5. $|E| = k(|V| - 1)$.

Proof. By (10) applied for $\{v\}_{v \in V}$, we obtain that $|E| \geq k(|V| - 1)$. Suppose for a contradiction that $|E| > k(|V| - 1)$. Hence (10) is strict for the partition $\{v\}_{v \in V}$.

If there is no partition of V, other than $\{V\}$, that satisfies (10) with equality, then we can delete any edge of G to obtain G' that also satisfies (10). Hence, by the minimality of G, there exists a packing of k spanning trees in G', and hence in G, which is a contradiction.

So there is a partition $\boldsymbol{\mathcal{P}} \neq \{V\}$ of V that satisfies (10) with equality. Note that $\mathcal{P} \neq \{v\}_{v \in V}$. Then there exists a member $\boldsymbol{X} \in \mathcal{P}$ such that $1 < |X| < |V|$.

We show that both $G[X]$ and G/X satisfy (10). First, if a partition $\mathcal{P}'$ of X violated (10) in $G[X]$, then $\mathcal{P}'' = (\mathcal{P} \backslash \{X\}) \cup \mathcal{P}'$ would violate (10) in G. Indeed, $e_E(\mathcal{P}'') = e_E(\mathcal{P}) + e_{E(G[X])}(\mathcal{P}') < k(|\mathcal{P}| - 1) + k(|\mathcal{P}'| - 1) = k(|\mathcal{P}''| - 1)$. Second, if a partition $\mathcal{P}'$ of $V(G/X)$ violated (10) in G/X, then $\mathcal{P}'' = (\mathcal{P}' \setminus \{Y\}) \cup \{(Y \setminus \{v_X\}) \cup X\}$ would violate (10) in G, where the contracted vertex v_X is in $Y \in \mathcal{P}'$. Indeed, $e_E(\mathcal{P}'') = e_{E(G/X)}(\mathcal{P}') < k(|\mathcal{P}'| - 1) = k(|\mathcal{P}''| - 1)$.

Since $G[X]$ and G/X satisfy (10) and (G, k) is a smallest counter-example, there exist a packing $T_1, \ldots, T_k$ of k spanning trees in $G[X]$ and a packing $T_1', \ldots, T_k'$ of k spanning trees in G/X. By replacing the vertex v_X in each T_i' by T_i, we obtain a packing of k spanning trees in G, which is a contradiction. □

For any $\emptyset \neq X \subseteq V$ and $\mathcal{P}_X = \{X\} \cup \{v\}_{v \in V \setminus X}$, by $|E| = k(|V| - 1)$ and (10), we get

$$i_E(X) - k(|X| - 1) = k|V \setminus X| - e_E(\mathcal{P}_X) = k(|\mathcal{P}_X| - 1) - e_E(\mathcal{P}_X) \leq 0. \tag{11}$$

Let $\boldsymbol{T} = (S, F)$ be a maximal tree in G satisfying (12). For $s \in V$, by (11), $T_s = (s, \emptyset)$ satisfies (12), so T exists.

$$i_{E \setminus F}(X) \quad \leq \quad k(|X| - 1) - |X \cap S| + 1 =: \boldsymbol{m(X)} \qquad \text{for every } \emptyset \neq X \subseteq V. \tag{12}$$

Lemma 3. *T is a spanning tree of G.*

Proof. Suppose that $S \neq V$. A vertex set X is called *tight* if (12) holds with equality. By (11), every tight set intersects S. A tight set X is *dangerous* if $X \setminus S \neq \emptyset$. Note that, by $|E| = k(|V| - 1)$ and $S \neq V$, V is dangerous. Thus, there exists a minimal dangerous set $\boldsymbol{X}$. Then, by (12) and (11), we have $d_{E \setminus F}(X \cap S, X \setminus S) = i_{E \setminus F}(X) - i_{E \setminus F}(X \cap S) - i_{E \setminus F}(X \setminus S) \geq k(|X| - 1) - |X \cap S| + 1 - (k - 1)(|X \cap S| - 1) - k(|X \setminus S| - 1) = k \geq 1$, so there exists an edge uv from $u \in X \cap S$ to $v \in X \setminus S$. Note that $T' = (S \cup \{v\}, F \cup \{uv\})$ is a tree. By the maximality of T, we get that there exists a set Y such that $i_{E \setminus (F \cup \{uv\})}(Y) > k(|Y| - 1) - |Y \cap (S \cup \{v\})| + 1$. Then, by (12), we get that $v \in Y$ and $u \in V \setminus Y$. Observe that $v \in (X \cap Y) \setminus S$. Then, by the tightness of X and Y, the modularity of m, $X \cap Y \neq \emptyset$, (12), and supermodularity of $i_{E \setminus F}$, we have $i_{E \setminus F}(X) + i_{E \setminus F}(Y) = m(X) + m(Y) = m(X \cap Y) + m(X \cup Y) \geq i_{E \setminus F}(X \cap Y) + i_{E \setminus F}(X \cup Y) \geq i_{E \setminus F}(X) + i_{E \setminus F}(Y)$, so equality holds everywhere. In particular, $X \cap Y$ is dangerous. Since $u \in X \setminus Y$, this contradicts the minimality of X, and the proof of the lemma is completed. □

By Lemma 3, T is a spanning tree of G, so $S = V$ and $|F| = |V| - 1$. Then, by $|E| = k(|V| - 1)$ and (12), for every partition $\mathcal{P}$ of V, we have $(k - 1)(|V| - 1) - e_{E \setminus F}(\mathcal{P}) = \sum_{X \in \mathcal{P}} i_{E \setminus F}(X) \leq \sum_{X \in \mathcal{P}} (k - 1)(|X| - 1) = (k - 1)(|V| - 1) - (k - 1)(|\mathcal{P}| - 1)$, that is $(G - F, k - 1)$ satisfies the condition (10). Hence, by the minimality of (G, k), there exists a packing of $k - 1$ spanning trees in $G - F$. By adding the spanning tree T of G, we get a packing of k spanning trees in G which is a contradiction. □

4.2 Matroid-based packing of rooted trees

A nice extension of Theorem 3 with some matroid constraint was proposed in [11].

