Packing mixed hyperarborescences

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Abstract

The aim of this paper is twofold. We first provide a new orientation theorem which gives a natural and simple proof of a result of Gao, Yang [11] on matroid-reachability-based packing of mixed arborescences in mixed graphs by reducing it to the corresponding theorem of Cs. Király [17] on directed graphs. Moreover, we extend another result of Gao, Yang [12] by providing a new theorem on mixed hypergraphs having a packing of mixed hyperarborescences such that their number is at least ℓ and at most ℓ', each vertex belongs to exactly k of them, and each vertex v is the root of least $f(v)$ and at most $g(v)$ of them.

Keywords: arborescence, mixed hypergraph, packing

1 Introduction

This paper is not a survey on packing arborescences. Such a survey is in preparation, see [21]. We only present here those theorems of the topic that are closely related to the new results of this paper. A preliminary version of the paper appeared in [20].

Edmonds [5] characterized digraphs having a packing of spanning arborescences with fixed roots. Frank [7] extended it for a packing of spanning arborescences whose roots are not fixed. The result of Frank [7], and independently Cai [3], answers the question when a digraph has an (f, g)-bounded packing of spanning arborescences, that is when each vertex v can be the root of at least $f(v)$ and at most $g(v)$ arborescences in the packing. Bérczi, Frank [2] extends it for an (f, g)-bounded k-regular (ℓ, ℓ')-limited packing of not necessarily spanning arborescences, where k-regular means that each vertex belongs to exactly k arborescences in the packing and (ℓ, ℓ')-limited means that the number of arborescences in the packing is at least ℓ and at most ℓ'. Kamiyama, Katoh, Takizawa [16] provided a different type of generalization of Edmonds' theorem in which they wanted to pack reachability arborescences in a digraph, that is each arborescence in the packing must contain all the vertices that can be reached from its root in the digraph. Durand de Gevigney, Nguyen, Szigeti [4] gave a generalization of Edmonds' theorem where a matroid constraint was added for the packing. More precisely, given a matroid M on a multiset of vertices of a digraph D, we wanted to have a matroid-based packing of arborescences, that is for every vertex v of D, the set of roots of the arborescences in the packing containing v must form a basis of M. In [17] Cs. Király proposed a common generalization of the previous two results. He characterized pairs (D, M) of a digraph and a matroid that have a matroid-reachability-based packing of arborescences, that is for every vertex v of D, the set of roots of the arborescences in the packing containing v must form a basis of the subset of the elements of M from which v is reachable in D.

All of these results hold for dypergraphs, see [10], [14], [21], [1], [6], and for mixed graphs, see [7], [11], [21], [19], [6], [12]. In fact, all of these results, except the one of Bérczi, Frank [2], are known to hold for mixed hypergraphs, see [6], [14], [15]. The present paper will fill in this gap by showing that this result also holds for mixed hypergraphs. More precisely, we will characterize mixed hypergraphs having an (f, g)-bounded k-regular (ℓ, ℓ')-limited packing of mixed hyperarborescences. Our theorem naturally generalizes a result of Gao, Yang [12] on (f, g)-bounded packing of k spanning mixed arborescences and will follow from the theory of generalized polymatroids. The other aim of this paper is to provide a new proof of another result of Gao, Yang [11] on matroid-reachability-based packing of mixed arborescences.

Our approach is to reduce the result to the result of Cs. Király [17] on matroid-reachability-based packing of arborescences via a new orientation theorem.

The organization of the paper is as follows. In Section 3 we consider problems related to matroid-reachability-based packings of mixed arborescences. In Section 4 we consider problems related to (f, g)-bounded k-regular (ℓ, ℓ')-limited packings of mixed hyperarborescences.

2 Definitions

A *multiset* of V may contain multiple occurrences of elements. For a multiset S of V and a subset X of V, $\boldsymbol{S_X}$ denotes the multiset consisting of the elements of X with the same multiplicities as in S and $\overline{\boldsymbol{X}}$ denotes $V - X$. A set of disjoint subsets of V is called a *subpartition* of V. For a subpartition $\mathcal{P}$ of V, $\cup\boldsymbol{\mathcal{P}}$ denotes the set of elements that belong to some member of $\mathcal{P}$. A subpartition $\mathcal{P}$ of V is a *partition* of V if $\cup\mathcal{P} = V$. For a function h on V and a subset X of V, $h(X) = \sum_{v\in X} h(v)$.

A set function p (b) on S is called *supermodular* (*submodular*) if (1) ((2), respectively) holds. The in-degree function of a digraph and the rank function of a matroid are well-known examples of submodular set functions. If p satisfies the supermodular inequality for all intersecting sets for then p is called *intersecting supermodular*.

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \text{ for all } X, Y \subseteq S, \tag{1}$$
$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \text{ for all } X, Y \subseteq S. \tag{2}$$

Let $\boldsymbol{D} = (V, A)$ be a directed graph, shortly *digraph*. For a subset X of V, the set of arcs in A *entering* X is denoted by $\boldsymbol{\rho_A(X)}$ and the *in-degree* of X is $\boldsymbol{d^-_A(X)} = |\rho_A(X)|$. For a subset X of V, we denote by $\boldsymbol{P^X_D}$ ($\boldsymbol{Q^X_D}$) the set of vertices from (to) which there exists a path to (from, respectively) at least one vertex of X. We say that D is an *arborescence with root s*, shortly *s-arborescence*, if $s \in V$ and there exists a unique path from s to v for every $v \in V$; or equivalently, if D contains no circuit and every vertex in $V - s$ has in-degree 1. We say that D is a *branching with root set S* if $S \subseteq V$ and there exists a unique path from S to v for every $v \in V$. A subgraph of D is called *spanning* if its vertex set is V. A subgraph of D is called a *reachability s-arborescence* if it is an s-arborescence and its vertex set is Q^s_D. By a *packing* of subgraphs in D, we mean a set of subgraphs that are arc-disjoint. A packing of subgraphs is called *k-regular* if every vertex belongs to exactly k subgraphs in the packing. For two functions $f, g : V \to \mathbb{Z}_+$, a packing of arborescences is called *(f, g)-bounded* if the number of v-arborescences in the packing is at least $f(v)$ and at most $g(v)$ for every $v \in V$. For $\ell, \ell' \in \mathbb{Z}_+$, a packing of arborescences is called *(ℓ, ℓ')-limited* if the number of arborescences in the packing is at least ℓ and at most ℓ'. For a multiset S of V and a matroid M on S, a packing of arborescences in D is called *matroid-based* (resp. *matroid-reachability-based*) if every $s \in S$ is the root of at most one arborescence in the packing and for every $v \in V$, the multiset of roots of arborescences containing v in the packing forms a basis of S ($S_{P^v_D}$, respectively) in M.

Let $\boldsymbol{F} = (V, E \cup A)$ be a *mixed graph*, where E is a set of edges and A is a set of arcs. A mixed subgraph F' of F is a *mixed path* if the edges in F' can be oriented in such a way that we obtain a directed path. For a subset X of V, we denote by $\boldsymbol{P^X_F}$ ($\boldsymbol{Q^X_F}$) the set of vertices from (to) which there exists a mixed path to (from, respectively) at least one vertex of X. We say that F is *strongly connected* if there exists a mixed path from s to t for all $(s, t) \in V^2$. The maximal strongly connected subgraphs of F are called the *strongly connected components* of F. A *mixed s-arborescence* is a mixed graph that has an orientation that is an s-arborescence. A mixed subgraph of F is called a *spanning (reachability) mixed s-arborescence* if it is a mixed s-arborescence and its vertex set is V (Q^s_F, respectively). By a *packing* of subgraphs in F, we mean a set of subgraphs that are edge- and arc-disjoint. All the packing problems considered in digraphs can also be considered in mixed graphs.

Let $\boldsymbol{\mathcal{D}} = (V, \mathcal{A})$ be a directed hypergraph, shortly *dypergraph*, where $\boldsymbol{\mathcal{A}}$ is the set of dyperedges of $\mathcal{D}$. A *dyperedge* e is an ordered pair (Z, z), where $z \in V$ is the *head* and $\emptyset \neq Z \subseteq V - z$ is the set of *tails* of e. For $X \subseteq V$, a dyperedge (Z, z) *enters* X if $z \in X$ and $Z \cap \overline{X} \neq \emptyset$. The set of dyperedges in $\mathcal{A}$ *entering* X is denoted by $\boldsymbol{\rho_{\mathcal{A}}(X)}$ and the *in-degree* of X is $\boldsymbol{d^-_{\mathcal{A}}(X)} = |\rho_{\mathcal{A}}(X)|$. By *trimming* a dyperedge (Z, z), we mean the operation that replaces (Z, z) by an arc yz where $y \in Z$. We say that $\mathcal{D}$ is a *hyperarborescence with root s*, shortly *s-hyperarborescence*, if $\mathcal{D}$ can be trimmed to an s-arborescence. We mention that we delete vertices that became isolated vertices during the trimming, that is the vertex set of the arborescence is not necessarily the vertex set of $\mathcal{D}$. We say that $\mathcal{D}$ is a *hyperbranching with root set S* if $\mathcal{D}$ can be trimmed to a branching with root set S (the resulting isolated vertices in $V - S$ are deleted). If $S = \{s\}$ then a hyperbranching with root set S is an s-hyperarborescence. A hyperbranching $(V, \mathcal{A}')$ with root set S is called *spanning* in $\mathcal{D}$ if $\mathcal{A}' \subseteq \mathcal{A}$ and $|\mathcal{A}'| = |V| - |S|$. A *packing* of subdypergraphs in $\mathcal{D}$ is a set of subdypergraphs that are dyperedge-disjoint. We say that $\mathcal{D}$ has a *matroid-based/(f, g)-bounded/k-regular/(ℓ, ℓ')-limited* packing of hyperarborescences if $\mathcal{D}$ can be trimmed to a digraph (V, A) that has a matroid-based/(f, g)-bounded/k-regular/(ℓ, ℓ')-limited packing of arborescences.

