Discrete Applied Mathematics 250 (2018) 183–185

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On minimally 2-T-connected directed graphs

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ARTICLE INFO

Article history:
Received 14 December 2017
Accepted 2 April 2018
Available online 26 April 2018

Keywords:
Arc-connectivity
Vertex-connectivity

ABSTRACT

We prove that in a minimally 2-T-connected directed graph, that contains no parallel arcs entering or leaving a vertex in T, there exists a vertex of in-degree and out-degree 2. This is a common generalization of two earlier results of Mader (1978), (2002).

1. Introduction

Let $D = (V, A)$ be a directed graph, or more briefly, a digraph. As usual, $\boldsymbol{\rho_D}$ and $\boldsymbol{\delta_D}$ denote the *in-* and *out-degree* functions of D. For $U, W \subset V$, $\overline{\boldsymbol{U}} = V \setminus U$, $\boldsymbol{D[U]}$ denotes the subgraph of D induced by U and $\boldsymbol{d_D(U, W)}$ denotes the number of arcs with tail in $U \setminus W$ and head in $W \setminus U$.

We say that D is *k-arc-connected* if $|V| \geq 2$ and for every ordered pair (u, v) of vertices, there exist k arc disjoint paths from u to v. We call D *minimally k-arc-connected* if D is k-arc-connected and the deletion of any arc destroys this property. Instead of 1-arc-connected we will use *strongly-connected*.

Mader [1] provided a constructive characterization of k-arc-connected digraphs. To prove that result he showed the following theorem. The special case of Theorem 1 when $k = 2$ will be generalized in this paper.

Theorem 1 (*Mader [1]*)**.** *Every minimally k-arc-connected digraph D contains a vertex v with* $\rho_D(v) = \delta_D(v) = k$.

The digraph D is said to be *k-vertex-connected* if $|V| \geq k + 1$ and for every ordered pair (u, v) of vertices, there exist k internally vertex disjoint paths from u to v. We say that D is *minimally k-vertex-connected* if D is k-vertex-connected and the deletion of any arc destroys this property.

Mader [2] conjectured that a result similar to Theorem 1 also holds for vertex-connectivity.

Conjecture 1 (*Mader [2]*)**.** *Every minimally k-vertex-connected digraph D contains a vertex v with* $\rho_D(v) = \delta_D(v) = k$.

Mader [3] settled Conjecture 1 for $k = 2$.

Theorem 2 (*Mader [3]*)**.** *Every minimally 2-vertex-connected digraph D contains a vertex v with* $\rho_D(v) = \delta_D(v) = 2$.

For $T \subseteq V$, the digraph D is called 2-T-*connected* if $|V| \geq 3$ and for every ordered pair (u, v) of vertices, there exist two paths from u to v that are arc disjoint and internally vertex disjoint in T. This notion generalizes both 2-arc-connectivity ($T = \emptyset$) and 2-vertex-connectivity ($T = V$). It is easy to see that D is 2-T-connected if and only if upon deleting any arc or

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https://doi.org/10.1016/j.dam.2018.04.003

any vertex in T, the remaining digraph is strongly-connected. We call D *minimally* 2-T-*connected* if D is 2-T-connected and the deletion of any arc destroys this property.

We provide a common generalization of Theorem 1 for $k = 2$ and Theorem 2. The proof will follow the ideas of Mader [3].

Theorem 3. *Every minimally* 2-T-*connected digraph* D, *that contains no parallel arcs entering or leaving a vertex in* T *contains a vertex* v *with* $\rho_D(v) = \delta_D(v) = 2$.

Note that Theorem 3 implies Theorem 1 for $k = 2$ (when $T = \emptyset$) and Theorem 2 (when $T = V$, since no parallel arc exists in a minimally 2-vertex-connected digraph).