Theorem 4 (Katoh, Tanigawa [11])**.** *Let $G = (V, E)$ be a graph, S a multiset of vertices in V and $\mathsf{M} = (S, \mathcal{I}_{\mathsf{M}})$ a matroid with rank function r_{M}. There exists a complete M-based packing of rooted trees in G if and only if*

$$\begin{aligned} S_v &\in \mathcal{I}_{\mathsf{M}} && \text{for every } v \in V, && (13)\\ e_E(\mathcal{P}) &\geq \sum_{X\in\mathcal{P}} (r_{\mathsf{M}}(S) - r_{\mathsf{M}}(S_X)) && \text{for every partition } \mathcal{P} \text{ of } V. && (14) \end{aligned}$$

If S is a multiset of vertices in V of size k and M is the free matroid on S, then Theorem 4 reduces to Theorem 3.

Theorem 4 was deduced from the following result in [11].

Theorem 5 (Katoh, Tanigawa [11])**.** *Let $G = (V, E)$ be a graph, S a multiset of vertices in V and $\mathsf{M} = (S, \mathcal{I}_{\mathsf{M}})$ a matroid with rank function r_{M} that satisfies (13). Let the function r_{KT} be defined on E as follows, for every $F \subseteq E$,*

$$r_{KT}(F) = r_{\mathsf{M}}(S)|V| - |S| + \min\{e_F(\mathcal{P}) - \sum_{X\in\mathcal{P}} (r_{\mathsf{M}}(S) - r_{\mathsf{M}}(S_X)) : \mathcal{P} \text{ partition of } V\}. \tag{15}$$

(a) Then r_{KT} is the rank function of a matroid M_{KT}.
(b) $F \subseteq E$ is the edge set of a complete M-based packing of rooted trees in G if and only if

$$F \text{ is independent in } \mathsf{M}_{KT}, \tag{16}$$
$$|F| = r_{\mathsf{M}}(S)|V| - |S|. \tag{17}$$

We propose the following more general result to be applied later.

Theorem 6. *Let $G = (V, E)$ be a graph, S a multiset of vertices in V, and $\mathsf{M} = (S, \mathcal{I}_{\mathsf{M}})$ a matroid with rank function r_{M}. Let the function $\boldsymbol{r'_{KT}}$ be defined on $E \cup S$ as follows, for every $F \subseteq E, T \subseteq S$,*

$$r'_{KT}(F \cup T) = r_{\mathsf{M}}(S)|V| + \min\{e_F(\mathcal{P}) - \sum_{X\in\mathcal{P}} (r_{\mathsf{M}}(S) - r_{\mathsf{M}}(T_X)) : \mathcal{P} \text{ partition of } V\}. \tag{18}$$

(a) Then r'_{KT} is the rank function of a matroid $\boldsymbol{\mathsf{M}'_{KT}}$.
(b) $F \subseteq E$ is the edge set and $T \subseteq S$ is the root set of an M-based packing of rooted trees in G if and only if

$$F \cup T \text{ is independent in } \mathsf{M}'_{KT}, \tag{19}$$
$$|F \cup T| = r_{\mathsf{M}}(S)|V|. \tag{20}$$

Proof. (a) It is clear that r'_{KT} is integer-valued. By (18), $e_.(\cdot) \geq 0$, and $r_{\mathsf{M}}(T_.) \geq 0$, we have $r'_{KT}(F \cup T) = \min\{e_F(\mathcal{P}) + \sum_{X\in\mathcal{P}}((|X|-1)r_{\mathsf{M}}(S) + r_{\mathsf{M}}(T_X)) : \mathcal{P}$ partition of $V\} \geq 0$ for every $F \subseteq E, T \subseteq S$. Since the functions $e_.(\mathcal{P})$ and $r_{\mathsf{M}}(\cdot_X)$ are non-decreasing for fixed $\mathcal{P}$ and X, r'_{KT} is also non-decreasing. For every $F \subseteq E, T \subseteq S$, by taking the partition $\{v\}_{v\in V}$ in (18), and by the subcardinality of r_{M}, we have

$$r'_{KT}(F \cup T) \leq r_{\mathsf{M}}(S)|V| + |F| - r_{\mathsf{M}}(S)|V| + \sum_{v\in V} r_{\mathsf{M}}(T_v) \leq |F| + \sum_{v\in V} |T_v| = |F \cup T|,$$

so r'_{KT} is subcardinal.

To show the submodularity of r'_{KT}, let $F_1, F_2 \subseteq E, T^1, T^2 \subseteq S$, and $\mathcal{P}_1, \mathcal{P}_2$ the partitions that provide $r'_{KT}(F_1 \cup T^1)$ and $r'_{KT}(F_2 \cup T^2)$. Then, by Lemma 1 applied for F_1 and F_2 and Lemma 2 applied for $b(\cdot) = r_{\mathsf{M}}(\cdot) - r_{\mathsf{M}}(S)$, we have

$$\begin{aligned}
r'_{KT}(F_1 \cup T^1) + r'_{KT}(F_2 \cup T^2) &= r_{\mathsf{M}}(S)|V| + e_{F_1}(\mathcal{P}_1) + \sum_{X \in \mathcal{P}_1} (r_{\mathsf{M}}(T^1_X) - r_{\mathsf{M}}(S)) \\
&+ r_{\mathsf{M}}(S)|V| + e_{F_2}(\mathcal{P}_2) + \sum_{Y \in \mathcal{P}_2} (r_{\mathsf{M}}(T^2_Y) - r_{\mathsf{M}}(S)) \\
&\geq r_{\mathsf{M}}(S)|V| + e_{F_1 \cap F_2}(\mathcal{P}_1 \sqcap \mathcal{P}_2) + \sum_{Z \in \mathcal{P}_1 \sqcap \mathcal{P}_2} (r_{\mathsf{M}}((T^1 \cap T^2)_Z) - r_{\mathsf{M}}(S)) \\
&+ r_{\mathsf{M}}(S)|V| + e_{F_1 \cup F_2}(\mathcal{P}_1 \sqcup \mathcal{P}_2) + \sum_{W \in \mathcal{P}_1 \sqcup \mathcal{P}_2} (r_{\mathsf{M}}((T^1 \cup T^2)_W) - r_{\mathsf{M}}(S)) \\
&\geq r'_{KT}((F_1 \cap F_2) \cup (T^1 \cap T^2)) + r'_{KT}((F_1 \cup F_2) \cup (T^1 \cup T^2)) \\
&= r'_{KT}((F_1 \cup T^1) \cap (F_2 \cup T^2)) + r'_{KT}((F_1 \cup T^1) \cup (F_2 \cup T^2)).
\end{aligned}$$

Thus r'_{KT} is submodular. From the previous arguments, it follows that r'_{KT} is the rank function of a matroid M'_{KT}.