Let $\boldsymbol{\mathcal{F}} = (V, \mathcal{E} \cup \mathcal{A})$ be a *mixed hypergraph*, where $\boldsymbol{\mathcal{E}}$ is the set of hyperedges and $\boldsymbol{\mathcal{A}}$ is the set of dyperedges of $\mathcal{F}$. A *hyperedge* is a subset of V of size at least two. A hyperedge X *enters* a subset Y of V if $X \cap Y \neq \emptyset \neq \overline{X} \cap Y$. By *orienting* a hyperedge X, we mean the operation that replaces the hyperedge X by a dyperedge $(X - x, x)$ for some $x \in X$. For $\vec{\mathcal{Z}} \subseteq \mathcal{A}$, $\mathcal{Z}$ denotes the set of underlying hyperedges of $\vec{\mathcal{Z}}$. For $\mathcal{Z} \subseteq \mathcal{E}$ and $X \subseteq V$, we denote by $\boldsymbol{V(\mathcal{Z})}$ the set of vertices that belong to at least one hyperedge in $\mathcal{Z}$ and by $\boldsymbol{\mathcal{Z}(X)}$ the set of hyperedges in $\mathcal{Z}$ that are contained in X. A *mixed s-hyperarborescence* is a mixed hypergraph that has an orientation that is an s-hyperarborescence. A mixed s-hyperarborescence $\mathcal{B} = (V, \mathcal{E}' \cup \mathcal{A}')$ is called *spanning* in $\mathcal{F}$ if $\mathcal{E}' \subseteq \mathcal{E}$, $\mathcal{A}' \subseteq \mathcal{A}$, and $|\mathcal{E}'| + |\mathcal{A}'| = |V| - 1$. For a family $\mathcal{P}$ of subsets of V, we denote by $\boldsymbol{e_{\mathcal{E} \cup \mathcal{A}}(\mathcal{P})}$ the number of hyperedges in $\mathcal{E}$ and dyperedges in $\mathcal{A}$ that enter some member of $\mathcal{P}$. For $X \subseteq V$, we use $e_{\mathcal{E} \cup \mathcal{A}}(X)$ for $e_{\mathcal{E} \cup \mathcal{A}}(\{X\})$. A *packing* of mixed subhypergraphs in $\mathcal{F}$ is a set of mixed subhypergraphs that are hyperedge- and dyperedge-disjoint. We say that $\mathcal{F}$ has an *(f, g)-bounded/k-regular/(ℓ, ℓ')-limited* packing of mixed hyperarborescences if $\mathcal{E}$ has an orientation $\vec{\mathcal{E}}$ such that the dypergraph $(V, \vec{\mathcal{E}} \cup \mathcal{A})$ has an (f, g)-bounded/k-regular/(ℓ, ℓ')-limited packing of hyperarborescences.

3 Packing mixed arborescences

In this section we list known results on packing mixed arborescences that are related to our first contribution. We propose a new approach to prove a result of Gao, Yang [11] on matroid-reachability-based packing of mixed arborescences via a new orientation theorem, and we provide its proof.

We start with the fundamental result of Edmonds [5] on packing spanning arborescences with fixed roots.

Theorem 1 (Edmonds [5]). *Let $D = (V, A)$ be a digraph and S a multiset of V. There exists a packing of spanning s-arborescences $(s \in S)$ in D if and only if*

$$d_A^-(X) \geq |S| - |S_X| \text{ for every } \emptyset \neq X \subseteq V.$$

Frank [7] extended Theorem 1 for mixed graphs. Here we present a seemingly more general version of it but it is equivalent to the original result.

Theorem 2 (Frank [7]). *Let $F = (V, E \cup A)$ be a mixed graph, S a multiset of V. There exists a packing of spanning mixed s-arborescences $(s \in S)$ in F if and only if*

$$e_{E \cup A}(\mathcal{P}) \quad \geq \quad |S||\mathcal{P}| - |S_{\cup \mathcal{P}}| \qquad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{3}$$

An elegant extension of Theorem 1 for packing reachability arborescences was provided in [16].

Theorem 3 (Kamiyama, Katoh, Takizawa [16]). *Let $D = (V, A)$ be a digraph and S a multiset of V. There exists a packing of reachability s-arborescences $(s \in S)$ in D if and only if*

$$d_A^-(X) \geq |S_{P_D^X}| - |S_X| \text{ for every } X \subseteq V.$$

When each vertex is reachable from every vertex of S, Theorem 3 reduces to Theorem 1. Theorem 3 can be proved by induction and using Edmonds' result on packing spanning branchings, see Hörsch, Szigeti [15].

Theorem 3 can also be generalized for mixed graphs follows. For convenience, we present not the original version of the result but one due to Gao, Yang [11] that fits better to our framework.

Theorem 4 (Matsuoka, Tanigawa [19]). *Let $F = (V, E \cup A)$ be a mixed graph and S a multiset of V. There exists a packing of reachability mixed s-arborescences $(s \in S)$ in F if and only if for every strongly connected component C of F and every set $\mathcal{P}$ of subsets of P_F^C such that $Z \cap C \neq \emptyset$ and $e_{E \cup A}(Z - C) = 0$ for every $Z \in \mathcal{P}$ and $Z \cap Z' \cap C = \emptyset$ for every $Z, Z' \in \mathcal{P}$,*

$$e_{E \cup A}(\mathcal{P}) \geq \sum_{Z \in \mathcal{P}} (|S_{P_F^C}| - |S_Z|).$$

If F is a digraph then Theorem 4 reduces to Theorem 3.

Another type of generalizations of Theorem 1 was obtained by adding a matroid constraint.

Theorem 5 (Durand de Gevigney, Nguyen, Szigeti [4]). *Let $D = (V, A)$ be a digraph, S a multiset of V and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists a M-based packing of arborescences in D if and only if*

$$d_A^-(X) \geq r_\mathsf{M}(S) - r_\mathsf{M}(S_X) \text{ for every } \emptyset \neq X \subseteq V.$$

For the free matroid M, Theorem 5 reduces to Theorem 1.

A common generalization of Theorems 3 and 5 was found by Cs. Király [17].

Theorem 6 (Cs. Király [17]). *Let $D = (V, A)$ be a digraph, S a multiset of V and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists a matroid-reachability-based packing of arborescences in D if and only if*

$$d_A^-(X) \quad \geq \quad r_\mathsf{M}(S_{P_D^X}) - r_\mathsf{M}(S_X) \quad \text{ for every } X \subseteq V. \tag{4}$$

For the free matroid M, Theorem 6 reduces to Theorem 3. When each vertex is reachable from a basis of M, Theorem 6 reduces to Theorem 5.

Gao, Yang [11] provided another characterization of the existence of a matroid-reachability-based packing of arborescences.

Theorem 7 (Gao, Yang [11]). *Let $D = (V, A)$ be a digraph, S a multiset of V and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists a matroid-reachability-based packing of arborescences in D if and only if for every strongly connected component C of D and every $X \subseteq P_D^C$ such that $X \cap C \neq \emptyset$ and $d_A^-(X - C) = 0$,*

$$d_A^-(X) \quad \geq \quad r_\mathsf{M}(S_{P_D^C}) - r_\mathsf{M}(S_X). \tag{5}$$

Let us show that Theorems 6 and 7 are equivalent.

Proof. We have to prove that (4) and (5) are equivalent.

(4) $\Longrightarrow$ (5): If (4) holds then let C be a strongly connected component of D and $X \subseteq P_D^C$ such that $X \cap C \neq \emptyset$ and $d_A^-(X - C) = 0$. Then, we have $P_D^X = P_D^C$ and hence (4) implies (5).

(5) $\Longrightarrow$ (4): Now if (5) holds then let X be a subset of V. Let $C_1, \ldots, C_k$ be the strongly connected components of D in a topological ordering that is if there exists an arc from C_i to C_j then $i < j$. Let

$$\begin{aligned} \boldsymbol{J} &= \{1 \leq j \leq k : X \cap C_j \neq \emptyset\}, \\ \boldsymbol{X_j} &= (X \cap C_j) \cup \bigcup_{\substack{i \in J - \{j\} \\ C_i \subseteq P_D^{C_j}}} P_D^{C_i} \qquad \text{for every } j \in J. \end{aligned}$$

Note that $X_j \subseteq P_D^{C_j}$, $X_j \cap C_j \neq \emptyset$ and $d_A^-(X_j - C_j) = 0$ for every $j \in J$.

Claim 1. $d_A^-(X) \geq \sum_{j \in J} d_A^-(X_j)$.

