Edge-connectivity of permutation hypergraphs

Neil Jami[*] Zoltán Szigeti[†]

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Abstract

In this note we provide a generalization of a result of Goddard, Raines and Slater [4] on edge-connectivity of permutation graphs for hypergraphs. A permutation hypergraph $\mathcal{G}_\pi$ is obtained from a hypergraph $\mathcal{G}$ by taking two disjoint copies of $\mathcal{G}$ and by adding a perfect matching between them. The main tool in the proof of the graph result was the theorem on partition constrained splitting off preserving k-edge-connectivity due to Bang-Jensen, Gabow, Jordán and Szigeti [1]. Recently, this splitting off theorem was extended for hypergraphs by Bernáth, Grappe and Szigeti [2]. This extension made it possible to find a characterization of hypergraphs for which there exists a k-edge-connected permutation hypergraph.

1 Definitions

Let $G = (V, E)$ be a graph. For a vertex set X of V, the set of edges between X and $V - X$ is called a **cut** of G. The size of this cut of G is denoted by $\boldsymbol{d_G(X)}$. For disjoint subsets X and Y of V, we denote by $\boldsymbol{d_G(X,Y)}$ the number of edges between X and Y. The minimum size of a cut of G is denoted by $\boldsymbol{\lambda(G)}$. The graph G is called $\boldsymbol{k}$**-edge-connected** if $\lambda(G) \geq k$. The **minimum degree** $\boldsymbol{\delta(G)}$ of G is defined as $\min\{d_G(v) : v \in V\}$. A graph $H = (V + s, E)$ is called $\boldsymbol{k}$**-edge-connected in** V if each cut, except eventually the one defined by s and V, contains at least k edges. The set of neighbors of the vertex s, that is the vertices adjacent to s, is denoted by $\boldsymbol{N_H(s)}$. The complete graph on n vertices is denoted by $\boldsymbol{K_n}$. By taking two disjoint copies of K_n we get the graph $\boldsymbol{2K_n}$.

Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph, where V is a finite set and $\mathcal{E}$ is a set of non-empty subsets of V, called **hyperedges**. A hyperedge of cardinality 2 is a **graph edge**. For a vertex set X of V, the set of hyperedges intersecting X and $V - X$ is called a **cut** and is denoted by $\boldsymbol{\delta_{\mathcal{G}}(X)}$. The size of a cut of $\mathcal{G}$ is denoted by $\boldsymbol{d_{\mathcal{G}}(X)}$. For disjoint subsets X and Y of V, we denote by $\boldsymbol{d_{\mathcal{G}}(X,Y)}$ the number of hyperedges intersecting both X and Y. The hypergraph $\mathcal{G}$ is called $\boldsymbol{k}$**-edge-connected** if each cut contains at least k hyperedges. A 1-edge-connected hypergraph is called **connected**. A maximal connected subhypergraph of $\mathcal{G}$ is called a **connected component** of $\mathcal{G}$. Let $\boldsymbol{\omega_k(\mathcal{G})}$ be defined as the maximum number of connected components of $\mathcal{G} - \mathcal{F}$ minus 1, where $\mathcal{F}$ is a set of $k - 1$ hyperedges in $\mathcal{E}$. A hypergraph $\mathcal{H} = (V + s, \mathcal{E})$ is called $\boldsymbol{k}$**-edge-connected in** V if each cut, except eventually the one defined by s and V, contains at least k hyperedges. The set of vertices adjacent to the vertex s in $\mathcal{H}$ is denoted by $\boldsymbol{N_{\mathcal{H}}(s)}$.

[*]Ensimag, Grenoble Institute of Technology, 681, rue de la Passerelle, Domaine Universitaire, Saint Martin d'Hères, France, 38402

[†]Laboratoire G-SCOP, CNRS, Grenoble INP, UJF, 46, Avenue Félix Viallet, Grenoble, France, 38000.