(b) To prove the **necessity**, suppose that $F \subseteq E$ is the edge set and $T \subseteq S$ is the root set of an M-based packing of rooted trees in G. Then, $T_v \in \mathcal{I}_{\mathsf{M}}$ for every $v \in V$. So, by Theorem 5 applied for T, $r_{\mathsf{M}}(S)|V| = |F| + |T| = r_{KT}(F) + |T| = r_{\mathsf{M}}(S)|V| + \min\{e_F(\mathcal{P}) - \sum_{X \in \mathcal{P}} (r_{\mathsf{M}}(S) - r_{\mathsf{M}}(T_X)) : \mathcal{P}$ partition of $V\} = r'_{KT}(F \cup T)$, hence (19) and (20) hold.

To prove the **sufficiency**, suppose that (19) and (20) hold for some $F \subseteq E$ and $T \subseteq S$. Then, by taking the partition $\mathcal{P} = \{V\}$ in (18) and by the monotonicity of r_{M}, we get that $0 = |F \cup T| - r_{\mathsf{M}}(S)|V| = r'_{KT}(F \cup T) - r_{\mathsf{M}}(S)|V| \leq r_{\mathsf{M}}(T) - r_{\mathsf{M}}(S) \leq 0$, so $r_{\mathsf{M}}(T) = r_{\mathsf{M}}(S)$. Note also that, by (19), T is independent in M'_{KT}. Then, by taking the partition $\{v\}_{v \in V}$ in (18) and by the subcardinality of r_{M}, we have $|T| = r'_{KT}(T) \leq r_{\mathsf{M}}(S)|V| - r_{\mathsf{M}}(S)|V| + \sum_{v \in V} r_{\mathsf{M}}(T_v) \leq \sum_{v \in V} |T_v| = |T|$, so equivality holds everywhere, that is $T_v \in \mathcal{I}_{\mathsf{M}}$ for every $v \in V$. Let $\mathsf{M}|_T$ be the matroid obtained from M by restricting it on T. Then $T_v \in \mathcal{I}_{\mathsf{M}|_T}$ for every $v \in V$. Let M^T_{KT} be the matroid of Theorem 5 with ground set T. Then, for its rank function, we have $r^T_{KT}(F) = r_{\mathsf{M}}(T)|V| - |T| + \min\{e_F(\mathcal{P}) - \sum_{X \in \mathcal{P}} (r_{\mathsf{M}}(T) - r_{\mathsf{M}}(T_X)) : \mathcal{P}$ partition of $V\} = r'_{KT}(F \cup T) - |T| = |F \cup T| - |T| = |F|$, so F is independent in M^T_{KT}. Then, by Theorem 5, there exists a complete $\mathsf{M}|_T$-based packing of rooted trees in G whose edge-set is F, and we are done. □

Let us clarify the relation between the matroids M_{KT} and M'_{KT}.

Claim 6. *If (13) holds, then S is independent in M'_{KT} and $\mathsf{M}'_{KT}/S = \mathsf{M}_{KT}$.*

Proof. For the partition $\mathcal{P}$ that provides $r'_{KT}(S)$ in (18), by the monotonicity of r_{M} and (13), we have

$$\begin{aligned}
|S| \geq r'_{KT}(S) &= r_{\mathsf{M}}(S)|V| - \sum_{X \in \mathcal{P}} (r_{\mathsf{M}}(S) - r_{\mathsf{M}}(S_X)) \\
&= \sum_{X \in \mathcal{P}} (r_{\mathsf{M}}(S_X) + (|X| - 1) r_{\mathsf{M}}(S)) \geq \sum_{X \in \mathcal{P}} \sum_{v \in X} r_{\mathsf{M}}(S_v) = \sum_{X \in \mathcal{P}} \sum_{v \in X} |S_v| = |S|.
\end{aligned}$$

Further, by (15) and (18), we have $r_{KT}(F) = r'_{KT}(F \cup S) - |S| = r'_{KT}(F \cup S) - r'_{KT}(S) = (r'_{KT})_{/S}(F)$ for every $F \subseteq E$. □

4.3 Matroid-based (f, g)-bounded packing of k rooted trees

The aim of this subsection is to extend Theorem 4 when we have two kinds of constraints on the roots of the rooted trees in the packing. In order to do so we need the following result that was introduced in [4] and proved in [9, Theorem 12].

Theorem 7 ([4], [9])**.** *Let $\{S_1,\dots,S_n\}$ be a partition of a set S, $\alpha_i,\beta_i\in\mathbb{Z}_+$ for all $i\in\{1,\dots,n\}$ and $\mu\in\mathbb{Z}_+$. Let*

$$\begin{aligned}\mathcal{B} &= \{Z\subseteq S:\alpha_i\le|Z\cap S_i|\le\beta_i \text{ for } i=1,\dots,n, |Z|=\mu\}, && (21)\\ r(Z) &= \min\{\sum_{i=1}^n\min\{\beta_i,|Z\cap S_i|\},\mu-\sum_{i=1}^n\max\{\alpha_i-|Z\cap S_i|,0\}\} \text{ for every } Z\subseteq S. && (22)\end{aligned}$$

There exists a matroid whose set of bases is $\mathcal{B}$ and rank function is r if and only if

$$\begin{aligned}\alpha_i &\le \min\{\beta_i,|S_i|\} \quad \text{for all } i\in\{1,\dots,n\}, && (23)\\ \sum_{i=1}^n\alpha_i\le\mu&\le\sum_{i=1}^n\min\{\beta_i,|S_i|\}. && (24)\end{aligned}$$

This matroid is called generalized partition matroid*.*

We also need the matroid intersection theorem of Edmonds [1].

Theorem 8 (Edmonds [1])**.** *Let $\mathsf{M}_1=(S,r_1)$ and $\mathsf{M}_2=(S,r_2)$ be two matroids on S, and $\mu\in\mathbb{Z}_+$. A common independent set of size μ of M_1 and M_2 exists if and only if*

$$r_1(Z)+r_2(S\setminus Z)\ge\mu \quad \text{for all } Z\subseteq S. \tag{25}$$

We are now able to present and prove an extension of Theorem 4.

