On minimally $2$-$T$-connected directed graphs

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**ABSTRACT**

We prove that in a minimally $2$-$T$-connected directed graph, that contains no parallel arcs entering or leaving a vertex in $T$, there exists a vertex of in-degree and out-degree $2$. This is a common generalization of two earlier results of Mader (1978, 2002).

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**1. Introduction**

Let $D = (V, A)$ be a directed graph, or more briefly, a digraph. As usual, $\rho_D$ and $\delta_D$ denote the in- and out-degree functions of $D$. For $U, W \subseteq V$, $\overline{U} = V \setminus U$, $D[U]$ denotes the subgraph of $D$ induced by $U$ and $d_D(U, W)$ denotes the number of arcs with tail in $U \setminus W$ and head in $W \setminus U$.

We say that $D$ is $k$-arc-connected if $|V| \geq 2$ and for every ordered pair $(u, v)$ of vertices, there exist $k$ arc disjoint paths from $u$ to $v$. We call $D$ minimally $k$-arc-connected if $D$ is $k$-arc-connected and the deletion of any arc destroys this property. Instead of $1$-arc-connected we will use strongly-connected.

Mader [1] provided a constructive characterization of $k$-arc-connected digraphs. To prove that result he showed the following theorem. The special case of Theorem 1 when $k = 2$ will be generalized in this paper.

**Theorem 1** (Mader [1]). Every minimally $k$-arc-connected digraph $D$ contains a vertex $v$ with $\rho_D(v) = \delta_D(v) = k$.

The digraph $D$ is said to be $k$-vertex-connected if $|V| \geq k + 1$ and for every ordered pair $(u, v)$ of vertices, there exist $k$ internally vertex disjoint paths from $u$ to $v$. We say that $D$ is minimally $k$-vertex-connected if $D$ is $k$-vertex-connected and the deletion of any arc destroys this property.

Mader [2] conjectured that a result similar to Theorem 1 also holds for vertex-connectivity.

**Conjecture 1** (Mader [2]). Every minimally $k$-vertex-connected digraph $D$ contains a vertex $v$ with $\rho_D(v) = \delta_D(v) = k$.


**Theorem 2** (Mader [3]). Every minimally $2$-vertex-connected digraph $D$ contains a vertex $v$ with $\rho_D(v) = \delta_D(v) = 2$.

For $T \subseteq V$, the digraph $D$ is called $2$-$T$-connected if $|V| \geq 3$ and for every ordered pair $(u, v)$ of vertices, there exist two paths from $u$ to $v$ that are arc disjoint and internally vertex disjoint in $T$. This notion generalizes both $2$-arc-connectivity ($T = \emptyset$) and $2$-vertex-connectivity ($T = V$). It is easy to see that $D$ is $2$-$T$-connected if and only if upon deleting any arc or

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any vertex in $T$, the remaining digraph is strongly-connected. We call $D$ minimally $2$-$T$-connected if $D$ is $2$-$T$-connected and the deletion of any arc destroys this property.

We provide a common generalization of Theorem 1 for $k = 2$ and Theorem 2. The proof will follow the ideas of Mader [3].

**Theorem 3.** Every minimally $2$-$T$-connected digraph $D$, that contains no parallel arcs entering or leaving a vertex in $T$ contains a vertex $v$ with $\rho_D(v) = \delta_D(v) = 2$.

Note that Theorem 3 implies Theorem 1 for $k = 2$ (when $T = \emptyset$) and Theorem 2 (when $T = V$, since no parallel arc exists in a minimally $2$-vertex-connected digraph).

We present a short proof of Theorem 3, which is due to an application of the language of bi-sets. For $X_f \subseteq X_0 \subseteq V$, $X = (X_f, X_0)$ is called a bi-set. The set $X_f$ is called the inner-set, $X_0$ is the outer-set and $W(X) = X_0 \setminus X_f$ is the wall of $X$. If $X_f = \emptyset$ or $X_0 = V$, the bi-set $X$ is called trivial. The complement of $X$ is defined by $\overline{X} = (\overline{X_f}, \overline{X_0})$. The intersection and the union of two bi-sets $X = (X_f, X_0)$ and $Y = (Y_f, Y_0)$ are defined as follows:

$$X \cap Y = (X_f \cap Y_f, X_0 \cap Y_0),$$
$$X \cup Y = (X_f \cup Y_f, X_0 \cup Y_0).$$

An arc $xy$ enters $X$ if $x \in V \setminus X_0$ and $y \in X_f$. The in-degree $\hat{\rho}_D(X)$ of $X$ is the number of arcs entering $X$.

