Packing mixed hyperarborescences

Zoltán Szigeti

Laboratory G-SCOP,
University Grenoble Alpes
Grenoble, France
zoltan.szigeti@grenoble-inp.fr

Abstract: The aim of this paper is twofold. We first provide a new orientation theorem which gives a natural and simple proof of a result of Gao, Yang [11] on matroid-reachability-based packing of mixed arborescences in mixed graphs by reducing it to the corresponding theorem of Cs. Király [17] on directed graphs. Moreover, we extend another result of Gao, Yang [12] by providing a new theorem on mixed hypergraphs having a packing of mixed hyperarborescences such that their number is at least ℓ and at most ℓ', each vertex belongs to exactly k of them, and each vertex v is the root of least $f(v)$ and at most $g(v)$ of them.

Keywords: arborescence, mixed hypergraph, packing

1 Introduction

This paper is not a survey on packing arborescences. Such a survey is in preparation, see [22]. We only present here those theorems of the topic that are closely related to the new results of this paper. A preliminary version of the paper appeared in [21].

Edmonds [5] characterized digraphs having a packing of spanning arborescences with fixed roots. Frank [7] extended it for a packing of spanning arborescences whose roots are not fixed. The result of Frank [7], and independently Cai [3], answers the question when a digraph has an (f, g)-bounded packing of spanning arborescences, that is when each vertex v can be the root of at least $f(v)$ and at most $g(v)$ arborescences in the packing. Bérczi, Frank [2] entends it for an (f, g)-bounded, k-regular, (ℓ, ℓ')-limited packing of not necessarily spanning arborescences, where k-regular means that each vertex belongs to exactly k arborescences in the packing and (ℓ, ℓ')-limited means that the number of arborescences in the packing is at least ℓ and at most ℓ'. Kamiyama, Katoh, Takizawa [16] provided a different type of generalization of Edmonds' theorem in which they wanted to pack reachability arborescences, that is each arborescence in the packing must contain all the vertices that can be reached from its root in the digraph. Durand de Gevigney, Nguyen, Szigeti [4] gave a generalization of Edmonds' theorem where a matroid constraint was added for the packing. More precisely, given a matroid M on a multiset of vertices of a digraph D, we wanted to have a matroid-based packing of arborescences, that is for every vertex v of D, the set of roots of the arborescences in the packing containing v must form a basis of M. In [17] Cs. Király proposed a common generalization of the previous two results. He characterized pairs (D, M) of a digraph and a matroid that have a matroid-reachability-based packing of arborescences, that is for every vertex v of D, the set of roots of the arborescences in the packing containing v must form a basis of the subset of the elements of M from which v is reachably in D.

All of these results hold for dypergraphs, see [10], [14], [22], [1], [6], and for mixed graphs, see [7], [11], [22], [20], [6], [12]. In fact, all of these results, except the one of Bérczi, Frank [2], are known to hold for mixed hypergraphs, see [6], [14], [15]. The present paper will fill in this gap by showing that this result also holds for mixed hypergraphs. More precisely, we will characterize mixed hypergraphs having an (f, g)-bounded, k-regular, (ℓ, ℓ')-limited packing of mixed hyperarborescences. Our result naturally generalizes a result of Gao, Yang [12] on (f, g)-bounded packing of k spanning mixed arborescences. The other aim of this paper is to provide a new proof of another result of Gao, Yang [11] on matroid-reachability-based packing of mixed arborescences. Our approach is to reduce the result to the result of Cs. Király [17] on matroid-reachability-based packing of arborescences via a new orientation theorem.

2 Definitions

A *multiset* of V may contain multiple occurrences of elements. For a multiset S of V and a subset X of V, $\boldsymbol{S_X}$ denotes the multiset consisting of the elements of X with the same multiplicities as in S.