Theorem 9. *Let $G=(V,E)$ be a graph, S a multiset of vertices in V, $k\in\mathbb{Z}_+$, $f,g:V\to\mathbb{Z}_+$ functions, and $\mathsf{M}=(S,r_\mathsf{M})$ a matroid. There exists an M-based (f,g)-bounded packing of k rooted trees in G if and only if*

$$\begin{aligned}f(v) &\le \min\{r_\mathsf{M}(S_v),g(v)\} && \text{for every } v\in V, && (26)\\ k &\le \sum_{v\in V}\min\{r_\mathsf{M}(S_v),g(v)\}, &&&& (27)\\ e_E(\mathcal{P}) &\ge \sum_{X\in\mathcal{P}}\max\{r_\mathsf{M}(S)-r_\mathsf{M}(S_Y)-g(X\setminus Y):Y\subseteq X\} && \text{for every partition } \mathcal{P} \text{ of } V, && (28)\\ e_E(\mathcal{P})+k &\ge \sum_{X\in\mathcal{P}}\max\{r_\mathsf{M}(S)-r_\mathsf{M}(S_Y)+f(Y):Y\subseteq X\} && \text{for every partition } \mathcal{P} \text{ of } V. && (29)\end{aligned}$$

Proof. To prove the **necessity**, let $\boldsymbol{\mathcal{B}}$ be an M-based (f,g)-bounded packing of k rooted trees with root set $\boldsymbol{T}$. Since r_M is non-decreasing and $\mathcal{B}$ is M-based, $r_\mathsf{M}(S_v)\ge r_\mathsf{M}(T_v)=|T_v|$ for every $v\in V$. Then, since $\mathcal{B}$ is (f,g)-bounded, we have $\min\{r_\mathsf{M}(S_v),g(v)\}\ge|T_v|\ge f(v)$, so (26) holds. Further, we get $\sum_{v\in V}\min\{r_\mathsf{M}(S_v),g(v)\}\ge\sum_{v\in V}|T_v|=|T|=k$, so (27) holds. Moreover, since $\mathcal{B}$ is M-based, we get $r_\mathsf{M}(S)=r_\mathsf{M}(T)$. So, by Theorem 4 applied for T, the submodularity and the subcardinality of r_M, we have

$$e_E(\mathcal{P})\ge\sum_{X\in\mathcal{P}}(r_\mathsf{M}(S)-r_\mathsf{M}(T_X))\ge\sum_{X\in\mathcal{P}}\max\{r_\mathsf{M}(S)-r_\mathsf{M}(T_Y)-|T_{X\setminus Y}|:Y\subseteq X\}. \tag{30}$$

By (30), the monotonicity of r_M and $g(v)\ge|T_v|$ for every $v\in V$, we get that (28) holds. By $|T_v|\ge f(v)$ for every $v\in V$, we have $|T_{X\setminus Y}|\le|T_X|-f(Y)$ for every $Y\subseteq X\subseteq V$. So, by (30), the monotonicity of r_M and $\sum_{X\in\mathcal{P}}|T_X|=|T|=k$, (29) holds.

To prove the **sufficiency**, let us suppose that (26)–(29) hold. We may suppose that (13) holds. Indeed, by taking a maximal $S^*\subseteq S$ such that $S^*_v\in\mathcal{I}_\mathsf{M}$ for every $v\in V$, we have $r_\mathsf{M}(S^*_X)=r_\mathsf{M}(S_X)$ for every $X\subseteq V$, so (26)–(29) still hold. From now on we suppose that (13) holds. We formulate our problem as the intersection of two matroids. The first matroid is M'_{KT} with rank function r'_{KT} given in (18). The second matroid M_2 is the direct sum of the uniform matroid on ground set E of rank $r_{\mathsf{M}}(S)|V|-k$ and the generalized partition matroid on ground set S with the partition $\{S_v\}_{v\in V}$ and the bounds $(\alpha_v,\beta_v)=(f(v),g(v))$ and $\mu=k$. The rank function of M_2 satisfies, by (22), for all $F\subseteq E, T\subseteq S$,

$$r_2(F\cup T)=\min\{|F|, r_{\mathsf{M}}(S)|V|-k\}+\min\{\sum_{v\in V}\min\{g(v),|T_v|\}, k-\sum_{v\in V}\max\{f(v)-|T_v|,0\}\}. \quad (31)$$

Note that, by Theorem 7 and $r_{\mathsf{M}}(S_v)=|S_v|$ for every $v\in V$ (by (13)), the generalized partition matroid exists if and only if (26) and (27) hold and $f(V)\leq k$ (which holds by (29)).

Claim 7. *There exists an* M*-based* (f,g)*-bounded packing of* k *rooted trees with edge set* F *and root set* T *if and only if* $F\cup T$ *is a common independent set of* M'_{KT} *and* M_2 *of size* $r_{\mathsf{M}}(S)|V|$.

Proof. To prove the **necessity**, let $\mathcal{B}$ be an M-based (f,g)-bounded packing of k rooted trees with root set T. By Theorem 6, $F\cup T$ is an independent set of M'_{KT} of size $r_{\mathsf{M}}(S)|V|$. Since $\mathcal{B}$ is an (f,g)-bounded packing of k rooted trees, we have $f(v)\leq|T_v|\leq g(v)$ and $|T|=k$. Then $|F|=r_{\mathsf{M}}(S)|V|-k$. It follows that $F\cup T$ is an independent set of M_2, and we are done.

To prove the **sufficiency**, let us suppose that $F\cup T$ is a common independent set of M'_{KT} and M_2 of size $r_{\mathsf{M}}(S)|V|$. By Theorem 6, there exists an M-based packing $\mathcal{B}$ of rooted trees with edge set F and root set T. Since $F\cup T$ is independent in M_2 of size $r_{\mathsf{M}}(S)|V|$, we have $|T|=k$ and $f(v)\leq|T_v|\leq g(v)$ for every $v\in V$. Hence $\mathcal{B}$ is an (f,g)-bounded packing of k rooted trees, and we are done. □

By Claim 7 and Theorem 8, there exists an M-based (f,g)-bounded packing of k rooted trees if and only if

$$\min\{r'_{KT}(F\cup T)+r_2(\overline{F}\cup\overline{T}) : F\subseteq E, T\subseteq S\}\geq r_{\mathsf{M}}(S)|V|. \quad (32)$$

Let $F\subseteq E$ and $T\subseteq S$ attain the minimum.

Case 1. If $|\overline{F}|\geq r_{\mathsf{M}}(S)|V|-k$. Then, $r_2(E\cup\overline{T})=r_2(\overline{F}\cup\overline{T})$. Hence, since r'_{KT} is non-decreasing, we have $r'_{KT}(T)+r_2(E\cup\overline{T})\leq r'_{KT}(F\cup T)+r_2(\overline{F}\cup\overline{T})$, so we may suppose that $F=\emptyset$. Since, by Claim 6, S is independent in M'_{KT}, so is T. Hence, we have

$$r'_{KT}(T) \;=\; |T| \;=\; \sum_{v\in V}|T_v|. \quad (33)$$

By (31), there are two cases to consider.