We present a short proof of Theorem 3, which is due to an application of the language of bi-sets. For $X_I \subseteq X_O \subseteq V$, $\mathsf{X} = (X_O, X_I)$ is called a *bi-set*. The set X_I is called the *inner-set*, X_O is the *outer-set* and $\boldsymbol{w}(\mathsf{X}) = X_O \setminus X_I$ is the *wall* of X. If $X_I = \emptyset$ or $X_O = V$, the bi-set X is called *trivial*. The *complement* of X is defined by $\overline{\mathsf{X}} = (\overline{X_I}, \overline{X_O})$. The *intersection* and the *union* of two bi-sets $\mathsf{X} = (X_O, X_I)$ and $\mathsf{Y} = (Y_O, Y_I)$ are defined as follows:

$$\mathsf{X} \sqcap \mathsf{Y} = (X_O \cap Y_O, X_I \cap Y_I),$$
$$\mathsf{X} \sqcup \mathsf{Y} = (X_O \cup Y_O, X_I \cup Y_I).$$

An arc xy *enters* X if $x \in V \setminus X_O$ and $y \in X_I$. The *in-degree* $\hat{\boldsymbol{\rho}}_{\boldsymbol{D}}(\mathsf{X})$ of X is the number of arcs entering X.

Let $\boldsymbol{T} \subseteq V$ and $\boldsymbol{g^T}$ be the modular function defined on subsets of V by $g^T(\emptyset) = 0$, $g^T(v) = 1$ for $v \in T$ and $g^T(v) = 2$ for $v \in V \setminus T$. Let us introduce the following function:

$$\boldsymbol{f_D^T}(\mathsf{X}) = \hat{\rho}_D(\mathsf{X}) + g^T(w(\mathsf{X})).$$

The following Menger-type result can be readily proved.

Claim 1. *D is* 2-T-*connected if and only if for all nontrivial bi-sets* X *of* $V(D)$,

$$f_D^T(\mathsf{X}) \geq 2. \tag{1}$$

A bi-set X is called *tight* if $f_D^T(\mathsf{X}) = 2$. It is easy to verify the following characterization of minimally 2-T-connected digraphs.

Claim 2. *D is minimally* 2-T-*connected if and only if* (1) *and the following condition are satisfied:*

Every arc of D *enters a tight bi-set of* D. (2)

The main contribution of the present note is to provide a compact proof simultaneously for Theorem 1 when $k = 2$ and for Theorem 2.

2. Proof of Theorem 3

Proof. Suppose that the theorem is false and let $D = (V, A)$ be a counterexample. Let us define the following set: $\boldsymbol{A_0} = \{xy \in A : \rho_D(y) > 2 \text{ and } \delta_D(x) > 2\}$.

Lemma 1. $A_0 \neq \emptyset$.

Proof. Suppose that $A_0 = \emptyset$. If an arc a enters a vertex u of in-degree 2 or leaves a vertex u of out-degree 2, then we say that u *covers* a. By $A_0 = \emptyset$, every arc is covered by at least one of its end-vertices. Since D is a counterexample of the theorem, a vertex can cover at most 2 arcs and, for all $v \in V$, $\rho_D(v) + \delta_D(v) \geq 5$. Hence, since $|V| \geq 3$, we have the following contradiction: $2|V| \geq |A| = \frac{1}{2}\sum_{v\in V}(\rho_D(v) + \delta_D(v)) \geq \frac{5}{2}|V|$. □

Let $\boldsymbol{\mathcal{T}}$ be the set of bi-sets T so that either T or $\overline{\mathsf{T}}$ is a tight bi-set entered by an arc of A_0. By Lemma 1 and (2), $\mathcal{T} \neq \emptyset$. Let $\mathsf{X} = (X_O, X_I)$ be an element of $\mathcal{T}$ such that $|X_O| + |X_I|$ is minimum. Without loss of generality we may assume that X is a tight bi-set entered by the arc $\boldsymbol{ab}$ of A_0. Indeed, if $\overline{\mathsf{X}}$ is a tight bi-set entered by an arc ab of A_0, then let us consider the reversed digraph $\overleftarrow{D} = (V, \overleftarrow{A})$. Then $\overleftarrow{D}$ is a counterexample to Theorem 3, $A'_0 = \{yx \in \overleftarrow{A} : \rho_{\overleftarrow{D}}(x) > 2 \text{ and } \delta_{\overleftarrow{D}}(y) > 2\} = \overleftarrow{A}_0$ and X is a tight bi-set entered by the arc ba of A'_0.

