Perfect matchings versus odd cuts

Zoltán Szigeti*

Abstract

We give a simple proof for an important result of Edmonds, Lovász and Pulleyblank, stating that a brick has no non-trivial tight cuts. Our proof relies on some results on almost critical graphs. The introduction of these graphs is the second aim of the present paper.

1 Introduction

In this paper we introduce almost critical graphs, graphs having a critical making edge, that is an edge whose contraction leaves a factor-critical graph. We shall characterize the subgraph induced by the critical making edges. Using this characterization we present a new proof of the Tight Cut Lemma of Edmonds, Lovász and Pulleyblank. We shall apply some earlier results in matching theory, they will be given together with short proofs.

Edmonds, Lovász and Pulleyblank [1] determined the dimension of the perfect matching polytope of an arbitrary graph. They used the so-called brick decomposition and the most complicated part was to show that the result is true for bricks. In this case the problem is equivalent to saying that no brick contains a non-trivial tight cut. This is the Tight Cut Lemma. Their proof for this lemma contains some linear programming arguments, the description of the perfect matching polytope, the uncrossing technique and some graph theoretic arguments. Here we provide a purely graph theoretic proof.

The organization of the paper is as follows. In Section 2 some easy (well-known) claims are presented. In Section 3 we recall two results on elementary graphs. The first one is due to Lovász and Plummer [4] and it easily follows from Tutte's Theorem, while the second one follows from earlier results on ear-decompositions. We do not want to use ear-decompositions in this paper, thus a short proof is given for Claim 6. In Section 4 we generalize a result of Lovász [3] and an earlier result of the author [6] concerning factor-critical graphs. In Section 5 we recall some results on strong subgraphs due to Frank [2]. The results of Section 6 can be found either in Frank [2] or in Szigeti [5]. We shall prove each result in this section to emphasize their simplicity. The main result here is the characterization of the subgraph defined by the critical making edges of an almost critical graph. Section 7 contains a new lemma on bricks, which will be used in Section 8 to prove the Tight Cut Lemma. (We mention that the first ten claims in the paper are known.)

2 Preliminaries

A set M of edges is called a *matching* if no two edges in M have a common end vertex. A matching M of a graph G is *perfect* if M covers all the vertices of G. An edge of a graph G is *allowed* if it lies in some perfect matching of G. If G has a perfect matching then $N(G)$ denotes the subgraph of G induced by the allowed edges of G. For a subgraph F of G and a matching M of G, the subset of M contained in F is denoted by $M(F)$. A matching is called *near perfect* if it

*Equipe Combinatoire, Université Paris 6, 75252 Paris, Cedex 05, France

covers all but one vertices of G. A graph G is *factor-critical* if for each vertex v of G the graph $G-v$ has a perfect matching. Note that the addition of some new edges to a factor-critical graph results in a factor-critical graph.

For a subset S of vertices, $\delta(S)$ denotes the set of edges leaving S. $\delta(S)$ is a *cut*. Note that if M is a perfect matching of G then

$$|\delta(S)\cap M| \equiv |S| \quad (2). \tag{1}$$

A connected component H of a graph G is called *odd* (*even*) if $|V(H)|$ is odd (even). For $X\subseteq V(G)$, $c_o(G-X)$ denotes the number of odd components in $G-X$. The vertex set of the union of the even components of $G-X$ is denoted by C_X. We shall use frequently Tutte's Theorem [8] on the existence of perfect matchings.

Theorem 1 *A graph G has a perfect matching if and only if*

$$c_o(G-X)\le |X| \textit{ for all } X\subseteq V(G). \tag{2}$$

Let G be a graph with a perfect matching. A vertex set X is called a *barrier* if $c_o(G-X)=|X|$.

The vertex set of a graph G will be denoted by $V(G)$. The degree of a vertex v is denoted by $d(v)$. For a set $S\subseteq V(G)$, $\overline{S}=V(G)-S$, $G[S]$ denotes the subgraph of G induced by the vertex set S. The graph obtained by contracting a subgraph H of G will be denoted by G/H. The new vertex of $G-H$ will be denoted by v_H. This notation will be used frequently without any reference.

A graph G is called *2-connected* if $|V(G)|\ge 3$ and for every vertex v of G, $G-v$ is connected. If G is not 2-connected, then the maximal 2-connected subgraphs are called *blocks* of G. A graph G is called *3-connected* if $|V(G)|\ge 4$ and for every subset X of vertices of G with $|X|\le 2$, $G-X$ is connected.

The following claim easily follows from Theorem 1.