2 Permutation graphs

Given a graph G on n vertices and a permutation π of $[n]$, Chartrand and Harary [3] defined the **permutation graph G_π** as follows: we duplicate the graph G and we add a perfect matching defined by the permutation π between the two copies of the graph, in other words :

1. we take 2 disjoint copies $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G,
2. for every vertex $v_i \in V_1$, we add an edge between v_i of G_1 and $v_{\pi(i)}$ of G_2, this edge set is denoted by E_3,
3. $G_\pi = (V_1 \cup V_2, E_1 \cup E_2 \cup E_3)$.

Since, for any graph, the minimum size of a cut is less than or equal to the minimum degree, we have

$$\lambda(G_\pi) \leq \delta(G_\pi) = \delta(G) + 1.$$

For simple graphs, the following result answers when this upper bound can be achieved.

Theorem 1 *[Goddard, Raines, Slater [4]] Let G be a simple graph without isolated vertices. Then there exists a permutation π such that $\lambda(G_\pi) = \delta(G) + 1$ if and only if $G \neq 2K_k$ for some odd k.*

The tool to prove this result is presented in the next section.

3 k-admissible $\mathcal{P}$-allowed complete splitting off in graphs

Let $H = (V + s, E)$ be a graph with a specified vertex s, $\mathcal{P} = \{P_1, P_2\}$ a partition of V and $k \geq 2$ an integer. **Splitting off** at s means taking two edges $\{su, sv\}$ incident to s and replacing them by a new edge uv. **Complete splitting off** at s is a sequence of splitting off isolating s. A complete splitting off is called ***k*-admissible** if the new graph without the isolated vertex s is k-edge-connected and it is **$\mathcal{P}$-allowed** if the new edges are between P_1 and P_2.

A partition $\{A_1, \dots, A_4\}$ of V is called a **C_4-obstacle** of H if there exists $j \in \{1, 2\}$ such that

$$d_H(A_i) = k \text{ for } i = 1, \dots, 4, \quad (1)$$
$$d_H(A_1, A_3) = d_H(A_2, A_4) = 0, \quad (2)$$
$$k \text{ is odd}, \quad (3)$$
$$d_H(s, P_1) = d_H(s, P_2), \quad (4)$$
$$(A_j \cup A_{j+2}) \cap N_H(s) = P_1 \cap N_H(s). \quad (5)$$

The following theorem is a special case of a general result on partition constrained k-edge-connected complete splitting off in graphs.

Theorem 2 *[Bang-Jensen, Gabow, Jordán, Szigeti [1]] Let $H = (V + s, E)$ be a graph, $\mathcal{P} = \{P_1, P_2\}$ a partition of V and $k \geq 2$ an integer. Then there exists a k-admissible $\mathcal{P}$-allowed complete splitting off at s if and only if*

$$H \text{ is } k\text{-edge-connected in } V, \quad (6)$$
$$d_H(s, P_1) = d_H(s, P_2), \quad (7)$$
$$H \text{ contains no } C_4\text{-obstacle}. \quad (8)$$

4 Sketch of the proof of Theorem 1

We only prove the sufficiency. The main idea is the following : instead of finding the required permutation in one step we will find it in two steps. First we make an extension and then we apply splitting off. The extended graph H is obtained from G by taking two disjoint copies G_1 and G_2 of G, adding a new vertex s and connecting it to every other vertex. Since G is simple, it is easy to see that H is k-edge-connected, where $k = \delta(G) + 1$. Let $\mathcal{P} := \{V(G_1), V(G_2)\}$.

Theorem 1 follows from the equivalence of the following conditions:

(a) there exists a permutation π such that $\lambda(G_\pi) = \delta(G) + 1$,

(b) there exists a k-admissible $\mathcal{P}$-allowed complete splitting off at s in H,

(c) H contains no C_4-obstacle,

(d) $G \neq 2K_k$ if k is odd.

It is easy to verify that (a) and (b) are equivalent. Theorem 2 implies that (b) and (c) are equivalent. An easy calculation shows that (c) and (d) are equivalent.