Proof. If uv enters X_j then $v \in X \cap C_j \subseteq X$ and $u \notin X_j$. If $u \in X$ then $u \in X \cap C_{j'}$ for some $j' \in J$. Since $C_{j'}$ is strongly connected, $u \in C_{j'}$ and $v \in X \cap C_j$, we have $C_{j'} \subseteq P_D^{C_j}$, so $u \in X_j$ which is a contradiction. It follows that $u \notin X$, so uv enters X. Since $(X \cap C_j) \cap (X \cap C_{j'}) = \emptyset$ for distinct $j, j' \in J$, the claim follows. ☐

Claim 2. $\sum_{j \in J} (r_\mathsf{M}(S_{P_D^{C_j}}) - r_\mathsf{M}(S_{X_j})) \geq r_\mathsf{M}(S_{P_D^X}) - r_\mathsf{M}(S_X)$.

Proof. We prove it by induction on $|J|$. For $|J| = 1$, say $J = \{j\}$, the claim follows from $P_D^{C_j} = P_D^{X_j}$. Suppose that the inequality holds for $|J| - 1$. Let ℓ be the largest value in J. Note that we have

$$\begin{aligned} P_D^{X_\ell} \cap (X_\ell \cup P_D^{X - X_\ell}) &\supseteq X_\ell, & P_D^{X - C_\ell} \cap X &\supseteq X - C_\ell, \\ P_D^{X_\ell} \cup (X_\ell \cup P_D^{X - X_\ell}) &\supseteq P_D^X, & P_D^{X - C_\ell} \cup X &\supseteq X_\ell \cup P_D^{X - C_\ell}. \end{aligned}$$

Then, by induction, submodularity of r_{M}, first for $S_{P_D^{X_\ell}}$ and $S_{X_\ell\cup P_D^{X-X_\ell}}$, then for $S_{P_D^{X-C_\ell}}$ and S_X, and monotonicity of r_{M}, we have

$$\begin{aligned}\sum_{j\in J}(r_{\mathsf{M}}(S_{P_D^{C_j}})-r_{\mathsf{M}}(S_{X_j})) &\geq (r_{\mathsf{M}}(S_{P_D^{X_\ell}})-r_{\mathsf{M}}(S_{X_\ell}))+(r_{\mathsf{M}}(S_{P_D^{X-C_\ell}})-r_{\mathsf{M}}(S_{X-C_\ell}))\\ &\geq (r_{\mathsf{M}}(S_{P_D^{X}})-r_{\mathsf{M}}(S_{X_\ell\cup P_D^{X-X_\ell}}))+(r_{\mathsf{M}}(S_{X_\ell\cup P_D^{X-C_\ell}})-r_{\mathsf{M}}(S_X))\\ &\geq r_{\mathsf{M}}(S_{P_D^{X}})-r_{\mathsf{M}}(S_X),\end{aligned}$$

and the claim follows. □

By Claim 1, (5) applied for all X_j, and Claim 2, we get that

$$d_A^-(X)\geq\sum_{j\in J}d_A^-(X_j)\geq\sum_{j\in J}(r_{\mathsf{M}}(S_{P_D^{C_j}})-r_{\mathsf{M}}(S_{X_j}))\geq r_{\mathsf{M}}(S_{P_D^X})-r_{\mathsf{M}}(S_X),$$

so (4) holds. □

A common generalization of Theorems 4 and 7 was provided by Gao, Yang [11].

Theorem 8 (Gao, Yang [11])**.** *Let $F=(V,E\cup A)$ be a mixed graph, S a multiset of V and $\mathsf{M}=(S,r_{\mathsf{M}})$ a matroid. There exists a matroid-reachability-based packing of mixed arborescences in F if and only if for every strongly connected component C of F and every set $\mathcal{P}$ of subsets of P_F^C such that $Z\cap C\neq\emptyset$ and $e_{E\cup A}(Z-C)=0$ for every $Z\in\mathcal{P}$ and $Z\cap Z'\cap C=\emptyset$ for every $Z,Z'\in\mathcal{P}$,*

$$e_{E\cup A}(\mathcal{P}) \quad\geq\quad \sum_{Z\in\mathcal{P}}(r_{\mathsf{M}}(S_{P_F^C})-r_{\mathsf{M}}(S_Z)). \tag{6}$$

For the *free* matroid M, that is every set of S is independent in M, Theorem 8 reduces to Theorem 4. For $E=\emptyset$, Theorem 8 reduces to Theorem 7. Hörsch, Szigeti [15] pointed out that Theorem 8 holds for mixed hypergraphs. That more general result was proved in [15] by induction using a result on matroid-based packing of mixed hyperbranchings in mixed hypergraphs from [6]. Here we propose another approach to prove Theorem 8. It will be derived from its directed version (Theorem 6) using a new orientation result (Theorem 10). To prove Theorem 10 we need a result of Frank, see Theorem 15.4.13 in [9].

Theorem 9 (Frank [9])**.** *Let $G=(V,E)$ be a graph and h an integer-valued intersecting supermodular set function such that $h(V)=0$. There exists an orientation $\vec{G}=(V,\vec{E})$ of G such that*

$$d_{\vec{E}}^-(X) \quad\geq\quad h(X) \qquad \textit{for every } X\subseteq V \tag{7}$$

if and only if

$$e_E(\mathcal{P}) \quad\geq\quad \sum_{X\in\mathcal{P}}h(X) \qquad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \tag{8}$$

We can now present and prove our first contribution, a new orientation theorem. It will allow us to reduce Theorem 8 to Theorem 6. The motivation of the use of $h(X)-h(P_F^X)$ in (9) is the following. In order to be able to apply Theorem 6 we want to find an orientation $\vec{F}=(V,\vec{E}\cup A)$ of a mixed graph $F=(V,E\cup A)$ such that (4) holds in $\vec{F}$, that is $d^-_{\vec{E}\cup A}(X)\geq r_{\mathsf{M}}(S_{P_{\vec{F}}^X})-r_{\mathsf{M}}(S_X)$ and $r_{\mathsf{M}}(S_{P_{\vec{F}}^X})=r_{\mathsf{M}}(S_{P_F^X})$ for every $X\subseteq V$ or equivalently $d^-_{\vec{E}\cup A}(X)\geq r_{\mathsf{M}}(S_{P_F^X})-r_{\mathsf{M}}(S_X)$ for every $X\subseteq V$ which is (9) for $h(X)=-r_{\mathsf{M}}(S_X)$.

Theorem 10. *Let $F=(V,E\cup A)$ be a mixed graph and h an integer-valued intersecting supermodular set function on V. There exists an orientation $\vec{E}$ of E such that*

$$d^-_{\vec{E}\cup A}(X) \quad\geq\quad h(X)-h(P_F^X) \qquad \textit{for every } X\subseteq V \tag{9}$$

if and only if for every strongly connected component C of F and every set $\mathcal{P}$ of subsets of P_F^C such that $Z\cap C\neq\emptyset, e_{E\cup A}(Z-C)=0$ for every $Z\in\mathcal{P}$; and $Z\cap Z'\cap C=\emptyset$ for every $Z,Z'\in\mathcal{P}$,

$$e_{E\cup A}(\mathcal{P}) \quad\geq\quad \sum_{Z\in\mathcal{P}}(h(Z)-h(P_F^C)). \tag{10}$$

Proof. To prove the **necessity**, let $\vec{\boldsymbol{E}}$ be an orientation of E such that (9) holds, $\boldsymbol{C}$ a strongly connected component of F and $\boldsymbol{\mathcal{P}}$ a set of subsets of P_F^C such that $Z \cap C \neq \emptyset, e_{E\cup A}(Z-C)=0$ for every $Z \in \mathcal{P}$; and $Z \cap Z' \cap C = \emptyset$ for all $Z, Z' \in \mathcal{P}$. It follows that $\rho_{\vec{E}\cup A}(Z) \cap \rho_{\vec{E}\cup A}(Z') = \emptyset$ and $P_F^Z = P_F^C$ for all $Z, Z' \in \mathcal{P}$. Then, by (9) applied for every $Z \in \mathcal{P}$, we obtain (10) because

$$e_{E\cup A}(\mathcal{P}) \geq e_{\vec{E}\cup A}(\mathcal{P}) = \sum_{Z\in\mathcal{P}} d_{\vec{E}\cup A}(Z) \geq \sum_{Z\in\mathcal{P}} (h(Z) - h(P_F^C)). \tag{11}$$

To prove the **sufficiency**, let $(\boldsymbol{F} = (\boldsymbol{V},\ \boldsymbol{E} \cup \boldsymbol{A}),\ \boldsymbol{h})$ be a counterexample for Theorem 10 that minimizes $|V|$. Let $\boldsymbol{C}$ be a strongly connected component of F such that $e_{E\cup A}(\overline{C}) = 0$. Let $(\boldsymbol{F_1} = (\boldsymbol{V_1},\ \boldsymbol{E_1} \cup \boldsymbol{A_1}),\ \boldsymbol{h_1})$ be obtained from (F,h) by deleting the elements in C. As $e_{E\cup A}(\overline{C}) = 0$, we have $e_{E_1\cup A_1}(X) = e_{E\cup A}(X)$, $P_{F_1}^X = P_F^X$ and $h_1(X) = h(X)$ for every $X \subseteq V_1$. Then, since (F,h) satisfies (10), so does (F_1,h_1). Hence, by the minimality of (F,h), there exists an orientation $\vec{\boldsymbol{E}}_{\boldsymbol{1}}$ of E_1 such that

$$d^-_{\vec{E}_1\cup A_1}(X) \quad \geq \quad h(X) - h(P_F^X) \qquad \text{for every } X \subseteq V_1. \tag{12}$$

Let us now consider the subgraph $\boldsymbol{F_2} = (\boldsymbol{C},\ \boldsymbol{E_2} \cup \boldsymbol{A_2})$ of F induced by C. Moreover, let us define $\boldsymbol{h_2}(X) = \max\{h(Y) - d_A^-(Y) : Y \subseteq P_F^C, Y \cap C = X, e_{E\cup A}(Y-C) = 0\}$ for every $\emptyset \neq X \subseteq C$. For any non-empty set X_i in C, let $\boldsymbol{Y_i}$ be a set that provides $h_2(X_i)$. Gao, Yang [11] proved that h_2 is intersecting supermodular.