Claim 2 *Let v be a vertex of a graph G so that $G-v$ has no perfect matching and G has an odd number of vertices.*

(2.1) Then there exists a vertex set X containing v such that $c_o(G-X)>|X|$ and all the components of $G-X$ are factor-critical.

(2.2) Moreover, if there exists a vertex u so that $G-u$ has a perfect matching then u lies in one of the factor-critical components of $G-X$ and $c_o(G-X)=|X|+1$.

□

Claim 3 *Let G be a graph with a perfect matching and let X be a barrier in G.*

(3.1) $G[C_X]$ has a perfect matching.

(3.2) The connected components of $N(G-C_X)$ are connected components of $N(G)$.

Proof. Since each perfect mathing M of G matches the vertices in X to the odd components of $G-X$, the restrictions of M in $G[C_X]$ and in $G-C_X$ are perfect matchings of $G[C_X]$ and $G-C_X$. This implies (3.1) and (3.2). □

Claim 4 *If for an edge e of G the graph G/e is factor-critical, then e is allowed in G.*

Proof. G/e is factor–critical, so $(G/e)-v_e$ has a perfect matching M'. Then $M:=M'\cup e$ is a perfect matching of G, that is e is an allowed edge of G. □

3 Elementary graphs

A connected graph G is *matching covered* if each edge of G is allowed. A graph G with a perfect matching is called *elementary* if the allowed edges form a connected subgraph. (Each matching-covered graph is elementary.) The following characterization of elementary graphs is obvious by Tutte's theorem, see [4].

Claim 5 *Let G be a graph with a perfect matching. Then G is elementary if and only if for each non-empty barrier X of G, $G - X$ has no even component.* $\square$

Claim 6 *If G is elementary then for each allowed edge e of G, G/e is factor-critical.*

Proof. Suppose that G/e is not factor-critical for an allowed edge e of G. Then, by Claim (2.1), there exists a set $X \neq \emptyset$ so that $c_o((G/e) - X) \geq |X| + 1$. Since e is allowed in G, $(G/e) - v_e$ has a perfect matching. Thus, by Claim (2.2), v_e lies in one of the odd components of $(G/e) - X$ (say in K) and $c_o((G/e) - X) = |X| + 1$. This component K corresponds to an even component of $G - X$, thus, by the above equality, $c_o(G - X) = |X|$ and hence X is a barrier of G. Then, by Claim 5, G is not elementary. This contradiction proves the claim. $\square$

4 Factor-critical graphs

We shall need the following claim due to Lovász [3].

Claim 7 *Let H be a subgraph of a graph G. If H and G/H are factor-critical, then G is factor-critical.* $\square$

Lemma 1.5 in [6] is similar to Claim 7 and it was proved there using ear-decompositions. Here we present a common generalization of Lemma 1.5 in [6] and Claim 7 that will be applied later. For a partition V_1, V_2 of the vertex set of a graph G, we shall denote by G_i the graph obtained from G by deleting all the edges with both ends in V_i and identifying all the vertices of V_i $(i = 1, 2)$. The contracted vertices will be denoted by v_1 and v_2 respectively.

Lemma 8 *Let $G = (V, E)$ be a graph and let V_1, V_2 be a partition of V. If*

(i) G_1 is factor–critical,

(ii) for each edge e incident to v_2, G_2/e is factor-critical, and

(iii) $G[V_2]$ is connected,

then G is factor–critical.

Proof. We have to show that for each vertex $v \in V(G)$, $G - v$ has a perfect matching.

a.) First assume that $v \in V_2$. Since G_1 is factor-critical by (i), $G_1 - v$ has a perfect matching M_1. Let e be the edge of M_1 incident to v_1. This edge corresponds to an edge incident to v_2 in G_2 (also denoted by e). Then e is allowed in G by (ii) and Claim 4 so there exists a perfect matching M_2 of G_2 containing e. Obviously, $M_1 \cup M_2$ is a perfect matching of $G - v$.

b.) Secondly, assume that $v \in V_1$. Note that $|V(G)|$ is odd by a.). Suppose that $G - v$ has no perfect matching. Then, by Claim 2, there exists a vertex set X containing v so that $c_o(G - X) > |X|$. Since for each vertex u of V_2, $G - u$ has a perfect matching by a.), Claim (2.2) implies that all the vertices of V_2 lie in the odd components of $G - X$, and $c_o(G - X) = |X| + 1$. Moreover, since $G[V_2]$ is connected by (iii), all the vertices of V_2 lie in one of the odd components of $G - X$. Let us denote this component of $G - X$ by T. Then the contraction of the vertex set V_2 changes neither X nor the odd components of $G - X$ different from T and, since $|V_2|$ is even by (i), T becomes an even component of $G_2 - X$. That is $c_o(G_2 - X) = c_o(G - X) - 1 = |X|$. Let e be an edge of G_2 incident to v_2. Then $c_0((G_2/e) - X) = c_0(G_2 - X) + 1 = |X| + 1$, that is, by Tutte's Theorem, $(G_2/e) - v$ contains no perfect matching (recall that $v \in X$), a contradiction, since G_2/e is factor-critical by (ii). $\square$