5 Permutation hypergraphs

We define permutation hypergraphs as a natural generalization of permutation graphs. Given a hypergraph $\mathcal{G}$ on n vertices and a permutation π of $[n]$, we define the **permutation hypergraph $\mathcal{G}_\pi$** as follows:

1. we take 2 disjoint copies $\mathcal{G}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (V_2, \mathcal{E}_2)$ of $\mathcal{G}$,
2. for every vertex $v_i \in V_1$, we add an edge between v_i of $\mathcal{G}_1$ and $v_{\pi(i)}$ of $\mathcal{G}_2$, this edge set is denoted by E_3,
3. $\mathcal{G}_\pi = (V_1 \cup V_2, \mathcal{E}_1 \cup \mathcal{E}_2 \cup E_3)$.

The main result of this paper characterizes hypergraphs that admit a k-edge-connected permutation hypergraph.

Theorem 3 *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph and $k \geq 2$ an integer. Then there exists a permutation π such that $\mathcal{G}_\pi$ is k-edge-connected if and only if*

$$d_{\mathcal{G}}(X) \geq k - |X| \text{ for all } \emptyset \neq X \subseteq V, \tag{9}$$

$\mathcal{G}$ is not composed of two connected components, both of k vertices, k being odd. (10)

Theorem 3 will be proved in Section 7 using the result presented in Section 6.

6 k-admissible $\mathcal{P}$-allowed complete splitting off in hypergraphs

Let $\mathcal{H} = (V + s, \mathcal{E})$ be a hypergraph with a specified vertex s, $\mathcal{P} = \{P_1, P_2\}$ a partition of V and $k \geq 1$ an integer. A partition $\{A_1, \dots, A_4\}$ of V is called a **$\mathcal{C}_4$-obstacle** of $\mathcal{H}$ if there exists $j \in \{1, 2\}$ such that

$$d_{\mathcal{H}}(A_i) = k, \text{ for } i = 1, \dots, 4, \tag{11}$$

$$\delta_{\mathcal{H}}(A_1) \cap \delta_{\mathcal{H}}(A_3) = \delta_{\mathcal{H}}(A_2) \cap \delta_{\mathcal{H}}(A_4), \tag{12}$$

$$k - |\delta_{\mathcal{H}}(A_1) \cap \delta_{\mathcal{H}}(A_3)| \neq 1 \text{ is odd}, \tag{13}$$

$$d_{\mathcal{H}}(s, P_1) = d_{\mathcal{H}}(s, P_2), \tag{14}$$

$$(A_j \cup A_{j+2}) \cap N_{\mathcal{H}}(s) = P_1 \cap N_{\mathcal{H}}(s). \tag{15}$$

The following theorem generalizes Theorem 2 and is a special case of a general result on partition constrained k-edge-connected complete splitting off in hypergraphs.

Theorem 4 *[Bernáth, Grappe, Szigeti [2]] Let $\mathcal{H} = (V + s, \mathcal{E})$ be a hypergraph, where s is incident only to graph edges, $\mathcal{P} = \{P_1, P_2\}$ a partition of V and $k \geq 1$ an integer. Then there exists a k-admissible $\mathcal{P}$-allowed complete splitting off at s if and only if*

$$\mathcal{H} \text{ is } k\text{-edge-connected in } V, \tag{16}$$

$$d_{\mathcal{H}}(s) \geq 2\omega_k(\mathcal{H} - s), \tag{17}$$

$$d_{\mathcal{H}}(s, P_1) = d_{\mathcal{H}}(s, P_2), \tag{18}$$

$$\mathcal{H} \text{ contains no } \mathcal{C}_4\text{-obstacle.} \tag{19}$$

7 Proof of Theorem 3

7.1 Proof of the necessity

Suppose that there exists a permutation π such that $\mathcal{G}_\pi$ is k-edge-connected. We prove that (9) and (10) are satisfied.

(9) Let X be an arbitrary non-empty subset of V and X_1 the corresponding vertex set in V_1. Then, by the k-edge-connectivity of $\mathcal{G}_\pi$, $k \leq d_{\mathcal{G}_\pi}(X_1) = d_{\mathcal{G}}(X) + |X|$, and (9) follows.