Claim 3. *h_2 is an intersecting supermodular set function on C.*

Proof. For intersecting sets X_1 and X_2 in C, let $\boldsymbol{X_3} = X_1 \cap X_2$, $\boldsymbol{X_4} = X_1 \cup X_2$, $\boldsymbol{Y'_3} = Y_1 \cap Y_2$ and $\boldsymbol{Y'_4} = Y_1 \cup Y_2$. Note that, for $i = 3,4$, we have $Y'_i \subseteq P_F^C, Y'_i \cap C = X_i$ and $e_{E\cup A}(Y'_i - C) = 0$, and hence $h(Y'_i) - d_A^-(Y'_i) \leq h_2(X_i)$. Then, by the intersecting supermodularity of h and $-d_A^-$, we get that

$$\begin{aligned} h_2(X_1) + h_2(X_2) &= h(Y_1) - d_A^-(Y_1) + h(Y_2) - d_A^-(Y_2) \\ &\leq h(Y'_3) - d_A^-(Y'_3) + h(Y'_4) - d_A^-(Y'_4) \\ &\leq h_2(X_3) + h_2(X_4) \\ &= h_2(X_1 \cap X_2) + h_2(X_1 \cup X_2), \end{aligned}$$

so h_2 is intersecting supermodular. □

Let $\boldsymbol{h'}$ be defined by $h'(X) = h_2(X) - h(P_F^C)$ for every $\emptyset \neq X \subseteq C$ and $h'(\emptyset) = 0$. By the Claim 3, h' is intersecting supermodular on C. Let $\boldsymbol{\mathcal{P}} = \{X_1, \dots, X_t\}$ be a subpartition of C and $\boldsymbol{\mathcal{P}'} = \{Y_i : X_i \in \mathcal{P}\}$. Then $\mathcal{P}'$ is a set of subsets of P_F^C such that $Y_i \cap C \neq \emptyset$ and $e_{E\cup A}(Y_i - C) = 0$ for $1 \leq i \leq t$ and $Y_i \cap Y_j \cap C = \emptyset$ for $1 \leq i < j \leq t$. It follows, by (10), that

$$\begin{aligned} e_{E_2}(\mathcal{P}) &= e_{E\cup A}(\mathcal{P}') - e_A(\mathcal{P}') \\ &\geq \sum_{Y_i\in\mathcal{P}'} (h(Y_i) - h(P_F^C) - d_A^-(Y_i)) \\ &= \sum_{Y_i\in\mathcal{P}'} (h_2(X_i) - h(P_F^C)) \\ &= \sum_{X_i\in\mathcal{P}} h'(X_i). \end{aligned}$$

Thus the graph (C, E_2) satisfies (8). In particular, we get that $0 = e_{E_2}(C) \geq h'(C)$. Moreover, $h'(C) = h_2(C) - h(P_F^C) \geq h(P_F^C) - h(P_F^C) = 0$. Hence $h'(C) = 0$. Then, by Theorem 9, there exists an orientation $\vec{\boldsymbol{E}}_{\boldsymbol{2}}$ of E_2 such that $d^-_{\vec{E}_2}(X) \geq h'(X) = h_2(X) - h(P_F^C)$ for every $X \subseteq C$. It follows that for every $Y \subseteq P_F^C$ with $Y \cap C \neq \emptyset$ and $e_{E\cup A}(Y - C) = 0$, we have

$$d^-_{\vec{E}_2}(Y) = d^-_{\vec{E}_2}(Y \cap C) \geq h(Y) - h(P_F^C) - d_A^-(Y). \tag{13}$$

Let $\vec{\boldsymbol{F}} = (V, \vec{E} \cup A)$, where $\vec{E} = \vec{E}_1 \cup \vec{E}_2$. To finish the proof we show that $\vec{F}$ satisfies (9). If $X \subseteq V_1$ then, by (12), (9) holds. If $X \subseteq C$ then, by (13) applied for X, (9) holds. We suppose from now on that $X \cap C \neq \emptyset \neq X - C$. Let $\boldsymbol{Z} = P_F^{X-C}$, $\boldsymbol{Y} = Z \cap P_F^C$ and $\boldsymbol{W} = Y \cup (X \cap C)$. Then

$X \cap Z = X - C, P_F^C \cap (X \cup Z) = W$ and $P_F^C \cup (X \cup Z) = P_F^X$, $e_{E \cup A}(Y) = 0$. Thus, by (12) for $X - C$, (13) for W and the intersecting supermodularity of h, first for X and Z, and then for P_F^C and $X \cup Z$, we have

$$\begin{aligned} d^-_{\vec{E} \cup A}(X) &\geq d^-_{\vec{E}_1 \cup A}(X - C) + d^-_{\vec{E}_2 \cup A}(W) \\ &\geq (h(X - C) - h(Z)) + (h(W) - h(P_F^C)) \\ &\geq (h(X) - h(X \cup Z)) + (h(X \cup Z) - h(P_F^X)) \\ &= h(X) - h(P_F^X), \end{aligned}$$

so (9) holds. □

We mention that Theorem 9 and hence Theorem 10 also work for mixed hypergraphs. This shows that the result of Hörsch, Szigeti [15] can also be obtained from a theorem of Fortier et al. [6] on matroid-reachability-based packing of hyperarborescences.

For the sake of completeness we show that Theorems 9 and 10 are in fact equivalent. We have just seen that Theorem 9 implies Theorem 10. Now let us see the other direction.

Proof. Let $(G = (V, E), h)$ be an instance of Theorem 9 such that $h(V) = 0$ and (8) holds. If G is connected then Theorem 10 for (G, h) reduces to Theorem 9 because $h(P_G^X) = h(V) = 0$ for every $\emptyset \neq X \subseteq V$. If the number $\boldsymbol{k}$ of connected components of G is larger than one then we need some more effort. Let $\boldsymbol{\ell} = \max\{h(X) + h(Y) - h(X \cup Y) : X, Y \subset V, X \cap Y = \emptyset\}$ and $\boldsymbol{m} = \max\{k, \ell\}$. Let $\boldsymbol{G'} = (V', E')$ be obtained from G by adding a new vertex $\boldsymbol{s}$ and by connecting s to V by m edges such that G' is connected. Since $m \geq k$, this is possible. Let $\boldsymbol{h'}$ be defined as follows: $h'(s) = m$ otherwise $h'(X) = h(X - s)$ for every $s \neq X \subseteq V'$. Then h' is an integer-valued intersecting supermodular set function on V'. Indeed, the only case that must be checked is for pairs $X' = X \cup s$ and $Y' = Y \cup s$ with $\emptyset \neq X, Y \subseteq V$ and $X \cap Y = \emptyset$. Then $h'(X') + h'(Y') - h'(X' \cup Y') = h(X) + h(Y) - h(X \cup Y) \leq \ell \leq m = h'(s) = h'(X' \cap Y')$. Note that since G' is connected and $h(V) = 0$, we have

$$h'(P_{G'}^X) = h'(V') = h(V) = 0 \text{ for every } \emptyset \neq X \subseteq V'. \tag{14}$$

We now show that (8) implies (10) for (G', h'). We only need to check (10) for subpartitions $\mathcal{P}$ of V' because G' is connected. If $\{s\} \in \mathcal{P}$ then, by (8) and (14), we have $e_{E'}(\mathcal{P}) = m + e_E(\mathcal{P} - \{s\}) \geq h'(s) + \sum_{X \in \mathcal{P} - \{s\}} h(X) = \sum_{X \in \mathcal{P}} h'(X) = \sum_{X \in \mathcal{P}} (h'(X) - h'(P_{G'}^{V'}))$. Otherwise, by (8) and (14), we have $e_{E'}(\mathcal{P}) \geq e_E(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} h(X - s) = \sum_{X \in \mathcal{P}} h'(X) = \sum_{X \in \mathcal{P}} (h'(X) - h'(P_{G'}^{V'}))$ so (10) holds for (G', h').

We can hence apply Theorem 10 for (G', h') to get an orientation $\vec{G'} = (V', \vec{E'})$ such that

$$d^-_{\vec{E'}}(X) \geq h'(X) - h'(P_{G'}^X) \text{ for every } \emptyset \neq X \subseteq V'. \tag{15}$$

Deleting s from $\vec{G'}$ we obtain an orientation $\vec{G} = (V, \vec{E})$ of G. We conclude by showing that $\vec{G}$ satisfies (7). By applying (15) for s and (14), we get that $m = d_E(s) \geq d^-_{\vec{E'}}(s) \geq h'(s) - h'(P_{G'}^s) = m$ so equality holds everywhere. In particular, $d_E(s) = d^-_{\vec{E'}}(s)$. Hence all the edges from s to V are oriented from V to s, thus $d^-_{\vec{E}}(X) = d^-_{\vec{E'}}(X)$ for every $X \subseteq V$. Then, by (15) and (14), we have $d^-_{\vec{E}}(X) = d^-_{\vec{E'}}(X) \geq h'(X) - h'(P_{G'}^X) = h(X)$ for every $\emptyset \neq X \subseteq V$, so (7) holds. This completes the proof. □

We conclude this section by showing that Theorem 8 easily follows from Theorems 6 and 10.