The reader can easily verify that Lemma 8 generalizes Claim 7. By Claim 6, it also generalizes Lemma 1.5 in [6].

5 Strong subgraphs

Let H be a graph with a perfect matching. A non-empty barrier X of H is said to be a *strong barrier* if $H - X$ has no even components, each of the odd components is factor–critical and the bipartite graph H_X obtained from H by deleting the edges spanned by X and by contracting each factor–critical component to a single vertex is matching covered. If H has a strong barrier then H is called *half–elementary*. Let $G = (V, E)$ be a graph and assume that the subgraph H of G induced by $U \subseteq V$ is half–elementary with a strong barrier X. Then H is said to be a *strong subgraph* of G with strong barrier X if X separates $U - X$ from $V - U$ or if $U = V$.

Claim 9 *Let G be a connected graph. Suppose that $G - X$ has at least $|X|$ factor–critical components for a non-empty vertex set X. Then there exists a strong subgraph H of G with strong barrier $Y \subseteq X$ such that all the factor–critical components of $H - Y$ are components of $G - X$ as well.*

Proof. Let us choose $|X|$ factor–critical components of $G - X$, say $K_1, \ldots, K_{|X|}$, and let $H' := G[X \cup V(K_1) \cup \ldots \cup V(K_{|X|})]$. One of the color classes of the bipartite graph H'_X is X while the other one corresponds to the components $K_i's$. This class will be denoted by Z. Note that $|\Gamma_{H'_X}(Z)| = |Z|$ because G is connected, where $\Gamma_{H'_X}(Z)$ denotes the set of neighbors of Z in the bipartite graph H'_X. Let W be a minimal non-empty set of Z so that $|\Gamma_{H'_X}(W)| = |W|$ in H'_X. Let $Y := \Gamma_{H'_X}(W)$. Then, by the minimality of W, for all $\emptyset \neq A \subset W$, $|\Gamma_{H'_X}(A)| \geq |A| + 1$. By a characterization of matching covered bipartite graphs (see [4]), the bipartite graph with color classes W and Y is matching covered. Thus the subgraph of G defined by Y and the factor-critical components of $G - X$ corresponding to the vertices in W is the desired strong subgraph. $\square$

Claim 10 *Let u be a vertex of a non factor-critical graph G so that $G - u$ has a perfect matching. Then there exists a strong subgraph H in G so that H does not contain u.*

Proof. By Claim (2.1), there exists a vertex set X such that $c_o(G - X) > |X|$ and all the components of $G - X$ are factor-critical. $G - u$ has a perfect matching, thus, by Claim (2.2), u lies in one of the factor-critical components of $G - X$, say in K. Let $G' := G - V(K)$. Then $c_o(G' - X) \geq |X|$ and all of these components of $G' - X$ are factor-critical. Thus, by Claim 9, G' contains a strong subgraph H with strong barrier $Y \subseteq X$ such that all the factor–critical components of $H - Y$ are components of $G - X$. Note that H does not contain u. Since all the neighbors of K are in X, H is a strong subgraph in G, and the claim is proved. $\square$

We shall need the following result of Frank [2].

Claim 11 *A connected graph G has a strong subgraph if and only if G is not factor-critical.*

Proof. If G contains a strong subgraph with strong barrier X, then for a vertex $x \in X$, $G - x$ has no perfect matching ($X - x$ violates the Tutte condition) thus G is not factor-critical.

On the other hand, suppose that G is not factor-critical. If G has an odd number of vertices, then by Claim (2.1) and Claim 9, we are done. If G has an even number of vertices and v is a vertex of G, then either $G - v$ is factor-critical and then G itself is a strong subgraph with strong barrier v, or $G - v$ is not factor-critical and then, by Claim (2.1), $G - v$ contains a set X' so that $c_o((G - v) - X') > |X'|$ and all components of $(G - v) - X'$ are factor-critical, and then, by Claim 9 applied for G and $X = X' \cup v$, we are done. □

6 Almost critical graphs

We say that G is *almost critical* if it contains an edge whose contraction leaves a factor-critical graph. We call such an edge *critical making*. Note that, by Claim 6, matching-covered graphs (and consequently elementary graphs) are almost critical. If G is almost critical, then $B(G)$ denotes the subgraph of G induced by the critical making edges of G. (The isolated vertices are deleted.)