Let $T \subseteq V$ and $g^T$ be the modular function defined on subsets of $V$ by $g^T(\emptyset) = 0$, $g^T(v) = 1$ for $v \in T$ and $g^T(v) = 2$ for $v \in V \setminus T$. Let $w$ be the following function:

$$f^T_\emptyset(X) = \hat{\rho}_D(X) + g^T(w(X)).$$

The following Menger-type result can be readily proved.

**Claim 1.** $D$ is $2$-$T$-connected if and only if for all nontrivial bi-sets $X$ of $V(D)$,

$$f^T_\emptyset(X) \geq 2. \tag{1}$$

A bi-set $X$ is called tight if $f^T_\emptyset(X) = 2$. It is easy to verify the following characterization of minimally $2$-$T$-connected digraphs.

**Claim 2.** $D$ is minimally $2$-$T$-connected if and only if (1) and the following condition are satisfied:

Every arc of $D$ enters a tight bi-set of $T$. \tag{2}

The main contribution of the present note is to provide a compact proof simultaneously for Theorem 1 when $k = 2$ and for Theorem 2.

2. Proof of Theorem 3

**Proof.** Suppose that the theorem is false and let $D = (V, A)$ be a counterexample. Let us define the following set: $A_0 = \{xy \in A : \rho_D(y) > 2$ and $\delta_D(x) > 2\}$.

**Lemma 1.** $A_0 \neq \emptyset$.

**Proof.** Suppose that $A_0 = \emptyset$. If an arc $a$ enters a vertex $u$ of in-degree $2$ or leaves a vertex $u$ of out-degree $2$, then we say that $u$ covers $a$. By $A_0 = \emptyset$, every arc is covered by at least one of its end-vertices. Since $D$ is a counterexample of the theorem, a vertex can cover at most $2$ arcs and, for all $v \in V$, $\rho_D(v) + \delta_D(v) \geq 5$. Hence, since $|V| \geq 3$, we have the following contradiction: $2|V| > |A| = \frac{1}{2} \sum_{v \in V}(\rho_D(v) + \delta_D(v)) \geq \frac{5}{2}|V|$. \hfill \Box

Let $T$ be the set of bi-sets $T$ so that either $T$ or $\overline{T}$ is a tight bi-set entered by an arc of $A_0$. By Lemma 1 and (2), $T \neq \emptyset$. Let $X = (X_f, X_0)$ be an element of $T$ such that $|X_f| + |X_0|$ is minimum. Without loss of generality we may assume that $X$ is a tight bi-set entered by the arc $ab$ of $A_0$. Indeed, if $\overline{X}$ is a tight bi-set entered by an arc $ab$ of $A_0$, then let us consider the reversed digraph $D' = (V, \overline{A})$. Then $D'$ is a counterexample to Theorem 3, $A_0' = \{xy \in \overline{A} : \rho_D^{-1}(x) > 2$ and $\delta_D^{-1}(y) > 2\} = \overline{A}_0$ and $X$ is a tight bi-set entered by the arc $ba$ of $A_0'$.

Note that either $w(X) = \emptyset$ and $\hat{\rho}_D(X) = 2$, or $w(X) \in T$ and $\hat{\rho}_D(X) = 1$.

**Lemma 2.** There exists no arc $xy$ in $A_0$ such that $y \in X_f$ and $x \in X_0$.

**Proof.** Suppose there exists an arc $xy$ in $A_0$ such that $y \in X_f$ and $x \in X_0$. By (2), there exists a tight bi-set $Y = (Y_0, Y_1)$ entered by $xy$, so $Y \in T$.

**Claim 3.** $X_0 \cup Y_0 = V$. 

Proof. If the claim is false, then $X \sqcup Y$ is a nontrivial bi-set. Since $y \in X \cap Y$, $X \cap Y$ is a nontrivial bi-set. Then, by the tightness of $X$ and $Y$, (1) applied for $X \sqcup Y$ and $X \cap Y$ and the submodularity of $f^D_T$ (since $\hat{\rho}_D$ is submodular and $g^T$ is modular), we have
\[
2 + 2 - 2 \geq f^D_T(X) + f^D_T(Y) - f^D_T(X \sqcup Y) \geq f^D_T(X \cap Y) \geq 2.
\]
Hence equality holds everywhere, so $X \cap Y$ is tight. Moreover, $X \cap Y$ is entered by $xy$, that is $X \cap Y \in T$ and, by $x \in X_0 \setminus Y_0$, we have $|X \cap Y||0| + |(|X \cap Y||0)|| < |X_0||0| + |X_0||, a contradiction. \hfill \Box

Claim 4. $X_i \cap Y_i = y$, $w(X \cap Y) = \emptyset$ and $|w(X)| = |w(Y)| = 1$.