Proof. Let (F, S, M) be an instance of Theorem 8 that satisfies (6). Then, for $h(X) = -r_{\mathsf{M}}(S_X)$ for all $X \subseteq V$, (10) holds, so by Theorem 10 applied for (F, h), there exists an orientation $\vec{E}$ of E such that in $\vec{F} = (V, \vec{E} \cup A)$ (9) holds. Let $X \subseteq V$. Since $P_{\vec{F}}^X \subseteq P_F^X$ and r_{M} is non-decreasing, we have $r_{\mathsf{M}}(S_{P_{\vec{F}}^X}) \leq r_{\mathsf{M}}(S_{P_F^X})$. By (9) applied for $P_{\vec{F}}^X$, we have $r_{\mathsf{M}}(S_{P_{\vec{F}}^X}) \geq r_{\mathsf{M}}(S_{P_F^X})$. Hence $r_{\mathsf{M}}(S_{P_{\vec{F}}^X}) = r_{\mathsf{M}}(S_{P_F^X})$. Thus (9) implies that (4) holds in $(\vec{F}, S, \mathsf{M})$. Then, by Theorem 6, there exists a matroid-reachability-based packing of arborescences in $(\vec{F}, S, \mathsf{M})$. Since $r_{\mathsf{M}}(S_{P_{\vec{F}}^X}) = r_{\mathsf{M}}(S_{P_F^X})$, by replacing the arcs in $\vec{E}$ by the edges in E, we obtain a matroid-reachability-based packing of mixed arborescences in (F, S, M). □

4 Packing mixed hyperarborescences

In this section we first list known results on mixed hyperarborescences that are related to our second contribution on (f, g)-bounded k-regular (ℓ, ℓ')-limited packings of mixed hyperarborescences and we provide the proof of the new result.

Theorem 1 was extended for the case when the roots of the arborescences are not fixed but the number of arborescences in the packing rooted at any vertex is bounded.

Theorem 11 (Frank [7], Cai [3]). *Let $D = (V, A)$ be a digraph, $f, g : V \to \mathbb{Z}_+$ functions and $k \in \mathbb{Z}_+$. There exists an (f, g)-bounded packing of k spanning arborescences in D if and only if*

$$g(v) \geq f(v) \quad \textit{for every } v \in V, \tag{16}$$

$$e_A(\mathcal{P}) \geq k|\mathcal{P}| - \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \tag{17}$$

If S is a multiset of V and $f(v) = g(v) = |S_v|$ for all $v \in V$ then Theorem 11 reduces to Theorem 1.

Theorem 11 can be generalized for the case when the arborescences are not necessarily spanning but every vertex must belong to the same number of arborescences in the packing. For $g : V \to \mathbb{Z}_+$ and $k \in \mathbb{Z}_+$, let $\boldsymbol{g_k}(v) = \min\{k, g(v)\}$ for every $v \in V$. For convenience, we present not the original version of the result of [2] which is about packing spanning branchings but one that fits better to our framework.

Theorem 12 (Bérczi, Frank [2]). *Let $D = (V, A)$ be a digraph, $f, g : V \to \mathbb{Z}_+$ functions and $k, \ell, \ell' \in \mathbb{Z}_+$. There exists an (f, g)-bounded k-regular (ℓ, ℓ')-limited packing of arborescences in D if and only if*

$$g_k(v) \geq f(v) \quad \textit{for every } v \in V, \tag{18}$$

$$\min\{g_k(V), \ell'\} \geq \ell \tag{19}$$

$$e_A(\mathcal{P}) \geq k|\mathcal{P}| - \min\{\ell' - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \tag{20}$$

For $k = \ell = \ell'$, Theorem 12 reduces to Theorem 11.

Theorem 1 was generalized for dypergraphs as follows. We only need the special case when the multiset S is equal to k times a vertex s.

Theorem 13 (Frank, T. Király, Z. Király [10]). *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, $s \in V$ and $k \in \mathbb{Z}_+$. There exists a packing of k spanning s-hyperarborescences in $\mathcal{D}$ if and only if*

$$d^-_{\mathcal{A}}(X) \geq k \quad \textit{for every } \emptyset \neq X \subseteq V - s.$$

Theorem 13 easily implies the following corollary that we will apply in the proof of our new result.

Corollary 1. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph and S a multiset of V. There exists a k-regular packing of s-hyperarborescences ($s \in S$) in $\mathcal{D}$ if and only if*

$$|S_v| \leq k \quad \textit{for every } v \in V, \tag{21}$$

$$d^-_{\mathcal{A}}(X) \geq k - |S_X| \quad \textit{for every } \emptyset \neq X \subseteq V. \tag{22}$$

A common extension of Theorems 2 and 11 was provided by Gao, Yang [12].

Theorem 14 (Gao, Yang [12]). *Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \to \mathbb{Z}_+$ functions, and $k \in \mathbb{Z}_+$. There exists an (f, g)-bounded packing of k spanning mixed arborescences in F if and only if* (16) *holds and*

$$e_{E\cup A}(\mathcal{P}) \geq k|\mathcal{P}| - \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \textit{ for every subpartition } \mathcal{P} \textit{ of } V.$$

If S is a multiset of V and $f(v) = g(v) = |S_v|$ for every $v \in V$ then Theorem 14 reduces to Theorem 2. If F is a digraph then Theorem 14 reduces to Theorem 11.

Theorem 14 can be generalized for mixed hypergraphs.

Theorem 15 (Hörsch, Szigeti [14]). *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}_+$ functions, and $k \in \mathbb{Z}_+$. There exists an (f, g)-bounded packing of k spanning mixed hyperarborescences in $\mathcal{F}$ if and only if* (16) *holds and*

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) \geq k|\mathcal{P}| - \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V.$$

If $\mathcal{F}$ is a mixed graph then Theorem 15 reduces to Theorem 14.

The main contribution of the present paper is a common generalization of Theorems 12 and 15.

Theorem 16. *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}_+$ functions, and $k, \ell, \ell' \in \mathbb{Z}_+ - \{0\}$. There exists an (f, g)-bounded k-regular (ℓ, ℓ')-limited packing of mixed hyperarborescences in $\mathcal{F}$ if and only if* (18) *and* (19) *hold and*

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) \quad \geq \quad k|\mathcal{P}| - \min\{\ell' - f(\overline{\cup\mathcal{P}}), g_k(\cup\mathcal{P})\} \qquad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \tag{23}$$

If $\mathcal{F}$ is a digraph then Theorem 16 reduces to Theorem 12. If $k = \ell = \ell'$ then Theorem 16 reduces to Theorem 15. Theorem 16 will be obtained from the theory of generalized polymatroids and some matroid construction for mixed hypergraphs. We now explain these concepts.

Generalized polymatroids were introduced by Hassin [13] and independently by Frank [8]. For a pair (p, b) of set functions on S and $\alpha, \beta \in \mathbb{R}$, let us introduce the polyhedra

$$\begin{aligned} \boldsymbol{Q(p,b)} &= \{x \in \mathbb{R}^S : p(Z) \leq x(Z) \leq b(Z) \ \text{ for all } Z \subseteq S\}, \\ \boldsymbol{K(\alpha,\beta)} &= \{x \in \mathbb{R}^S : \alpha \leq x(S) \leq \beta\}. \end{aligned}$$

If $p(\emptyset) = b(\emptyset) = 0$, p is supermodular, b is submodular and $b(X) - p(Y) \geq b(X - Y) - p(Y - X)$ for all $X, Y \subseteq S$, the polyhedron $Q(p, b)$ is called a *generalized-polymatroid*, shortly *g-polymatroid*. The polyhedron $K(\alpha, \beta)$ is called a *plank*. The Minkowski sum of the n g-polymatroids $Q(p_i, b_i)$ is denoted by $\boldsymbol{\sum_1^n Q(p_i, b_i)}$. We will need the following results on g-polymatroids, for more details see [9].