The following two claims are due to Frank [2].

Claim 12 *Let H be a strong subgraph with strong barrier X in a graph G.*

(12.1) If e is an edge of H leaving X, then e is critical making in H. Whence, H is almost critical.

(12.2) If G is almost critical, then $G[X \cup C_X]$ contains no critical making edge of G. Whence, H contains all critical making edges of G.

Proof. H_X is matching-covered by definition, thus, by Claim 6, H_X/e is factor-critical. Using Claim 7, $|X|$ times, we can "blow up" the factor-critical components of $H - X$, one by one, showing that H/e is factor-critical, that is e is a critical making edge of H.

To show (12.2) assume that f is an edge in $G[X \cup C_X]$. Let us denote by X' the vertex set in $G' := G/f$ corresponding to X. Then $G' - X'$ has at least $|X| \geq |X'|$ factor-critical components. Since G is factor-critical, it is connected, so G' is connected. By Claim 9, G' has a strong subgraph, thus by Claim 11, $G' = G/f$ is not factor-critical, that is f is not critical making in G, which was to be proved. □

Claim 13 *The following are equivalent.*

(a) G is almost critical,

(b) G has a perfect matching and there exist no two vertex disjoint strong subgraphs in G,

(c) G has a strong subgraph and G/H is factor-critical for each strong subgraph H of G.

Proof. (a) implies (b): G is almost critical, that is it contains a critical making edge e. By Claim 4, e is allowed in G so G has a perfect matching. By Claim (12.2), e is contained in each strong subgraph, and we are done.

(b) implies (c): G has a perfect matching so G has a strong subgraph by Claim 11.

Let H be a strong subgraph of G with strong barrier X. Let $G' := G/H$. Suppose that G' is not factor-critical. G has a perfect matching by (b) so X is a barrier of G. By Claim (3.1), $G[C_X]$ has a perfect matching M. Then M is a perfect matching of $G' - v_H$, and hence, by Claim 10, G' contains a strong subgraph H' so that H' does not contain v_H. Since H' is also a strong subgraph in G, H and H' contradict (b).

(c) implies (a): G has a strong subgraph H by (c). Let e be an edge of H leaving the strong barrier X of H. By Claim (12.1), H/e is factor-critical. $(G/e)/(H/e) = G/H$ thus $(G/e)/(H/e)$ is factor-critical. Whence, by Claim 7, G/e is factor-critical, that is G is almost critical. □

In the following lemma we generalize Claim (12.1).

Lemma 14 *Let G be an almost critical graph. Let H be a strong subgraph of G with strong barrier X. Let $F_1, F_2, ..., F_{|X|}$ be the factor–critical components of $H - X$. Let $F_1^* := G/(G - V(F_1))$ and let $v := v_{G-V(F_1)}$. Then, for an edge $e \in E(F_1^*)$, the following are equivalent:*

(a) e is a critical making edge of F_1^*,

(b) e is a critical making edge of H,

(c) e is a critical making edge of G.

Proof. (a) implies (b): We may suppose that e not incident to v_{F_1}, otherwise, by Claim (12.1), we are done. Let us define the following sequence of graphs. Let $H_1 := H$ and let $H_i := H_{i-1}/F_i$ $(2 \leq i \leq |X|)$. Let H' be the graph obtained from $H_{|X|}/e$ by deleting the edges spanned by X. Let $V_2 := V(F_1^*/e) - v_{F_1}, V_1 := V(H') - V_2$. Then $H_1' := H'/H'[V_1] = F_1^*/e$ is factor-critical by assumption, $H_2' := H'/H'[V_2] = H_X$ is a matching covered bipartite graph because H is a strong subgraph, and obviously $H'[V_2]$ is connected. Thus, by Claim 6 and Lemma 8, H' is factor-critical. It follows that $H_{|X|}/e$ is factor-critical. By Claim 7, applied for H_i/e and F_i, $i = |X|, |X| - 1, ..., 2$, step by step, it follows that $H/e = H_1/e$ is factor-critical. Thus e is a critical making edge of H.

(b) implies (c): By assumption H/e is factor-critical. By Claim 13/(c), $(G/e)/(H/e) = G/H$ is also factor-critical. Thus, by Claim 7, G/e is factor-critical, that is e is critical making in G.