(10) Suppose that (10) is not satisfied that is $\mathcal{G}$ has exactly two connected components on vertex sets V^1 and V^2 and $|V^1| = |V^2| = k$ is odd. Then the vertex set of $\mathcal{G}_\pi$ is partitioned into 4 sets $V_1^1, V_1^2, V_2^1, V_2^2$ of size k, where $\{V_i^1, V_i^2\}$ corresponds to $\{V^1, V^2\}$ for $i = 1, 2$. Since $\mathcal{G}[V^1]$ and $\mathcal{G}[V^2]$ are connected components of $\mathcal{G}$, no hyperedge exists between V_i^1 and V_i^2 in $\mathcal{G}_\pi$ for $i = 1, 2$. Then, by $d_{\mathcal{G}_\pi}(V_1^1, V_2^1) + d_{\mathcal{G}_\pi}(V_1^1, V_2^2) = d_{\mathcal{G}_\pi}(V_1^1) = |V_1^1| = k$ and k is odd, one of them, say $d_{\mathcal{G}_\pi}(V_1^1, V_2^1)$, is larger than $\frac{k}{2}$. Since only graph edges exist between V_1^1 and V_2^1 in $\mathcal{G}_\pi$ and $\mathcal{G}_\pi$ is k-edge-connected, we have $k \leq d_{\mathcal{G}_\pi}(V_1^1 \cup V_2^1) = d_{\mathcal{G}_\pi}(V_1^1) + d_{\mathcal{G}_\pi}(V_2^1) - 2d_{\mathcal{G}_\pi}(V_1^1, V_2^1) < k + k - 2\frac{k}{2} = k$. This contradiction shows that (10) is satisfied.

7.2 Proof of the sufficiency

Suppose that the conditions (9) and (10) are satisfied for the hypergraph $\mathcal{G}$ and for the integer k. As for the graphic case, we extend first the hypergraph and then we apply splitting off. The extended hypergraph $\mathcal{H}$ is obtained from $\mathcal{G}$ by taking two disjoint copies $\mathcal{G}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (V_2, \mathcal{E}_2)$ of $\mathcal{G}$, adding a new vertex s and connecting it by the edge set E' to all the other vertices. Then $\mathcal{H} = (V_1 \cup V_2 \cup \{s\}, \mathcal{E}_1 \cup \mathcal{E}_2 \cup E')$. Note that for all $X \subseteq V_1 \cup V_2$, $d_{\mathcal{H}}(s, X) = |X|$. We define the partition $\mathcal{P}$ of the vertex set of $\mathcal{H} - s$ to be $\{V_1, V_2\}$. We show that there exists a k-admissible $\mathcal{P}$-allowed complete splitting off at s. After executing this complete splitting off at s, we get the permutation hypergraph $\mathcal{G}_\pi$ that is k-edge-connected and the theorem is proved. By Theorem 4, we must verify that the conditions (16)–(19) are satisfied for $\mathcal{H}, \mathcal{P}$ and k.

(16) Let $\emptyset \neq X \subset V_1 \cup V_2$. Let $X_1 := X \cap V_1$ and $X_2 := X \cap V_2$. Then one of them, say X_1, is not empty. Let $X' \subseteq V$ be the vertex set of $\mathcal{G}$ that corresponds to X_1 of $\mathcal{G}_1$. Then, by the construction of $\mathcal{H}$ and (9) applied for X', $d_{\mathcal{H}}(X) = d_{\mathcal{H}}(X_1) + d_{\mathcal{H}}(X_2) \geq d_{\mathcal{H}}(X_1) = d_{\mathcal{G}_1}(X_1) + |X_1| = d_{\mathcal{G}}(X') + |X'| \geq k$, and (16) follows.

(17) Let $\mathcal{F}$ be a set of $k-1$ hyperedges in $\mathcal{E}$ such that the number m of connected components of $\mathcal{H}' := \mathcal{H} - s - \mathcal{F}$ minus 1 to be $\omega_k(\mathcal{H} - s)$. We distinguish two cases :

Case 1. Suppose first that $\mathcal{H}'$ contains no isolated vertices. Then each connected component K_i' of $\mathcal{H}'$ contains at least 2 vertices and hence $\omega_k(\mathcal{H} - s) + 1 = m = \frac{1}{2}\sum_{i=1}^{m} 2 \leq \frac{1}{2}\sum_{i=1}^{m} |V(K_i')| = \frac{1}{2}|V(\mathcal{H}')| = \frac{1}{2}d_{\mathcal{H}}(s)$.