Case (a) If $r_2(E\cup\overline{T})=r_{\mathsf{M}}(S)|V|-k+\sum_{v\in V}\min\{g(v),|\overline{T}_v|\}$, then, by (33) and (27), we have

$$\begin{aligned} r'_{KT}(T)+r_2(E\cup\overline{T})-r_{\mathsf{M}}(S)|V| &= \sum_{v\in V}|T_v|-k+\sum_{v\in V}\min\{g(v),|\overline{T}_v|\} \\ &\geq -k+\sum_{v\in V}\min\{g(v),|S_v|\} \;\geq\; 0. \end{aligned}$$

Case (b) If $r_2(E\cup\overline{T})=r_{\mathsf{M}}(S)|V|-k+k-\sum_{v\in V}\max\{f(v)-|\overline{T}_v|,0\}$, then, by (33) and (26), we have

$$\begin{aligned} r'_{KT}(T)+r_2(E\cup\overline{T})-r_{\mathsf{M}}(S)|V| &= \sum_{v\in V}(|T_v|-\max\{f(v)-|\overline{T}_v|,0\}) \\ &= \sum_{v\in V}\min\{|T_v|,|S_v|-f(v)\} \;\geq\; 0. \end{aligned}$$

Case 2. If $|\overline{F}|<r_{\mathsf{M}}(S)|V|-k$. Then, $r_2(\overline{T})=r_2(\overline{F}\cup\overline{T})-|\overline{F}|$. Hence, by the submodularity and the subcardinality of r_{M}, we have $r'_{KT}(E\cup T)+r_2(\overline{T})\leq r'_{KT}(F\cup T)+|\overline{F}|+r_2(\overline{F}\cup\overline{T})-|\overline{F}|$, so we may suppose that $F=E$. Note that

$$r'_{KT}(E\cup T)=r_{\mathsf{M}}(S)|V|+\min\{e_E(\mathcal{P})-\sum_{X\in\mathcal{P}}(r_{\mathsf{M}}(S)-r_{\mathsf{M}}(T_X)) : \mathcal{P} \text{ partition of } V\}. \quad (34)$$

By (31), there are two cases to consider.

Case (a) If $r_2(\overline{T}) = \sum_{v\in V} \min\{g(v), |\overline{T}_v|\}$. By (13), the modularity of g, the monotonicity and submodularity of r_{M}, for every $X \subseteq V$, there exists $\boldsymbol{Y_X} \subseteq X$ such that we have

$$\begin{aligned} r_{\mathsf{M}}(T_X) + \sum_{v\in X} \min\{g(v), |\overline{T}_v|\} &= r_{\mathsf{M}}(T_X) + \sum_{v\in X\setminus Y_X} g(v) + \sum_{v\in Y_X} r_{\mathsf{M}}(\overline{T}_v) \\ &\geq r_{\mathsf{M}}(T_{Y_X}) + g(X\setminus Y_X) + r_{\mathsf{M}}(\overline{T}_{Y_X}) \geq r_{\mathsf{M}}(S_{Y_X}) + g(X\setminus Y_X). \end{aligned} \tag{35}$$

Then, by (34), (35), and (28), we have

$$\begin{aligned} & r'_{KT}(E\cup T) + r_2(\overline{T}) - r_{\mathsf{M}}(S)|V| \\ = \;& \min\{e_E(\mathcal{P}) + \sum_{X\in\mathcal{P}} (r_{\mathsf{M}}(T_X) - r_{\mathsf{M}}(S) + \sum_{v\in X} \min\{g(v), |\overline{T}_v|\}) : \mathcal{P} \text{ partition of } V\} \\ \geq \;& \min\{e_E(\mathcal{P}) + \sum_{X\in\mathcal{P}} (r_{\mathsf{M}}(S_{Y_X}) - r_{\mathsf{M}}(S) + g(X\setminus Y_X)) : \mathcal{P} \text{ partition of } V\} \geq 0. \end{aligned}$$

Case (b) If $r_2(\overline{T}) = k - \sum_{v\in V} \max\{f(v) - |\overline{T}_v|, 0\}$. By (13), the modularity of f, the monotonicity and submodularity of r_{M}, for every $X \subseteq V$, there exists $\boldsymbol{Y_X} \subseteq X$ such that we have

$$\begin{aligned} r_{\mathsf{M}}(T_X) - \sum_{v\in X} \max\{f(v) - |\overline{T}_v|, 0\} &= r_{\mathsf{M}}(T_X) + \sum_{v\in Y_X} (r_{\mathsf{M}}(\overline{T}_v) - f(v)) \\ &\geq r_{\mathsf{M}}(T_{Y_X}) + r_{\mathsf{M}}(\overline{T}_{Y_X}) - f(Y_X) \geq r_{\mathsf{M}}(S_{Y_X}) - f(Y_X). \end{aligned} \tag{36}$$

Then, by (34), (36), and (29), we have

$$\begin{aligned} & r'_{KT}(E\cup T) + r_2(\overline{T}) - r_{\mathsf{M}}(S)|V| \\ = \;& \min\{e_E(\mathcal{P}) + \sum_{X\in\mathcal{P}} (r_{\mathsf{M}}(T_X) - r_{\mathsf{M}}(S) - \sum_{v\in X} \max\{f(v) - |\overline{T}_v|, 0\}) + k : \mathcal{P} \text{ partition of } V\} \\ \geq \;& \min\{e_E(\mathcal{P}) + k + \sum_{X\in\mathcal{P}} (r_{\mathsf{M}}(S_{Y_X}) - r_{\mathsf{M}}(S) - f(Y_X)) : \mathcal{P} \text{ partition of } V\} \geq 0. \end{aligned}$$

It follows that in every case (32) holds, and hence the required packing exists. □

If $f(v) = 0$ and $g(v) = \infty$ for every $v \in V$ and $k = |S|$, then Theorem 9 reduces to Theorem 4.

4.4 Matroid-based (f, g)-bounded (α, β)-limited packing of rooted trees

Theorem 9 can easily be extended to packings where the number of rooted trees is not given but is lower and upper bounded.