(c) implies (a): By assumption G/e is factor–critical. Since H is a strong subgraph of G with strong barrier X, for every vertex $u \in X \cup (V(F_1^*/e) - v_e)$, the restrictions of the perfect matchings of $G/e - u$ on F_1^*/e are near–perfect matchings of F_1^*/e. This implies that F_1^*/e is factor–critical, that is the edge e is a critical making edge in F_1^*. □

The main result of this section is Theorem 15. It characterizes the subgraph $B(G)$ defined by the critical making edges of an almost critical graph G.

Theorem 15 *Let G be an almost critical graph. Then*

(15.1) $B(G)$ coincides with one of the connected components of $N(G)$.

(15.2) $V(B(G)) = \bigcap \{V(H) : H \text{ is a strong subgraph in } G\}$.

Proof. We prove Theorem (15.1) by induction on the number of vertices. By Claim 4, G has a perfect matching. First assume that, for each barrier X, $G - X$ contains no even component. Then, by Claim 5, G is elementary. Thus $N(G)$ is connected and, by Claim 6 and Claim 4, $E(B(G)) = E(N(G))$ so we are done. Secondly, assume that there exists a barrier X for which $G - X$ contains an even component. Then, by Claim 9, there exists a strong subgraph H with strong barrier $Y \subseteq X$ and clearly $|V(H)| < |V(G)|$. Claim (12.2) and Lemma 14, an edge of G is critical making in G if and only if it is critical making in H. Using the induction hypothesis for H (H is almost critical by Claim (12.1)) and Claim (3.2), Theorem (15.1) follows.

Let us denote by U the vertex set on the right hand side in Theorem (15.2). By Claim (12.2), $V(B(G)) \subseteq U$. To show the other direction let $v \in U$. By Claim 4, G has a perfect matching M. Let e be an edge of M incident to v. Let $G' := G/e$. Then $G' - v_e$ has a perfect matching, namely $M - e$. If G' was not factor-critical, then by Claim 10, G' contains a strong subgraph H so that H does not contain v_e. It follows that H is a strong subgraph of G not containing v, which is a contradiction since $v \in U$. Thus G' is factor-critical, that is e is a critical making edge of G, thus $v \in V(B(G))$, and we are done. □

Theorem (15.1) implies immediately the following claim.

Claim 16 *Let G be an almost critical graph. Let $u, v \in V(B(G))$ and let $T \subset V(G)$ so that $u \in T$ and $v \notin T$. Then $\delta(T)$ contains an allowed edge of G.* □

Claim 17 *Let G be an almost critical graph. Let $T \subset V(G)$ so that $G[T]$ is connected and $T \cap V(B(G)) = \emptyset$. Then there exists a strong subgraph H such that $T \cap V(H) = \emptyset$.*

Proof. Let $v \in T$ be an arbitrary vertex of T. Since, by assumption, $v \notin V(B(G))$, it follows by Theorem (15.2) that there exists a strong subgraph H with strong barrier X so that $v \notin V(H)$, that is v belongs to an even component C of $G - X$. By Claim (12.1) and Lemma 14, $X \subseteq V(B(G))$, thus $X \cap T = \emptyset$. Since $G[T]$ is connected, $T \subseteq C$. □

For more details about almost critical graphs see [7].

7 A lemma on bricks

A graph G is called *bicritical* if for each pair of vertices u and v of G the graph $G - u - v$ has a perfect matching. A *brick* is a 3-connected bicritical graph. Note that

$$\text{every bicritical graph is matching covered.} \tag{3}$$

We shall use the following well-known characterization of bicritical graphs, see Lovász, Plummer [4].

Claim 18 *Let G be a graph with a perfect matching. Then G is bicritical if and only if G contains no barrier of size at least two.* □

Lemma 19 *Let v be an arbitrary vertex of a brick G. Let $e_i = vu_i$ $1 \leq i \leq k$ be $1 \leq k \leq d(v)-2$ edges incident to v. Let $G' := G - e_1 - \ldots - e_k$. Then*

(19.1) G' is almost critical.

(19.2) $v \in V(B(G'))$.

(19.3) For some $1 \leq i \leq k, u_i \in V(B(G'))$.

Proof. $G - v$ is factor-critical because G is bicritical. It follows that each edge of G' incident to v is critical making. Thus G' is almost critical and $v \in V(B(G'))$.