Theorem 17 (Frank [9])**.** *The following hold:*

1. *Let $Q(p, b)$ be a g-polymatroid, $K(\alpha, \beta)$ a plank and $M = Q(p, b) \cap K(\alpha, \beta)$.*
   - *(i) $M \neq \emptyset$ if and only if $p \leq b$, $\alpha \leq \beta$, $p(S) \leq \beta$ and $\alpha \leq b(S)$.*
   - *(ii) M is a g-polymatroid.*
   - *(iii) If $M \neq \emptyset$ then $M = Q(p^\alpha_\beta, b^\alpha_\beta)$ with*

$$\boldsymbol{p^\alpha_\beta(Z)} = \max\{p(Z), \alpha - b(S - Z)\}, \quad \boldsymbol{b^\alpha_\beta(Z)} = \min\{b(Z), \beta - p(S - Z)\}. \tag{24}$$

2. *Let $Q(p_1, b_1)$ and $Q(p_2, b_2)$ be two non-empty g-polymatroids and $M = Q(p_1, b_1) \cap Q(p_2, b_2)$.*
   - *(i) $M \neq \emptyset$ if and only if $p_1 \leq b_2$ and $p_2 \leq b_1$.*
   - *(ii) If p_1, b_1, p_2, b_2 are integral and $M \neq \emptyset$ then M contains an integral element.*
3. *Let $Q(p_i, b_i)$ be n non-empty g-polymatroids. Then $\sum_1^n Q(p_i, b_i) = Q(\sum_1^n p_i, \sum_1^n b_i)$.*

Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$, let $\boldsymbol{\mathcal{I}_\mathcal{H}} = \{\mathcal{Z} \subseteq \mathcal{E} : |V(\mathcal{Z}')| > |\mathcal{Z}'| \text{ for all } \emptyset \neq \mathcal{Z}' \subseteq \mathcal{Z}\}$. Lorea [18] showed that $\mathcal{I}_\mathcal{H}$ is the set of independent sets of a matroid $\boldsymbol{\mathsf{M}_\mathcal{H}}$ on $\mathcal{E}$, called the *hypergraphic matroid* of the hypergraph $\mathcal{H}$. We also need the *k-hypergraphic matroid* $\boldsymbol{\mathsf{M}^k_\mathcal{H}}$ of $\mathcal{H}$ which is the k-sum matroid of $\mathsf{M}_\mathcal{H}$, that is the matroid on ground set $\mathcal{E}$ in which a subset of $\mathcal{E}$ is independent if it can be partitioned into k independent sets of $\mathsf{M}_\mathcal{H}$. Hörsch, Szigeti [14] extended the previous construction for mixed hypergraphs as follows. Let $\mathcal{F} = (V, \mathcal{A} \cup \mathcal{E})$ be a mixed hypergraph. For a subpartition $\mathcal{P}$ of V, $\boldsymbol{\mathcal{A}(\mathcal{P})}$ and $\boldsymbol{\mathcal{E}(\mathcal{P})}$ denote the set of dyperedges and the set of hyperedges that enter some member of $\mathcal{P}$. Let $\boldsymbol{\mathcal{H}_\mathcal{F}} = (V, \boldsymbol{\mathcal{E}_\mathcal{A}} \cup \mathcal{E})$ the underlying hypergraph of $\mathcal{F}$ and $\boldsymbol{\mathcal{D}_\mathcal{F}} = (V, \mathcal{A} \cup \mathcal{A}_\mathcal{E})$ the *directed extension* of $\mathcal{F}$ where $\boldsymbol{\mathcal{A}_\mathcal{E}} = \bigcup_{e\in\mathcal{E}} \mathcal{A}_e$ and for $e \in \mathcal{E}$, $\boldsymbol{\mathcal{A}_e} = \{(e - x, x) : x \in e\}$. The *extended k-hypergraphic matroid* $\boldsymbol{\mathsf{M}^k_\mathcal{F}}$ of $\mathcal{F}$ on $\mathcal{A} \cup \mathcal{A}_\mathcal{E}$ is obtained from $\mathsf{M}^k_{\mathcal{H}_\mathcal{F}}$ by replacing every $e \in \mathcal{E}$ by $|e|$ parallel copies of itself, associating these elements to the dyperedges in $\mathcal{A}_e$ and associating every hyperedge of $\mathcal{E}_\mathcal{A}$ to the corresponding dyperedge in $\mathcal{A}$. It is shown in [14] that the rank function of the extended k-hypergraphic matroid $\mathsf{M}^k_\mathcal{F}$ satisfies for all $\mathcal{Z} \subseteq \mathcal{A} \cup \mathcal{A}_\mathcal{E}$,

$$r_{\mathsf{M}^k_\mathcal{F}}(\mathcal{Z}) = \min\{|\mathcal{Z} \cap \mathcal{A}(\mathcal{P})| + |\{e \in \mathcal{E}(\mathcal{P}) : \mathcal{Z} \cap \mathcal{A}_e \neq \emptyset\}| + k(|V| - |\mathcal{P}|) : \mathcal{P} \text{ partition of } V\}. \tag{25}$$

Theorem 16 will follow from the following lemma.

Lemma 1. *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}_+$ functions, and $k, \ell, \ell' \in \mathbb{Z}_+ - \{0\}$. Let $\mathsf{M}_{\boldsymbol{v}} = (\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v), r_v)$ be the free matroid for all $v \in V$ and $\mathsf{M}^{\boldsymbol{k}}_{\boldsymbol{\mathcal{F}}}$ the extended k-hypergraphic matroid of $\mathcal{F}$ on $\mathcal{A} \cup \mathcal{A}_\mathcal{E}$. Let us define the following polyhedron*

$$\boldsymbol{T} = (\sum_{v\in V}(Q(0, r_v) \cap K(k - g_k(v), k - f(v)))) \cap K(k|V| - \ell', k|V| - \ell) \cap Q(0, r_{\mathsf{M}^k_\mathcal{F}}).$$

(a) The characteristic vectors of the dyperedge sets of the (f, g)-bounded k-regular (ℓ, ℓ')-limited packings of hyperarborescences in orientations of $\mathcal{F}$ are exactly the integer points of T.

(b) $T \neq \emptyset$ if and only if (18) *and* (19) *hold and for every $\mathcal{Z} \subseteq \mathcal{A} \cup \mathcal{A}_\mathcal{E}$,*

$$\sum_{v\in V} \max\{0, k - g_k(v) - d^-_\mathcal{Z}(v)\} \leq r_{\mathsf{M}^k_\mathcal{F}}(\overline{\mathcal{Z}}), \tag{26}$$

$$k|V| - \ell' - \sum_{v\in V} \min\{d^-_\mathcal{Z}(v), k - f(v)\} \leq r_{\mathsf{M}^k_\mathcal{F}}(\overline{\mathcal{Z}}). \tag{27}$$

(c) (26) *and* (27) *are equivalent to* (23).

Proof. **(a)** To prove the **necessity**, let $\boldsymbol{\mathcal{B}}$ be an (f, g)-bounded k-regular (ℓ, ℓ')-limited packing of hyperarborescences in an orientation $\vec{\boldsymbol{\mathcal{F}}}$ of $\mathcal{F}$. Let $\boldsymbol{S}$ be the root set of the hyperarborescences in $\mathcal{B}$ and $\vec{\boldsymbol{\mathcal{Z}}}$ the dyperedge set of $\mathcal{B}$. Since the packing is (f, g)-bounded k-regular (ℓ, ℓ')-limited, we have

$$f(v) \leq |S_v| \leq g_k(v) \quad \text{for all } v \in V, \tag{28}$$

$$k - d^-_{\vec{\mathcal{Z}}}(v) = |S_v| \quad \text{for all } v \in V, \tag{29}$$

$$\ell \leq |S| \leq \ell'. \tag{30}$$

By (29), we get

$$k|V| - |\vec{\mathcal{Z}}| = \sum_{v\in V}(k - d^-_{\vec{\mathcal{Z}}}(v)) = \sum_{v\in V} |S_v| = |S|. \tag{31}$$

Let $\boldsymbol{m}$ be the characteristic vector of $\vec{\mathcal{Z}}$ and $\boldsymbol{m_v}$ the restriction of m on $\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)$ for all $v \in V$. Then m_v is a characteristic vector, so

$$m_v \in Q(0, r_v) \text{ for all } v \in V. \tag{32}$$

By (28), (29) and $d^-_{\vec{\mathcal{Z}}}(v) = m_v(\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v))$ for all $v \in V$, we obtain that

$$m_v \in K(k - g_k(v), k - f(v)) \text{ for all } v \in V. \tag{33}$$

It follows, by (32) and (33), that

$$m \in \sum_{v\in V}(Q(0, r_v) \cap K(k - g_k(v), k - f(v))). \tag{34}$$

By (30), (31), and $|\vec{\mathcal{Z}}| = m(\mathcal{A} \cup \mathcal{A}_\mathcal{E})$, we obtain that

$$m \in K(k|V| - \ell', k|V| - \ell). \tag{35}$$

Claim 4. *$\vec{\mathcal{Z}}$ is independent in $\mathsf{M}^k_\mathcal{F}$.*

Proof. We first show that $\vec{\mathcal{Z}}$ is the dyperedge set of a packing of k spanning hyperbranchings in $\vec{\mathcal{F}}$. Indeed, let $\vec{\boldsymbol{\mathcal{G}}}$ be the dypergraph with vertex set $V \cup \{s\}$ where $\boldsymbol{s}$ is a new vertex, and dyperedge set $\vec{\mathcal{Z}} \cup A$ where $\boldsymbol{A} = \{ss' : s' \in S\}$. Since $\vec{\mathcal{Z}}$ is the dyperedge set of a k-regular packing of hyperarborescences in $\vec{\mathcal{F}}$, by Corollary 1, we have $d^-_{\vec{\mathcal{Z}}\cup A}(X) = d^-_{\vec{\mathcal{Z}}}(X) + d^-_A(X) = d^-_{\vec{\mathcal{Z}}}(X) + |S_X| \geq k$. Then, by Theorem 13, there exists a packing of k spanning s-hyperarborescences in $\vec{\mathcal{G}}$. By deleting the vertex s from each s-hyperarborescence in the packing we obtain a packing of k spanning hyperbranchings in $\vec{\mathcal{F}}$ with dyperedge set $\vec{\mathcal{Z}}'$. Since $\vec{\mathcal{Z}} \supseteq \vec{\mathcal{Z}}'$ and $|\vec{\mathcal{Z}}'| \geq k|V| - |S| = |\vec{\mathcal{Z}}|$, we get that $\vec{\mathcal{Z}} = \vec{\mathcal{Z}}'$. Hence $\vec{\mathcal{Z}}$ is the dyperedge set of a packing of k spanning hyperbranchings in $\vec{\mathcal{F}}$.