Note that either $w(\mathsf{X}) = \emptyset$ and $\hat{\rho}_D(\mathsf{X}) = 2$, or $w(\mathsf{X}) \in T$ and $\hat{\rho}_D(\mathsf{X}) = 1$.

Lemma 2. *There exists no arc* xy *in* A_0 *such that* $y \in X_I$ *and* $x \in X_O$.

Proof. Suppose there exists an arc xy in A_0 such that $y \in X_I$ and $x \in X_O$. By (2), there exists a tight bi-set $\mathsf{Y} = (Y_O, Y_I)$ entered by xy, so $\mathsf{Y} \in \mathcal{T}$.

Claim 3. $X_O \cup Y_O = V$.

Proof. If the claim is false, then $\mathsf{X} \sqcup \mathsf{Y}$ is a nontrivial bi-set. Since $y \in X_I \cap Y_I$, $\mathsf{X} \sqcap \mathsf{Y}$ is a nontrivial bi-set. Then, by the tightness of X and Y, (1) applied for $\mathsf{X} \sqcup \mathsf{Y}$ and $\mathsf{X} \sqcap \mathsf{Y}$ and the submodularity of f_D^T (since $\hat{\rho}_D$ is submodular and g^T is modular), we have

$$2 + 2 - 2 \geq f_D^T(X) + f_D^T(Y) - f_D^T(\mathsf{X} \sqcup \mathsf{Y}) \geq f_D^T(\mathsf{X} \sqcap \mathsf{Y}) \geq 2.$$

Hence equality holds everywhere, so $\mathsf{X} \sqcap \mathsf{Y}$ is tight. Moreover, $\mathsf{X} \sqcap \mathsf{Y}$ is entered by xy, that is $\mathsf{X} \sqcap \mathsf{Y} \in \mathcal{T}$ and, by $x \in X_O \setminus Y_O$, we have $|(\mathsf{X} \sqcap \mathsf{Y})_O| + |(\mathsf{X} \sqcap \mathsf{Y})_I| < |X_O| + |X_I|$, a contradiction. □

Claim 4. *$X_I \cap Y_I = y$, $w(\mathsf{X} \sqcap \mathsf{Y}) = \emptyset$ and $|w(\mathsf{X})| = |w(\mathsf{Y})| = 1$.*

Proof. By $\overline{\mathsf{Y}} = (\overline{Y_I}, \overline{Y_O}) \in \mathcal{T}$ and the minimality of X , we have

$$|\overline{Y_I}| + |\overline{Y_O}| \geq |X_O| + |X_I|. \tag{3}$$

Since $\mathsf{X}, \mathsf{Y} \in \mathcal{T}$, $1 \geq |w(\mathsf{X})|$ and $1 \geq |w(\mathsf{Y})|$. Then, by (3), Claim 3 and $y \in X_I \cap Y_I$, we have

$$2 \geq |\overline{Y_O} \cap w(\mathsf{X})| + |w(\mathsf{Y}) \cap \overline{X_O}| \geq |X_I \cap w(\mathsf{Y})| + 2|X_I \cap Y_I| + |w(\mathsf{X}) \cap Y_I| \geq 2.$$