We prove (19.3) by induction on k. If $k = 1$, then we may change the role of v and u_1, and the statement follows from (19.2). Suppose (19.3) is true for each $1 \leq l \leq k-1$ but not for k. By (19.1) and Theorem (15.2), it follows that there exists a strong subgraph H of G' with strong barrier X so that u_k belongs to an even component C of $G' - X$.

Claim 20 *v is in an odd component F_1 of $G' - X$ and $|X| \neq 1$.*

Proof. If $v \in X$, then X is a barrier in G, so, by Claim 18, $X = v$ and, since G is 3-connected, there is no even component of $G - X$, a contradiction. If v was in an even component of $G' - X$, then $X \cup v$ would be a barrier in G', and also in G which contradicts Claim 18. Thus v is in an odd component of $G' - X$.

If $|X| = 1$, then v has a neighbor in F_1 thus $|V(F_1)| \geq 3$. It follows that $G - (X \cup v)$ has at least two connected components (C and $F_1 - v$). This is a contradiction because $|X \cup v| = 2$ and G is 3-connected. □

Claim 21 *Let $R := \{u_i : u_i \in V(H) - X - V(F_1)\}$ and let T be the union of those connected components of $N(G')$ which contain the vertices in R. Then*

(21.1) $1 \leq |R| \leq k-1$.

(21.2) $T \cap V(B(G')) = \emptyset$.

(21.3) $T \subseteq V(H) - X - V(F_1)$.

Proof. There exists $1 \le j \le k-1$ so that u_j belongs to an odd component of $H - X$ different from F_1, because by Claim 20 and Claim 18, X is not a barrier in G. Since $u_k \in C$, $|R| \le k-1$, and (21.1) is proved.

By Theorem (15.1), $B(G')$ is equal to one of the connected components of $N(G)$, say K. By assumption, $R \cap V(B(G')) = \emptyset$, so the connected components of $N(G')$ containing the vertices of R are different from K thus $T \cap V(B(G')) = \emptyset$ and (21.2) is proved.

(21.3) follows from the facts that each connected component of T contains a vertex in $V(H) - X - V(F_1)$ (some u_i), $T \cap X = \emptyset$ (since, by (21.2), $T \cap V(B(G')) = \emptyset$ and, by Claim (12.1) and Lemma 14, $X \subseteq V(B(G'))$), and X is a cutset in G'. □

Let $G'' := G - \{e_i : u_i \in R\}$. Using (21.1), the induction hypothesis implies that there exists $u_j \in R$ so that $u_j \in V(B(G''))$.

Claim 22 *There exists an allowed edge of G' leaving T.*

Proof. By Lemma (19.1), G'' is almost critical. By Claim 20, $v \in V(F_1)$ and, by Claim (21.3), $T \cap V(F_1) = \emptyset$ so $v \notin T$. By Lemma (19.3), $v \in V(B(G''))$. By assumption, $u_j \in T \cap V(B(G''))$. Then, by Claim 16, there exists an allowed edge $f = st$ of G'' so that $t \in T, s \notin T$. Let M_1 be a perfect matching of G'' which contains f.

To prove the claim we show that f is an allowed edge of G'. There exists an odd component $F^* \neq F_1$ of $G' - X$ which contains t by Claim (21.2). By adding some edges to G' between F_1 and some even components of $G' - X$, the odd components of $G' - X$ different from F_1 do not change. Consequently, X is a barrier in G'' and F^* is an odd component of $G'' - X$. Thus there exists exactly one edge g of M_1 leaving F^* in G''. Since g is not incident to v, $g \in E(G')$. By Claim (12.1), g is critical making in H so, by Lemma 14, g is critical making in G' and hence, by Theorem (15.1), g is an allowed edge in G'. Let M_2 be a perfect matching of G' containing g. Then g is the only edge of M_2 leaving F^* because X is a barrier in G'. Thus $M_1(F^*) \cup M_2(G' - V(F^*)) \cup g$ is a perfect matching of G' containing f. □

Claim 22 gives a contradiction because, by the definition of T, there is no allowed edge leaving T in G'. □

8 The tight cut lemma

For a subset S of vertices, $\delta(S)$ denotes the set of edges leaving S. $\delta(S)$ is a *cut*, and a cut $\delta(S)$ is called *odd* if $|S|$ is odd. An odd cut $\delta(S)$ is *trivial* if $|S| = 1$ or $|\overline{S}| = 1$, otherwise *non-trivial*. An odd cut is called *tight* if it contains exactly one edge of each perfect matching. For an odd cut $\delta(S)$, a perfect matching M is called *S-fat* if $|\delta(S) \cap M| > 1$. Note that, by (1), $|\delta(S) \cap M| \ge 3$.