For any hyperbranching, the number of its vertices is at least the number of heads of its dyperedges plus the number of its roots and hence strictly larger than the number of its dyperedges. Thus the hyperedge set of the underlying hypergraph of each hyperbranching is independent in $\mathsf{M}_{\mathcal{H}_\mathcal{F}}$. As the hyperbranchings in the packing are dyperedge disjoint, it follows that $\mathcal{Z}$ is independent in $\mathsf{M}^k_{\mathcal{H}_\mathcal{F}}$. Then, since $\vec{\mathcal{Z}}$ is in the orientation $\vec{\mathcal{F}}$ of $\mathcal{F}$, $\vec{\mathcal{Z}}$ is independent in $\mathsf{M}^k_{\vec{\mathcal{F}}}$. □

By Claim 4 and since m is the characteristic vector of $\vec{\mathcal{Z}}$, we get that

$$m \in Q(0, r_{\mathsf{M}^k_{\vec{\mathcal{F}}}}). \tag{36}$$

It follows, by (34), (35), and (36), that m is an integer point of T.

To prove the **sufficiency**, let $\boldsymbol{m} = (m_v)_{v\in V}$ be an integer point of T, that is $m_v \in Q(0, r_v) \cap K(k - g_k(v), k - f(v))$ for all $v \in V$ and $m \in K(k|V| - \ell', k|V| - \ell) \cap Q(0, r_{\mathsf{M}^k_{\vec{\mathcal{F}}}})$. Since m_v is an integer point in $Q(0, r_v)$, m_v is the characteristic vector of a subset $\vec{\boldsymbol{\mathcal{Z}}}_{\boldsymbol{v}}$ of $\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)$. Since $m_v \in K(k - g_k(v), k - f(v))$, we have

$$k - g_k(v) \leq m_v(\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)) = |\vec{\mathcal{Z}}_v| = m_v(\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)) \leq k - f(v). \tag{37}$$

Let $\vec{\boldsymbol{\mathcal{Z}}} = \bigcup_{v\in V} \vec{\mathcal{Z}}_v$. Note that $d^-_{\vec{\mathcal{Z}}}(v) = |\vec{\mathcal{Z}}_v|$ for all $v \in V$. Then, by $f \geq 0$, we have $k - d^-_{\vec{\mathcal{Z}}}(v) \geq f(v) \geq 0$ for all $v \in V$. Since $m \in K(k|V| - \ell', k|V| - \ell)$, we have

$$k|V| - \ell' \leq m(\mathcal{A} \cup \mathcal{A}_\mathcal{E}) = |\vec{\mathcal{Z}}| = m(\mathcal{A} \cup \mathcal{A}_\mathcal{E}) \leq k|V| - \ell. \tag{38}$$

Since $m \in Q(0, r_{\mathsf{M}^k_{\vec{\mathcal{F}}}})$, we get that $\vec{\mathcal{Z}}$ is independent in $\mathsf{M}^k_{\vec{\mathcal{F}}}$. It follows that $\vec{\mathcal{Z}}$ is a subset of the dyperedge set of an orientation $\vec{\boldsymbol{\mathcal{F}}}$ of $\mathcal{F}$ and in the hypergraph $\boldsymbol{\mathcal{H}_\mathcal{F}} = (V, \mathcal{E}_\mathcal{A} \cup \mathcal{E})$ we have for all $X \subseteq V$,

$$|\mathcal{Z}(X)| \leq r_{\mathsf{M}^k_{\mathcal{H}_\mathcal{F}}}(\mathcal{Z}(X)) \leq k(|X| - 1). \tag{39}$$

Let $\boldsymbol{S}$ be the multiset of V such that $|S_v| = k - d^-_{\vec{\mathcal{Z}}}(v)$ for all $v \in V$. Since $k - d^-_{\vec{\mathcal{Z}}}(v) \geq 0$ for all $v \in V$, S exists. As $d^-_{\vec{\mathcal{Z}}} \geq 0$, (21) holds. Since for all $X \subseteq V$, by (39), we have

$$d^-_{\vec{\mathcal{Z}}}(X) = \sum_{v\in X} d^-_{\vec{\mathcal{Z}}}(v) - |\vec{\mathcal{Z}}(X)| = \sum_{v\in X}(k - |S_v|) - |\mathcal{Z}(X)| \geq k|X| - |S_X| - k(|X| - 1) = k - |S_X|,$$

so (22) holds for $\vec{\boldsymbol{\mathcal{F}}}' = (V, \vec{\mathcal{Z}})$. Then, by Corollary 1, there exists a k-regular packing of s-hyperarborescences ($s \in S$) in $\vec{\mathcal{F}}'$ and hence in $\vec{\mathcal{F}}$. Since the number of dyperedges in the packing is $k|V| - |S| = \sum_{v\in V}(k - |S_v|) = \sum_{v\in V} d^-_{\vec{\mathcal{Z}}}(v) = |\vec{\mathcal{Z}}|$, the dyperedge set of the packing is $\vec{\mathcal{Z}}$. As for all $v \in V$, by (37), we have

$$f(v) \leq k - |\vec{\mathcal{Z}}_v| = k - d^-_{\vec{\mathcal{Z}}}(v) = |S_v| = k - d^-_{\vec{\mathcal{Z}}}(v) = k - |\vec{\mathcal{Z}}_v| \leq g_k(v) \leq g(v),$$

so the packing is (f, g)-bounded. Since, by (38), we have

$$\ell \leq k|V| - |\vec{\mathcal{Z}}| = |S| = k|V| - |\vec{\mathcal{Z}}| \leq \ell',$$

so the packing is (ℓ, ℓ')-limited. Finally, as $\vec{\mathcal{F}}$ is an orientation of $\mathcal{F}$, the proof is complete.

(b) By Theorem 17.1, for all $v \in V$, $Q(0, r_v) \cap K(k - g_k(v), k - f(v)) \neq \emptyset$ if and only if $0 \leq r_v$ (that always holds), $k - g_k(v) \leq k - f(v)$ (that is (18) holds), $0 \leq k - f(v)$ (that holds by the previous inequality) and $k - g_k(v) \leq r_v(\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)) = d^-_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)$. Then $Q(0, r_v) \cap K(k - g_k(v), k - f(v)) = Q(p_v, b_v)$ where, by (24), we have for all $\mathcal{Z}_v \subseteq \rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)$,

$$\boldsymbol{p_v}(\mathcal{Z}_v) = \max\{0, k - g_k(v) - d^-_{\overline{\mathcal{Z}}_v}(v)\}, \quad \boldsymbol{b_v}(\mathcal{Z}_v) = \min\{d^-_{\mathcal{Z}_v}(v), k - f(v)\}. \tag{40}$$

By Theorem 17.3, $\sum_{v\in V} Q(p_v, b_v) = Q(p_\Sigma, b_\Sigma)$ where

$$\boldsymbol{p_\Sigma} = \sum_{v\in V} p_v \text{ and } \boldsymbol{b_\Sigma} = \sum_{v\in V} b_v. \tag{41}$$

By Theorem 17.1, $Q(p_\Sigma, b_\Sigma) \cap K(k|V| - \ell', k|V| - \ell) \neq \emptyset$ if and only if $Q(p_v, b_v) \neq \emptyset$ for all $v \in V$, $k|V| - \ell' \leq k|V| - \ell$ (which is equivalent to one of the conditions in (19)), $p_\Sigma(\mathcal{A} \cup \mathcal{A}_\mathcal{E}) \leq k|V| - \ell$ (which, by $p_\Sigma(\mathcal{A} \cup \mathcal{A}_\mathcal{E}) = \sum_{v \in V} p_v(\rho_{\mathcal{A} \cup \mathcal{A}_\mathcal{E}}(v)) = \sum_{v \in V} \max\{0, k - g_k(v) - d^-_{\overline{\mathcal{A} \cup \mathcal{A}_\mathcal{E}}}(v)\} = \sum_{v \in V}(k - g_k(v)) = k|V| - g_k(V)$, is equivalent to the other condition in (19)) and $b_\Sigma(\mathcal{A} \cup \mathcal{A}_\mathcal{E}) \geq k|V| - \ell'$. Then the intersection is equal to a generalized polyhedron $Q(p, b)$ where, by (24), (40), and (41), for all $\mathcal{Z} \subseteq \mathcal{A} \cup \mathcal{A}_\mathcal{E}$, we have

$$\boldsymbol{p}(\mathcal{Z}) = \max\left\{\sum_{v \in V} \max\{0, k - g_k(v) - d^-_{\overline{\mathcal{Z}}}(v)\}, k|V| - \ell' - \sum_{v \in V} \min\{d^-_{\overline{\mathcal{Z}}}(v), k - f(v)\}\right\}, \quad (42)$$

$$\boldsymbol{b}(\mathcal{Z}) = \min\left\{\sum_{v \in V} \min\{d^-_{\mathcal{Z}}(v), k - f(v)\}, k|V| - \ell - \sum_{v \in V} \max\{0, k - g_k(v) - d^-_{\mathcal{Z}}(v)\}\right\}. \quad (43)$$

By Theorem 17.2, $T = Q(p, b) \cap Q(0, r_{\mathsf{M}^k_\mathcal{F}}) \neq \emptyset$ if and only if $Q(p, b) \neq \emptyset$, $p \leq r_{\mathsf{M}^k_\mathcal{F}}$ (which, by (42), is equivalent to (26) and (27)), and $b \geq 0$ (which holds by $b \geq p \geq 0$). Note that $k - g_k(v) \leq d^-_{\mathcal{A} \cup \mathcal{A}_\mathcal{E}}(v)$ for all $v \in V$ and $b_\Sigma(\mathcal{A} \cup \mathcal{A}_\mathcal{E}) \geq k|V| - \ell'$ follow from $p \leq r_{\mathsf{M}^k_\mathcal{F}}$ applied for $\mathcal{Z} = \emptyset$ and the proof is complete.