Theorem 10. *Let $G = (V, E)$ be a graph, S a multiset of vertices in V, $\alpha, \beta \in \mathbb{Z}_+$, $f, g : V \to \mathbb{Z}_+$ functions, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. There exists an M-based (f, g)-bounded (α, β)-limited packing of rooted trees in G if and only if* (26) *and* (28) *hold and*

$$\alpha \leq \beta, \tag{37}$$

$$\alpha \leq \sum_{v\in V} \min\{r_{\mathsf{M}}(S_v), g(v)\}, \tag{38}$$

$$e_E(\mathcal{P}) + \beta \geq \sum_{X\in\mathcal{P}} \max\{r_{\mathsf{M}}(S) - r_{\mathsf{M}}(S_Y) + f(Y) : Y \subseteq X\} \quad \textit{for every partition } \mathcal{P} \textit{ of } V. \tag{39}$$

Proof. To prove the **necessity**, let $\boldsymbol{\mathcal{B}}$ be an M-based (f, g)-bounded (α, β)-limited packing of rooted trees with root set $\boldsymbol{T}$. Let $k = |T|$. Since $\mathcal{B}$ is (α, β)-limited, we have $\alpha \leq k \leq \beta$. Hence, (37) holds. Further, by Theorem 9, we get that (26), (27) (and hence (38)), (28) and (29) (and hence (39)) hold.

To prove the **sufficiency**, let us suppose that (26), (28), (37), (38), and (39) hold. We show that there exists an integer k that satisfies $\alpha \leq k \leq \beta$, (27) and (29). By (37), (38), and (39), it is enough to prove that for every partition $\mathcal{P}$ of V, we have

$$\sum_{v \in V} \min\{r_{\mathsf{M}}(S_v), g(v)\} \quad \geq \quad \sum_{X \in \mathcal{P}} \max\{r_{\mathsf{M}}(S) - r_{\mathsf{M}}(S_Y) + f(Y) : Y \subseteq X\} - e_E(\mathcal{P}). \tag{40}$$

Let $\mathcal{P}$ be a partition of V. For every $X \in \mathcal{P}$, let $\boldsymbol{Y_X, Y'_X} \subseteq X$ such that

$$\sum_{v \in X} \min\{r_{\mathsf{M}}(S_v), g(v)\} \quad = \quad \sum_{v \in Y_X} r_{\mathsf{M}}(S_v) + g(X \setminus Y_X), \tag{41}$$

$$\max\{r_{\mathsf{M}}(S) - r_{\mathsf{M}}(S_Y) + f(Y) : Y \subseteq X\} \quad = \quad r_{\mathsf{M}}(S) - r_{\mathsf{M}}(S_{Y'_X}) + f(Y'_X). \tag{42}$$

Then, by (41), (42), the submodularity of r_{M}, the modularity of g and f, (26) applied for $Y_X \cap Y'_X$ and for $Y'_X \setminus Y_X$, and (28), we have

$$\begin{aligned}
& \sum_{v \in V} \min\{r_{\mathsf{M}}(S_v), g(v)\} - \sum_{X \in \mathcal{P}} \max\{r_{\mathsf{M}}(S) - r_{\mathsf{M}}(S_Y) + f(Y) : Y \subseteq X\} \\
= \quad & \sum_{X \in \mathcal{P}} \Big(\sum_{v \in Y_X} r_{\mathsf{M}}(S_v) + g(X \setminus Y_X) + r_{\mathsf{M}}(S_{Y'_X}) - f(Y'_X) - r_{\mathsf{M}}(S) \Big) \\
\geq \quad & \sum_{X \in \mathcal{P}} \Big(\sum_{v \in Y_X \cap Y'_X} r_{\mathsf{M}}(S_v) + r_{\mathsf{M}}(S_{Y_X \cup Y'_X}) + g(X \setminus (Y_X \cup Y'_X)) + g(Y'_X \setminus Y_X) \\
& \qquad - f(Y_X \cap Y'_X) - f(Y'_X \setminus Y_X) - r_{\mathsf{M}}(S) \Big) \\
\geq \quad & \sum_{X \in \mathcal{P}} \big(r_{\mathsf{M}}(S_{Y_X \cup Y'_X}) + g(X \setminus (Y_X \cup Y'_X)) - r_{\mathsf{M}}(S) \big) \\
\geq \quad & \sum_{X \in \mathcal{P}} \min\{r_{\mathsf{M}}(S_Y) + g(X \setminus Y) - r_{\mathsf{M}}(S) : Y \subseteq X\} \\
= \quad & -\sum_{X \in \mathcal{P}} \max\{r_{\mathsf{M}}(S) - r_{\mathsf{M}}(S_Y) - g(X \setminus Y) : Y \subseteq X\} \geq -e_E(\mathcal{P}),
\end{aligned}$$

so (40) holds. Then, by Theorem 9, there exists an M-based (f,g)-bounded packing of k rooted trees in G. Since $\alpha \leq k \leq \beta$, the packing is (α,β)-limited, and we are done. $\square$

If $\alpha = \beta = k$, then Theorem 10 reduces to Theorem 9.

4.5 Matroid-based (f,g)-bounded (α,β)-limited packing of rooted hypertrees

Theorem 3 was generalized to hypergraphs in [6].

Theorem 11 (Frank, Király, Kriesell [6]). *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph and $k \in \mathbb{Z}_+$. There exists a packing of k spanning hypertrees in $\mathcal{G}$ if and only if*

$$e_{\mathcal{E}}(\mathcal{P}) \quad \geq \quad k(|\mathcal{P}| - 1) \qquad \textit{for every partition } \mathcal{P} \textit{ of } V. \tag{43}$$

If $\mathcal{G}$ is a graph, then Theorem 11 reduces to Theorem 3. In fact, Theorem 11 can easily be proved by Theorems 2 and 3.

We will now exploit the fact that in Theorem 2 we can treat not only one but two supermodular functions on partitions. We can hence generalize Theorem 10 to hypergraphs.

Theorem 12. *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph, S a multiset of vertices in V, $\alpha, \beta \in \mathbb{Z}_+$, $f, g : V \to \mathbb{Z}_+$ functions, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists an M-based (f, g)-bounded (α, β)-limited packing of rooted hypertrees in $\mathcal{G}$ if and only if* (26)*,* (37)*, and* (38) *hold and*

$$e_\mathcal{E}(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} \max\{r_\mathsf{M}(S) - r_\mathsf{M}(S_Y) - g(X \setminus Y) : Y \subseteq X\} \quad \text{for every partition } \mathcal{P} \text{ of } V, \tag{44}$$

$$\beta + e_\mathcal{E}(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} \max\{r_\mathsf{M}(S) - r_\mathsf{M}(S_Y) + f(Y) : Y \subseteq X\} \quad \text{for every partition } \mathcal{P} \text{ of } V. \tag{45}$$

Proof. To prove the **necessity**, suppose that there exists an M-based (f, g)-bounded (α, β)-limited packing of rooted hypertrees in $\mathcal{G}$. Then, by definition, the hypertrees in the packing can be trimmed to get an M-based (f, g)-bounded (α, β)-limited packing of rooted trees. Then, by Theorem 10, we get that (26), (28), (37), (38), and (39) hold. Then (44) and (45) hold in $\mathcal{G}$.