The aim of this section is to present a simple proof for the so-called tight cut lemma of Edmonds, Lovász, Pulleyblank [1].

Theorem 23 *Let $\delta(S)$ be a non-trivial odd cut in a brick G. Then there exists an S-fat perfect matching in G. In other words, a brick does not contain non-trivial tight cuts.*

Proof. Suppose there exists a non-trivial tight cut $\delta(S)$.

Claim 24 *$G[S]$ ($G[\overline{S}]$) is connected.*

Proof. Suppose not, and let S_1, S_2 be a partition of S so that there is no edge between S_1 and S_2. We may suppose that $|S_1|$ is even. Let $e \in \delta(S_1)$. (Such an edge exists because G is connected.) By (3), there exists a perfect matching M of G containing e. M is a perfect matching and $|S_2|$ is odd so there exists an edge $f \in \delta(S_2) \cap M$. Since $\delta(S_1) \cap \delta(S_2) = \emptyset$, $e, f \in \delta(S) \cap M$, thus M is S-fat and we are done. $\square$

Let us chose S so that $\delta(S)$ is a non-trivial tight cut and $|S|$ is minimal.

Two cases will be distinguished. Either there exists a vertex $v \in V(G)$ so that $|\delta(v) \cap \delta(S)| \geq 2$ (Case 1.) or for each edge uv with $u \in S, v \in \overline{S}$ we have $\delta(v) \cap \delta(S) = uv = \delta(u) \cap \delta(S)$ (Case 2). In the first case we will not use the minimality of S thus we may suppose that $v \in \overline{S}$.

CASE 1. Since $G[\overline{S}]$ is connected by Claim 24, v has at least one neighbor in $\overline{S}$. Let us denote the neighbors of v in $\overline{S}$ by $e_1 = vu_1, \ldots, e_k = vu_k$. Then by assumption, $1 \leq k \leq d(v) - 2$. Let $G' = G - e_1 - \ldots - e_k$. Let $T := S \cup v$. By Lemma (19.1), G' is almost critical; by Lemma (19.2), $v \in V(B(G'))$; by Lemma (19.3) for some neighbor u_i of v, $u_i \in V(B(G'))$; thus, by Claim 16, $\delta(T)$ contains an allowed edge f of G'. By (3), there exists a perfect matching M of G' containing f. Since v has no neighbor in $\overline{S}$ in G', M matches v to a vertex in S. Hence M is an S-fat perfect matching in G and we are done.

CASE 2. The following claim is the only place where we need the minimality of S.

Claim 25 *$G[S - u]$ is connected for each edge uv with $u \in S, v \in \overline{S}$.*

Proof. Suppose not, that is there exists an edge uv with $u \in S, v \in \overline{S}$ and a partition of $S - u$ into two sets $S_1 \neq \emptyset \neq S_2$ so that there is no edge between S_1 and S_2.

First suppose that $|S_1|$ and $|S_2|$ are odd. By (3), there exists a perfect matching M of G containing uv. M is a perfect matching and $|S_1|$ is odd so there exists an edge $f \in \delta(S_1) \cap M$. Since $\delta(S_1) \cap \delta(S_2) = \emptyset$, $e, f \in \delta(S) \cap M$, thus M is S-fat and we are done.

Now suppose that $|S_1|$ and $|S_2|$ are even. Then $S' := S_1 \cup v$ is a non-trivial odd cut and $|S'| < |S|$. Then, by the minimality of S, there exists an S'-fat perfect matching M of G. Since $|\delta(S') \cap M| \geq 3$ and there is no edge between S_1 and S_2, $|\delta(S) \cap M| \geq 2$ so M is S-fat and we are done. $\square$

Claim 26 *$G[\overline{S} - v]$ is connected for some edge uv with $u \in S, v \in \overline{S}$.*

Proof. Let us consider the blocks of $G[\overline{S}]$. If it has only one block, then $G[\overline{S} - v]$ is connected for each vertex $v \in \overline{S}$. If it has more blocks, then, because of the tree structure of the blocks, there exists a block B that contains only one cut vertex. G contains no cut vertex so there exists a vertex $v \in B$ so that v has a neighbor u in S and $B - v$ is connected. In both cases the lemma is proved. $\square$

By Claim 26, by Claim 25 and since we are in Case 2, there exists an edge uv so that $u \in S, v \in \overline{S}$, v has only one neighbor in S (namely u) and u has only one neighbor in $\overline{S}$ (namely v), and $G[S - u]$ and $G[\overline{S} - v]$ are connected. From now on, the minimality of S will not be used, that is S and $\overline{S}$ play the same role.