Thus we have equality everywhere and the claim follows. □

By $xy \in A_0$, Claim 4 and the tightness of X and Y, we have

$$\begin{aligned} 2 &< \rho_D(y) = \rho_D(X_I \cap Y_I) = \hat{\rho}_D(\mathsf{X} \sqcap \mathsf{Y}) \leq \hat{\rho}_D(\mathsf{X}) + \hat{\rho}_D(\mathsf{Y}) \\ &= (f_D^T(\mathsf{X}) - g^T(w(\mathsf{X}))) + (f_D^T(\mathsf{Y}) - g^T(w(\mathsf{Y}))) \leq (2-1) + (2-1) = 2, \end{aligned}$$

a contradiction that completes the proof of Lemma 2. □

Lemma 3. *$D[X_I]$ is strongly-connected.*

Proof. Suppose there exists $\emptyset \neq U \subset X_I$ with $\rho_{D[X_I]}(U) = 0$. Then, by (1) applied for $\mathsf{Z} = (Z_O, Z_I) = (U \cup w(\mathsf{X}), U)$, $w(\mathsf{Z}) = w(\mathsf{X})$ and the tightness of X, we have

$$2 \leq \hat{\rho}_D(\mathsf{Z}) + g^T(w(\mathsf{Z})) \leq \hat{\rho}_D(\mathsf{X}) + g^T(w(\mathsf{X})) = 2.$$

Hence, equality holds everywhere, so Z is a tight bi-set with $\hat{\rho}_D(\mathsf{Z}) = \hat{\rho}_D(\mathsf{X})$ thus entered by ab, that is $\mathsf{Z} \in \mathcal{T}$. By $Z_I \subset X_I$ and $w(\mathsf{X}) = w(\mathsf{Z})$, we have $|Z_O| + |Z_I| < |X_O| + |X_I|$, a contradiction. □

Lemma 4. *The following statements hold for $\boldsymbol{V}_+ = \{v \in V : \rho_D(v) > 2 = \delta_D(v)\}$:*

(a) *If $\rho_D(v) > 2$ and $uv \in A \setminus A_0$, then $u \in V_+$.*
(b) *If $X_I \neq b$, then $X_I \subseteq V_+$.*
(c) *If $X_I \neq b$ and $w(\mathsf{X}) \neq \emptyset$, then $w(\mathsf{X}) \subseteq V_+$.*

Proof. (a) By $\rho_D(v) > 2$ and $uv \in A \setminus A_0$, we have $\delta(u) = 2$, and then, since D is a counterexample, $\rho_D(u) > 2$ and hence $u \in V_+$.

(b) By $\rho_D(b) > 2$ and (a), all vertices from which b is reachable in $D - A_0$ by a nontrivial path are in V_+. Thus, by Lemmas 2 and 3, $X_I - b \subseteq V_+$. By $X_I \neq b$ and Lemma 3, there exists an arc bc in $D[X_I]$. By Lemma 2, $c \in V_+$ and (a), we get $b \in V_+$.

(c) If $w(\mathsf{X}) \neq \emptyset$, then, by $\hat{\rho}_D(\mathsf{X}) = 1$ and (1) applied for (X_I, X_I), we have $d_D(w(\mathsf{X}), X_I) \geq 1$, so, by Lemma 2, (b) and (a), we obtain $w(\mathsf{X}) \subseteq V_+$. □

We finish the proof by considering the in-degree of X_I. We distinguish two cases.

Case 1. If $X_I = b$, then, by $ab \in A_0$, the assumption of the theorem and the fact that X is tight, we have the following contradiction:

$$2 < \rho_D(b) = \hat{\rho}_D(\mathsf{X}) + d_D(w(\mathsf{X}), b) \leq \hat{\rho}_D(\mathsf{X}) + g^T(w(\mathsf{X})) = 2.$$

Case 2. If $X_I \neq b$, then, by the fact that X is a tight bi-set entered by ab, Lemma 4(c), (1) applied for $(\overline{X_I}, \overline{X_I})$ and Lemma 4(b), we have the following contradiction.

$$\begin{aligned} 3 - 2 &\geq \hat{\rho}_D(\mathsf{X}) + 2|w(\mathsf{X})| - 2 \geq \hat{\rho}_D(\mathsf{X}) + d_D(w(\mathsf{X}), X_I) - \delta_D(X_I) \\ &= \rho_D(X_I) - \delta_D(X_I) = \sum_{v \in X_I} (\rho_D(v) - \delta_D(v)) \geq |X_I| \geq 2. \end{aligned}$$

These contradictions complete the proof of the theorem. □

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