To prove the **sufficiency**, suppose that (26), (37), (38), (44), and (45) hold. Note that (44) and (45) are equivalent to $e_\mathcal{E}(\mathcal{P}) \geq p_1(\mathcal{P})$ and $e_\mathcal{E}(\mathcal{P}) \geq p_2(\mathcal{P})$ for every partition $\mathcal{P}$ of V, where the functions p_1 and p_2 are defined in (3) and (4). By Claim 2, p_1 and p_2 are supermodular on partitions of V. Thus, by Theorem 2, $\mathcal{G}$ can be trimmed to a graph G that satisfies (28) and (39). Since (26), (37), and (38) hold by assumption, Theorem 10 implies that there exists an M-based (f, g)-bounded (α, β)-limited packing of rooted trees in G. By replacing each edge of G by the hyperedge that was trimmed to it, we obtain an M-based (f, g)-bounded (α, β)-limited packing of rooted hypertrees in $\mathcal{G}$. ☐

If $\mathcal{G}$ is a graph, then Theorem 12 reduces to Theorem 10. If S is a multiset of vertices in V of size k, M is the free matroid on S, $\alpha = \beta = k$, $f(v) = 0$ and $g(v) = \infty$ for every $v \in V$, then Theorem 12 reduces to Theorem 11.

4.6 Augmentation for matroid-based (f, g)-bounded (α, β)-limited packing of rooted hypertrees

Frank [3] solved the augmentation version of Theorem 3 in which a minimum number of edges must be added to a graph to have a packing of k spanning trees.

Theorem 13 (Frank [3])**.** *Let $G = (V, E)$ be a graph and $k, \gamma \in \mathbb{Z}_+$, We can add γ edges to G to have a packing of k spanning trees if and only if*

$$\gamma + e_E(\mathcal{P}) \geq k(|\mathcal{P}| - 1) \quad \text{for every partition } \mathcal{P} \text{ of } V. \tag{46}$$

If $\gamma = 0$, then Theorem 13 reduces to Theorem 3. Theorem 13 can be easily proved by Theorem 1 applied for $p_1(\mathcal{P}) = p_2(\mathcal{P}) = k(|\mathcal{P}| - 1) - e_E(\mathcal{P})$. Note that, by Claim 1(c) and Lemma 1, $p_1 = p_2$ is a supermodular function on partitions.

We will now exploit the fact that in Theorem 1 we can treat two different supermodular functions on partitions. We can hence propose a common generalization of Theorems 12 and 13.

Theorem 14. *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph, S a multiset of vertices in V, $\alpha, \beta, \gamma \in \mathbb{Z}_+$, $f, g : V \to \mathbb{Z}_+$ functions, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. We can add γ edges to $\mathcal{G}$ to have an M-based (f, g)-bounded (α, β)-limited packing of rooted hypertrees if and only if* (26)*,* (37)*, and* (38) *hold and*

$$r_\mathsf{M}(S) - r_\mathsf{M}(S_Y) \leq \min\{\beta - f(Y), g(\overline{Y})\} \quad \text{for every } Y \subseteq V, \tag{47}$$

$$g(V) + \gamma + e_\mathcal{E}(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} \max\{r_\mathsf{M}(S) - r_\mathsf{M}(S_Y) + g(Y) : Y \subseteq X\} \text{ for every partition } \mathcal{P} \text{ of } V, \tag{48}$$

$$\beta + \gamma + e_\mathcal{E}(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} \max\{r_\mathsf{M}(S) - r_\mathsf{M}(S_Y) + f(Y) : Y \subseteq X\} \text{ for every partition } \mathcal{P} \text{ of } V. \tag{49}$$

Proof. To prove the **necessity**, suppose that we can add an edge set F of size γ to $\mathcal{G}$ to have an M-based (f,g)-bounded (α,β)-limited packing of rooted hypertrees in $\mathcal{G}+F=(V,\mathcal{E}')$. Then, by Theorem 12, we get that (26), (37), (38), (44), and (45) hold for $\mathcal{E}'$. Applying (44) and (45) for $\mathcal{P}=\{V\}$, we get (47). Since $e_{\mathcal{E}'}(\mathcal{P})\le e_{\mathcal{E}}(\mathcal{P})+\gamma$, (44) and (45) imply (48) and (49).

To prove the **sufficiency**, suppose that (26), (37), (38), (47), (48), and (49) hold. Let the functions $\boldsymbol{p_1'}$ and $\boldsymbol{p_2'}$ be defined as follows. For every partition $\mathcal{P}$ of V, $p_1'(\mathcal{P})=p_1(\mathcal{P})-e_{\mathcal{E}}(\mathcal{P})$ and $p_2'(\mathcal{P})=p_2(\mathcal{P})-e_{\mathcal{E}}(\mathcal{P})$, where p_1 and p_2 are defined in (3) and (4). By Claim 2 and Lemma 1, p_1' and p_2' are supermodular on partitions of V. By (47), we get that (6) holds for p_1' and p_2'. By (48) and (49), we get that (7) holds for p_1' and p_2'. Hence Theorem 1 implies that there exists an edge set F on V of size γ such that $e_F(\mathcal{P})\ge\max\{p_1'(\mathcal{P}),p_2'(\mathcal{P})\}$ for every partition $\mathcal{P}$ of V. This means that in the hypergraph $\mathcal{G}'=(V,\mathcal{E}'=\mathcal{E}\cup F)$, (44) and (45) hold for $\mathcal{E}'$. Since (26), (37), and (38) also hold, by Theorem 12, there exists an M-based (f,g)-bounded (α,β)-limited packing of rooted hypertrees in $\mathcal{G}'$, which completes the proof of Theorem 14. $\square$

If $\gamma=0$, then Theorem 14 reduces to Theorem 12. If $\mathcal{G}$ is a graph, S is a multiset of vertices in V of size k and M is the free matroid on S, $f(v)=0$ and $g(v)=\infty$ for every $v\in V$ and $\alpha=\beta=k$, then Theorem 14 reduces to Theorem 13.