Let $G'' := G - u - v$. Then, since $G''[S - u] = G[S - u]$ and $G''[\overline{S} - v] = G[\overline{S} - v]$, we have

Claim 27 *$G''[S - u]$ ($G''[\overline{S} - v]$) is connected.* $\square$

Claim 28 *G'' has a perfect matching but there exists no allowed edge of G'' between $S - u$ and $\overline{S} - v$.*

Proof. By (3) there exists a perfect matching M of G containing uv and then $M - uv$ is a perfect matching of G''.

If there was an allowed edge f of G'' between $S - u$ and $\overline{S} - v$, then for a perfect matching M'' of G'' containing f, $M'' \cup uv$ would be an S-fat perfect matching of G, contradiction. $\square$

Claim 29 *For each strong subgraph H of G'', $(S - u) \cap V(H) \neq \emptyset$ and $(\overline{S} - v) \cap V(H) \neq \emptyset$.*

Proof. Let H be an arbitrary strong subgraph of G'' with non-empty strong barrier X. If say $(S-u)\cap V(H)=\emptyset$, then $X\cup v$ is a barrier of G because all neighbors of u are in $S\cup v$. Since $|X\cup v|\geq 2$ this contradicts Claim 18. □

Claim 30 *For each strong subgraph H of G'' with strong barrier X, either $X\subseteq S-u$ or $X\subseteq \overline{S}-v$.*

Proof. By Claim (12.1), H is almost critical and $X\subseteq V(B(H))$, so, by Theorem (15.1), X belongs to a connected component of $N(H)$. By Claim 28, G'' has a perfect matching thus, by Claim (3.2), X belongs to a connected component of $N(G'')$. Then, by Claim 28, X is disjoint either from $(S-u)$ or from $\overline{S}-v$, which was to be proved. □

Claim 31 *G'' is almost critical.*

Proof. By Claim 28, G'' has a perfect matching. If G'' is not almost critical, then, by Claim 13, there exist two disjoint strong subgraphs H_1 and H_2 in G'' with strong barriers X_1 and X_2. By Claim 30, we may suppose that $X_1\subseteq \overline{S}-v$. By Claim 29, $(S-u)\cap V(H_1)\neq\emptyset$, thus by Claim 27, $S-u\subset V(H_1)$. Hence $(S-u)\cap V(H_2)=\emptyset$, contradicting Claim 29. □

Claim 32 *$(S-u)\cap V(B(G''))\neq\emptyset$ and $(\overline{S}-v)\cap V(B(G''))\neq\emptyset$.*

Proof. By Claim 31, G'' is almost critical and, by Claim 27, $G''[S-u]$ is connected. If $(S-u)\cap V(B(G''))=\emptyset$, then, by Claim 17 with $T=S-u$, there exists a strong subgraph of G'' disjoint from $S-u$. This contradicts Claim 29. □

By Claim 31, G'' is almost critical. Let $T:=S-u$. By Claim 32, $T\cap V(B(G''))\neq\emptyset$ and $V(B(G''))-T\neq\emptyset$. Then, by Claim 16, there exists an allowed edge f of G'' between $S-u$ and $\overline{S}-v$. This contradicts Claim 28. □

Acknowledgement I thank Zoltán Király for some simplification in the proof. I also thank William McCuaig for the very careful reading of the paper. This research was done while the author visited the Research Institute for Discrete Mathematics, Lennéstrasse 2, 53113. Bonn, Germany by an Alexander von Humboldt fellowship.

References

[1] J. Edmonds, L. Lovász, W. R. Pulleyblank, Brick decompositions and the matching rank of graphs, *Combinatorica* **2** (1982) 247-274.

[2] A. Frank, Conservative weightings and ear-decompositions of graphs, *Combinatorica* **13** (1) (1993) 65-81.

[3] L.Lovász, A note on factor-critical graphs, Studia Sci. Math. Hungar. 7 (1972) 279-280.

[4] L. Lovász, M. D. Plummer, "Matching Theory," North Holland, Amsterdam, (1986).

[5] Z. Szigeti, On Lovász's Cathedral Theorem, Proceedings of the Third IPCO Conference, eds.: G. Rinaldi, L.A. Wolsey, (1993) 413-423.

[6] Z. Szigeti, On a matroid defined by ear-decompositions of graphs, *Combinatorica* **16** (2) (1996) 233-241.

[7] Z. Szigeti, On generalizations of matching-covered graphs, *European J. Combin.* **22** (2001) 865-877.

[8] W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.* **22** (1947) 107-111.