Case 2. Suppose next that $\mathcal{H}'$ contains some isolated vertices, let v be one of them. Then, by $|\mathcal{F}| = k-1$ and by (9) applied for v, $0 = d_{\mathcal{H}'}(v) \geq d_{\mathcal{G}}(v) - |\mathcal{F}| = d_{\mathcal{G}}(v) - (k-1) \geq 0$. Hence we have equality everywhere, that is all the hyperedges of $\mathcal{F}$ contain the vertex v. Thus all the hyperedges of $\mathcal{F}$ belong to the same connected component of $\mathcal{H} - s$, say K_1^1 of $\mathcal{G}_1$. Note that, by the above argument, all the isolated vertices of $\mathcal{H}'$ belong to K_1^1. Let $K_2^1, \ldots, K_t^1$ be the other connected components of $\mathcal{G}_1$. Note that $\mathcal{G}_2$ has also t connected components. By $2 \leq |V(K_i^1)|$ for $i = 2, \ldots, t$, $\omega_k(\mathcal{H}-s) = m-1 \leq 2t - 2 + |V(K_1^1)| \leq \sum_{i=1}^{t} |V(K_i^1)| = |V_1| = \frac{1}{2}d_{\mathcal{H}}(s)$.

In both cases (17) is satisfied.

(18) $d_{\mathcal{H}}(s, P_1) = |V_1| = |V_2| = d_{\mathcal{H}}(s, P_2)$ and (18) is satisfied.

(19) Let us suppose that a $\mathcal{C}_4$-obstacle exists in $\mathcal{H}$, let $\{A_1, \ldots, A_4\}$ be the partition of $V_1 \cup V_2$ satisfying (11)–(15) with say $j = 1$. By (15) and $\mathcal{P} = \{V_1, V_2\}$, $V_1 = A_1 \cup A_3$ et $V_2 = A_2 \cup A_4$. By (12), all hyperedges intersecting both A_1 and A_3 also intersect A_2 and A_4. By construction, no such hyperedge exists, and then by (13), $k \neq 1$ is odd. It also follows by (11), that $|A_i| = d_{\mathcal{H}}(A_i) = k$. By (9), all connected components of $\mathcal{G}$ contains at least k vertices, so $\mathcal{G}$ has exactly two connected components, $\mathcal{G}[A_1]$ and $\mathcal{G}[A_3]$, both of k vertices and k is odd, that is (10) is violated. This contradiction finishes the proof of Theorem 3.

8 Application

We show in this section that Theorem 3 is a generalization of Theorem 1.

Let G be a graph satisfying the conditions of Theorem 1. Let us consider G as a hypergraph and let $k := \delta(G)+1$. Since G contains no isolated vertices, $k = \delta(G)+1 \geq 2$. Let X be an arbitrary non-empty vertex set in V. Since G is simple, for any vertex $v \in X$, $d_G(v, X-v) \leq |X| - 1$. Then $d_G(X) \geq d_G(v, V-X) = d_G(v) - d_G(v, X-v) \geq \delta(G) - (|X|-1) = k - |X|$, so (9) is satisfied. Suppose that (10) is not satisfied, that is G has exactly two connected components, both of k vertices and k is odd. Then, since the graph is simple, each vertex has degree at most $k-1$. But $k = \delta(G)+1$, so each vertex has degree at least $k-1$. It follows that $G = 2K_k$ and k is odd. This contradiction shows that G satisfies all the conditions of Theorem 3, so by this theorem, there exists a permutation π such that G_π is k-edge-connected, hence $\delta(G) + 1 = k \leq \lambda(G_\pi) \leq \delta(G) + 1$ and Theorem 1 is proved.

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