4.7 Augmentation for matroid-based (f,g)-bounded (α,β)-limited packing of rooted hyperforests

We conclude by mentioning that Theorem 14 can be generalized for rooted hyperforests. In order to present this result we need the following definitions.

A *forest* of G is a subgraph of G that contains no cycle. A couple (S,F) is a *rooted forest* of G if F is a forest of G and $S\subseteq V(F)$ contains exactly one vertex of each connected component of F. The set S is called the *root set* of the rooted forest. The couple $(S,\mathcal{F})$ is a *rooted hyperforest* if $\mathcal{F}$ can be trimmed to a forest F such that for the graph F' obtained from F by deleting the isolated vertices not in S, (S,F') is a rooted forest. By a *packing* of rooted forests in G, we mean a set $\mathcal{B}$ of rooted forests of G that are edge disjoint. For two functions $f,g:V\to\mathbb{Z}_+$, we say that the packing $\mathcal{B}$ is (f,g)-*bounded* if for every vertex v of G, v is a root in at least $f(v)$ and at most $g(v)$ rooted forests in $\mathcal{B}$. For two non-negative integers α and β, we say that the packing $\mathcal{B}$ is (α,β)-*limited* if the total number of roots in the rooted forests in $\mathcal{B}$ is at least α and at most β. For a family $\mathcal{S}$ of subsets of V and a matroid M on $\mathcal{S}$, a packing of rooted forests in G is called M-*based* if there exists $S'\subseteq S$ for every $S\in\mathcal{S}$ such that $\{S':S\in\mathcal{S}\}$ is the set of the root sets of the rooted forests in the packing and for every vertex v of G, $\{S\in\mathcal{S}:$ rooted forests (S',F) in the packing contains $v\}$ is a basis of M. A set $\mathcal{B}$ of rooted hyperforests in $\mathcal{G}$ is called a *packing* if $\mathcal{B}$ can be trimmed to a packing $\mathcal{B}'$ of rooted forests. Furthermore, $\mathcal{B}$ is said to be (f,g)-*bounded*, (α,β)-*limited* and M-*based* if $\mathcal{B}'$ is (f,g)-bounded, (α,β)-limited and M-based.

The argument in [10] showing that Theorem 4 in [10] implies Theorem 5 in [10] can be applied here as well. Hence Theorem 14 implies the following result.

Theorem 15. *Let $\mathcal{G}=(V,\mathcal{E})$ be a hypergraph, $\mathcal{S}$ a family of subsets of V, $\alpha,\beta,\gamma\in\mathbb{Z}_+$, $f,g:V\to\mathbb{Z}_+$ functions, and $\mathsf{M}=(\mathcal{S},r_{\mathsf{M}})$ a matroid. We can add γ edges to $\mathcal{G}$ to have an M-based (f,g)-bounded (α,β)-limited packing of rooted hyperforests if and only if* (37) *and* (47) *hold and*

$$f(v)\le\min\{r_{\mathsf{M}}(\mathcal{S}_v),g(v)\}\quad\text{for every } v\in V, \tag{50}$$

$$\alpha\le\sum_{v\in V}\min\{r_{\mathsf{M}}(\mathcal{S}_v),g(v)\}, \tag{51}$$

$$g(V)+\gamma+e_{\mathcal{E}}(\mathcal{P})\ge\sum_{X\in\mathcal{P}}\max\{r_{\mathsf{M}}(\mathcal{S})-r_{\mathsf{M}}(\mathcal{S}_Y)+g(Y):Y\subseteq X\}\quad\text{for every partition } \mathcal{P} \text{ of } V, \tag{52}$$

$$\beta+\gamma+e_{\mathcal{E}}(\mathcal{P})\ge\sum_{X\in\mathcal{P}}\max\{r_{\mathsf{M}}(\mathcal{S})-r_{\mathsf{M}}(\mathcal{S}_Y)+f(Y):Y\subseteq X\}\quad\text{for every partition } \mathcal{P} \text{ of } V. \tag{53}$$

If $\mathcal{S} = \{S_v\}_{v \in V}$, then Theorem 15 reduces to Theorem 14. Note that Theorem 15 implies all the results of this section.

References

[1] J. Edmonds, Submodular functions, matroids, and certain polyhedra, in: eds. R. Guy, H. Hanani, N. Sauer, and J. Schönheim, Combinatorial Structures and their Applications, Gordon and Breach, New York, 69–70, 1970.

[2] J. Edmonds, *Edge-disjoint branchings*, in Combinatorial Algorithms, B. Rustin ed., Academic Press, New York, (1973) 91–96.

[3] A. Frank, *Edge-connection of graphs, digraphs and hypergraphs*, More sets, graphs and numbers, (E. Győri, G. Katona, L. Lovász, eds.) Bolyai Mathematical Society Math. Studies 15, Springer Verlag (2006) 93–142.

[4] A. Frank, Connections in Combinatorial Optimization, Oxford University Press, 2011.

[5] A. Frank, T. Király, A Survey on Covering Supermodular Functions, Bonn Workshop of Combinatorial Optimization (2008) 87–126.

[6] A. Frank, T. Király, M. Kriesell, *On decomposing a hypergraph into k connected sub-hypergraphs,* Discret. Appl. Math. 131 (2) (2003) 373–383.

[7] P. Hoppenot, M. Martin, Z. Szigeti, *Regular packing of rooted hyperforests with root constraints in hypergraph*, Math. Program. 213 (2025) 1211–1233.

[8] P. Hoppenot, Z. Szigeti, *On arborescence packing augmentation in hypergraphs*, submitted, (2024)

[9] F. Hörsch, Z. Szigeti, *Packing of mixed hyperarborescences with flexible roots via matroid intersection*, Electronic Journal of Combinatorics 28 (3) (2021) P3.29.

[10] F. Hörsch, Z. Szigeti, *Reachability in arborescence packings*, Discret. Appl. Math. 320 (2022) 170–183.

[11] N. Katoh, S. Tanigawa, *Rooted-tree decomposition with matroid constrains and the infinitesimal rigidity of frameworks with boundaries*, SIAM J. Discret. Math. 27 (1) (2013) 155–185.

[12] L. Lovász, *On two minimax theorems in graphs*, J. Comb. Theory, Ser. B 21 (1976) 96–103.

[13] C. St. J. A. Nash-Williams, *Edge-disjoints spanning trees of finite graphs*, Journal of the London Mathematical Society 36 (1961) 445–450.

[14] W.T. Tutte, *On the problem of decomposing a graph into n connected factors*, Journal of the London Mathematical Society 36 (1961) 221–230.