Proof. By $\overline{V} = (\overline{V_1}, \overline{V_2}) \in T$ and the tightness of $X_i$, we have
\[
|\overline{V_1}| + |\overline{V_2}| \geq |X_0||0| + |X_0||1|.
\]
Since $X, Y \in T$, $1 \geq |w(X)|$ and $1 \geq |w(Y)|$. Then, by (3), Claim 3 and $y \in X_i \cap Y_i$, we have
\[
2 \geq |\overline{V_0} \cap w(X)| + |w(Y) \cap \overline{V_2}| \geq |X_i \cap w(Y)| + 2|X_i \cap Y_i| + |w(X) \cap Y_i| \geq 2.
\]
Thus we have equality everywhere and the claim follows. \hfill \Box

Lemma 3. $D[X_i]$ is strongly-connected.

Proof. Suppose there exists $\emptyset \neq U \subset X_i$ with $\rho_D(U)(U) = 0$. Then, by (1) applied for $Z = (Z_0, Z_1) = (U \cup w(X), U)$, $w(Z) = w(X)$ and the tightness of $X$, we have
\[
2 \leq \hat{\rho}_D(Z) + g^T(w(Z)) \leq \hat{\rho}_D(X) + g^T(w(X)) = 2.
\]
Hence, equality holds everywhere, so $Z$ is a tight bi-set with $\hat{\rho}_D(Z) = \hat{\rho}_D(X)$ thus entered by $ab$, that is $Z \in T$. By $Z_i \subset X_i$ and $w(X) = w(Z)$, we have $|Z_0||0| + |Z_1||1| < |X_0||0| + |X_1||1|$, a contradiction that completes the proof of Lemma 2. \hfill \Box

Lemma 4. The following statements hold for $V_+ = \{v \in V : \rho_D(v) > 2 = \delta_D(v)\}$:

(a) If $\rho_D(v) > 2$ and $uv \in A \setminus A_0$, then $u \in V_+$.
(b) If $X_i \neq b$, then $X_i \subseteq V_+$.
(c) If $X_i \neq b$ and $w(X) \neq \emptyset$, then $w(X) \subseteq V_+$.

Proof. (a) By $\rho_D(v) > 2$ and $uv \in A \setminus A_0$, we have $\delta(u) = 2$, and then, since $D$ is a counterexample, $\rho_D(u) > 2$ and hence $u \in V_+$. (b) By $\rho_D(b) > 2$ and (a), all vertices from which $b$ is reachable in $D - A_0$ by a nontrivial path are in $V_+$. Thus, by Lemmas 2 and 3, $X_i - b \subseteq V_+$. By $X_i \neq b$ and Lemma 3, there exists an arc $bc$ in $D[X_i]$. By Lemma 2, $c \in V_+$ and (a), we get $b \in V_+$. (c) If $w(X) \neq \emptyset$, then, by $\hat{\rho}_D(X) = 1$ and (1) applied for $(X_i, X_i)$, we have $d_D(w(X), X_i) \geq 1$, so, by Lemma 2, (b) and (a), we obtain $w(X) \subseteq V_+$. \hfill \Box

We finish the proof by considering the in-degree of $X_i$. We distinguish two cases.

Case 1. If $X_i = b$, then, by $ab \in A_0$, the assumption of the theorem and the fact that $X$ is tight, we have the following contradiction:
\[
2 < \rho_D(b) = \hat{\rho}_D(X) + d_D(w(X), b) \leq \hat{\rho}_D(X) + g^T(w(X)) = 2.
\]

Case 2. If $X_i \neq b$, then, by the fact that $X$ is a tight bi-set entered by $ab$, Lemma 4(c), (1) applied for $(\overline{X_i}, \overline{X_i})$ and Lemma 4(b), we have the following contradiction.
\[
3 - 2 \geq \hat{\rho}_D(X_i) + 2|w(X)| - 2 \geq \hat{\rho}_D(X) + d_D(w(X), X_i) - \delta_D(X_i)
\]
\[
= \rho_D(X_i) - \delta_D(X_i) = \sum_{v \in X_i} (\rho_D(v) - \delta_D(v)) \geq |X_i| \geq 2.
\]
These contradictions complete the proof of the theorem. \hfill \Box

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