(c) We note that (26) is equivalent to

$$k|V| - g_k(V) - \sum_{v \in V} \min\{d^-_{\mathcal{Z}}(v), k - g_k(v)\} \leq r_{\mathsf{M}^k_\mathcal{F}}(\overline{\mathcal{Z}}). \quad (44)$$

First we show that (26) and (27) imply (23). Let $\boldsymbol{\mathcal{P}}$ be a subpartition of V. Let $\boldsymbol{\mathcal{Z}} = \bigcup_{v \in \overline{\cup \mathcal{P}}} \rho_\mathcal{A}(v) \cup \bigcup_{e \in \mathcal{E}(\mathcal{F}(\overline{\cup \mathcal{P}}))} \mathcal{A}_e$ and $\boldsymbol{\mathcal{P}'} = \mathcal{P} \cup \{v\}_{v \in \overline{\cup \mathcal{P}}}$. Note that $d^-_{\mathcal{Z}}(v) = 0$ for all $v \in \cup \mathcal{P}$,

$$\sum_{v \in V} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} \leq k|\overline{\cup \mathcal{P}}| - h(\overline{\cup \mathcal{P}}) \text{ for } h \in \{g_k, f\}, \quad (45)$$

$\mathcal{P}'$ is a partition of V, and, by (25), we have

$$r_{\mathsf{M}^k_\mathcal{F}}(\overline{\mathcal{Z}}) \leq |\overline{\mathcal{Z}} \cap \mathcal{A}(\mathcal{P}')| + |\{e \in \mathcal{E}(\mathcal{P}') : \overline{\mathcal{Z}} \cap \mathcal{A}_e \neq \emptyset\}| + k(|V| - |\mathcal{P}'|) = e_{\mathcal{A} \cup \mathcal{A}_\mathcal{E}}(\mathcal{P}) + k(|V| - |\mathcal{P}| - |\overline{\cup \mathcal{P}}|). \quad (46)$$

Then (44), (45) applied for $h = g_k$ and (46) imply $e_{\mathcal{E} \cup \mathcal{A}}(\mathcal{P}) \geq k|\mathcal{P}| - g_k(\cup \mathcal{P})$. Similarly, (27), (45) applied for $h = f$ and (46) imply $e_{\mathcal{E} \cup \mathcal{A}}(\mathcal{P}) \geq k|\mathcal{P}| - \ell' + f(\overline{\cup \mathcal{P}})$. Hence (23) follows.

We now show that (23) implies (27) and (44) (and hence (26)). Let $\boldsymbol{\mathcal{Z}} \subseteq \mathcal{A} \cup \mathcal{A}_\mathcal{E}$. By (25), there exists a partition $\boldsymbol{\mathcal{P}}$ of V such that for $\boldsymbol{\mathcal{K}} = \{e \in \mathcal{E}(\mathcal{P}) : \overline{\mathcal{Z}} \cap \mathcal{A}_e \neq \emptyset\}$, we have

$$r_{\mathsf{M}^k_\mathcal{F}}(\overline{\mathcal{Z}}) = |\overline{\mathcal{Z}} \cap \mathcal{A}(\mathcal{P})| + |\mathcal{K}| + k(|V| - |\mathcal{P}|). \quad (47)$$

For $h \in \{g_k, f\}$, let $\boldsymbol{\mathcal{P}_h} = \{X \in \mathcal{P} : d^-_{\mathcal{Z}}(v) \leq k - h(v) \text{ for all } v \in X\}$. Note that $\mathcal{P}_h$ is a subpartition of V and for every $X \in \mathcal{P} - \mathcal{P}_h$, there exists a vertex $v_X \in X$ such that $d^-_{\mathcal{Z}}(v_X) > k - h(v_X)$. By the definition of $\mathcal{K}$, we have

$$\mathcal{A}_{\mathcal{E}(\mathcal{P}_h) - \mathcal{K}} \subseteq \mathcal{Z} \cap \mathcal{A}_{\mathcal{E}(\mathcal{P}_h)}. \quad (48)$$

Claim 5. $r_{\mathsf{M}^k_\mathcal{F}}(\overline{\mathcal{Z}}) + \sum_{v \in V} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} \geq e_{\mathcal{E} \cup \mathcal{A}}(\mathcal{P}_h) - k|\mathcal{P}_h| - h(\overline{\cup \mathcal{P}_h}) + k|V|$.

Proof. By (47), the definitions of $\mathcal{P}_h$ and v_X, $d^-_{\mathcal{Z}} \geq 0, k - h \geq 0$, (48), and $h \geq 0$, we have

$$\begin{aligned}
& r_{\mathsf{M}^k_\mathcal{F}}(\overline{\mathcal{Z}}) + \sum_{v \in V} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} \\
= \; & |\overline{\mathcal{Z}} \cap \mathcal{A}(\mathcal{P})| + |\mathcal{K}| + k(|V| - |\mathcal{P}|) + \sum_{v \in \cup \mathcal{P}_h} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} + \sum_{v \in \overline{\cup \mathcal{P}_h}} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} \\
\geq \; & |\overline{\mathcal{Z}} \cap \mathcal{A}(\mathcal{P}_h)| + \sum_{v \in \cup \mathcal{P}_h} d^-_{\mathcal{Z}}(v) + \sum_{X \in \mathcal{P} - \mathcal{P}_h} \sum_{v \in X} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} + |\mathcal{K}| + k(|V| - |\mathcal{P}|) \\
\geq \; & |\overline{\mathcal{Z}} \cap \mathcal{A}(\mathcal{P}_h)| + |\mathcal{Z} \cap \mathcal{A}(\mathcal{P}_h)| + |\mathcal{Z} \cap \mathcal{A}_{\mathcal{E}(\mathcal{P}_h)}| + \sum_{X \in \mathcal{P} - \mathcal{P}_h} (k - h(v_X)) + |\mathcal{K}| + k(|V| - |\mathcal{P}|) \\
\geq \; & |\mathcal{A}(\mathcal{P}_h)| + |\mathcal{A}_{\mathcal{E}(\mathcal{P}_h) - \mathcal{K}}| + \sum_{X \in \mathcal{P} - \mathcal{P}_h} (k - h(X)) + |\mathcal{K}| + k(|V| - |\mathcal{P}|) \\
\geq \; & e_{\mathcal{E} \cup \mathcal{A}}(\mathcal{P}_h) - |\mathcal{K}| + k(|\mathcal{P}| - |\mathcal{P}_h|) - h(\overline{\cup \mathcal{P}_h}) + |\mathcal{K}| + k(|V| - |\mathcal{P}|) \\
\geq \; & e_{\mathcal{E} \cup \mathcal{A}}(\mathcal{P}_h) - k|\mathcal{P}_h| - h(\overline{\cup \mathcal{P}_h}) + k|V|,
\end{aligned}$$

and the claim follows. ◻

Claim 5, applied for $h=f$, and (23) provide that $r_{\mathsf{M}^k_{\mathcal{F}}}(\overline{\mathcal{Z}})+\sum_{v\in V}\min\{d^-_{\mathcal{Z}}(v),k-f(v)\}\geq k|V|-\ell'$, so (27) holds. Similarly, Claim 5, applied for $h=g_k$, and (23) provide that $r_{\mathsf{M}^k_{\mathcal{F}}}(\overline{\mathcal{Z}})+\sum_{v\in V}\min\{d^-_{\mathcal{Z}}(v),k-g_k(v)\}\geq k|V|-g_k(V)$, so (44) holds. The proof of the theorem is complete. ◻

We finish the paper by showing that Theorem 17 and Lemma 1 imply Theorem 16.

Proof. Let $(\mathcal{F}=(V,\mathcal{E}\cup\mathcal{A}),f,g,k,\ell,\ell')$ be an instance of Theorem 16 that satisfies (18), (19) and (23). Since (23) holds, by Lemma 1(c), (26) and (27) hold. Since (18) and (19) also hold, by Lemma 1(b), the polyhedron T, defined in Lemma 1, is not empty. We have seen in the proof of Lemma 1(b) that T is the intersection of two generalized polymatroids $Q(p,b)$ and $Q(0,r_{\mathsf{M}^k_{\mathcal{F}}})$. Then, by Theorem 17.2(ii), T contains an integer point x. By Lemma 1(b), x is the characteristic vector of the dyperedge set of an (f,g)-bounded k-regular (ℓ,ℓ')-limited packing of hyperarborescences in an orientation $\vec{F}=(V,\vec{\mathcal{E}}\cup\mathcal{A})$ of $\mathcal{F}$. By replacing the dyperedges in $\vec{\mathcal{E}}$ by the underlying hyperedges in $\mathcal{E}$, we obtain the required packing. ◻

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