Let $\boldsymbol{D} = (V, A)$ be a directed graph, shortly *digraph*. For a subset X of V, the set of arcs in A *entering* X is denoted by $\boldsymbol{\rho_A(X)}$ and the *in-degree* of X is $\boldsymbol{d_A^-(X)} = |\rho_A(X)|$. For a subset X of V, we denote by $\boldsymbol{P_D^X}$ ($\boldsymbol{Q_D^X}$) the set of vertices from (to) which there exists a path to (from, respectively) at least one vertex of X. We say that D is an *arborescence with root s*, shortly *s-arborescence*, if $s \in V$ and there exists a unique path from s to v for every $v \in V$; or equivalently, if D contains no circuit and every vertex in $V - s$ has in-degree 1. We say that D is a *branching with root set S* if $S \subseteq V$ and there exists a unique path from S to v for every $v \in V$. A subgraph of D is called *spanning* if its vertex set is V. A subgraph of D is called a *reachability s-arborescence* if it is an s-arborescence and its vertex set is Q_D^s. By a *packing* of subgraphs in D, we mean a set of subgraphs that are arc-disjoint. A packing of subgraphs is called *k-regular* if every vertex belongs to exactly k subgraphs in the packing. For two functions $f, g : V \to \mathbb{Z}_+$, a packing of arborescences is called *(f, g)-bounded* if the number of v-arborescences in the packing is at least $f(v)$ and at most $g(v)$ for every $v \in V$. For $\ell, \ell' \in \mathbb{Z}_+$, a packing of arborescences is called *(ℓ, ℓ')-limited* if the number of arborescences in the packing is at least ℓ and at most ℓ'. For a multiset S of V and a matroid M on S, a packing of arborescences in D is called *matroid-based* (resp. *matroid-reachability-based*) if every $s \in S$ is the root of at most one arborescence in the packing and for every $v \in V$, the multiset of roots of arborescences containing v in the packing forms a basis of S (resp. $S_{P_D^v}$) in M.

Let $\boldsymbol{F} = (V, E \cup A)$ be a *mixed graph*, where E is a set of edges and A is a set of arcs. A mixed subgraph F' of F is a *mixed path* if the edges in F' can be oriented in such a way that we obtain a directed path. For a subset X of V, we denote by $\boldsymbol{P_F^X}$ ($\boldsymbol{Q_F^X}$) the set of vertices from (to) which there exists a mixed path to (from, respectively) at least one vertex of X. We say that F is *strongly connected* if there exists a mixed path from s to t for all $(s, t) \in V^2$. The maximal strongly connected subgraphs of F are called the *strongly connected components* of F. A *mixed s-arborescence* is a mixed graph that has an orientation that is an s-arborescence. A mixed subgraph of F is called a *spanning (resp. reachability) mixed s-arborescence* if it is a mixed s-arborescence and its vertex set is V (resp. Q_F^s). By a *packing* of subgraphs in F, we mean a set of subgraphs that are edge- and arc-disjoint. All the packing problems considered in digraphs can also be considered in mixed graphs.

Let $\boldsymbol{\mathcal{D}} = (V, \mathcal{A})$ be a directed hypergraph, shortly *dypergraph*, where $\boldsymbol{\mathcal{A}}$ is the set of dyperedges of $\mathcal{D}$. A *dyperedge* e is an ordered pair (Z, z), where $z \in V$ is the *head* and $\emptyset \neq Z \subseteq V - z$ is the set of *tails* of e. For $X \subseteq V$, a dyperedge (Z, z) *enters* X if $z \in X$ and $Z \cap \overline{X} \neq \emptyset$. The set of dyperedges in $\mathcal{A}$ *entering* X is denoted by $\boldsymbol{\rho_{\mathcal{A}}(X)}$ and the *in-degree* of X is $\boldsymbol{d_{\mathcal{A}}^-(X)} = |\rho_{\mathcal{A}}(X)|$. By *trimming* a dyperedge $e = (Z, z)$, we mean the operation that replaces e by an arc yz where $y \in Z$. We say that $\mathcal{D}$ is a *hyperarborescence with root s*, shortly *s-hyperarborescence*, if $\mathcal{D}$ can be trimmed to an s-arborescence. We say that $\mathcal{D}$ is a *hyperbranching with root set S* if $\mathcal{D}$ can be trimmed to a branching with root set S. A *packing* of subdypergraphs in $\mathcal{D}$ is a set of subdypergraphs that are dyperedge-disjoint. We say that $\mathcal{D}$ has a *matroid-based/(f, g)-bounded/k-regular/(ℓ, ℓ')-limited* packing of hyperarborescences if $\mathcal{D}$ can be trimmed to a digraph that has a matroid-based/(f, g)-bounded/k-regular/(ℓ, ℓ')-limited packing of arborescences.

Let $\boldsymbol{\mathcal{F}} = (V, \mathcal{E} \cup \mathcal{A})$ be a *mixed hypergraph*, where $\boldsymbol{\mathcal{E}}$ is the set of hyperedges and $\boldsymbol{\mathcal{A}}$ is the set of dyperedges of $\mathcal{F}$. A *hyperedge* is a subset of V of size at least two. A hyperedge e *enters* a subset Y of V if $e \cap Y \neq \emptyset \neq \overline{e} \cap Y$. By *orienting* a hyperedge e, we mean the operation that replaces the hyperedge e by a dyperedge $(e - x, x)$ for some $x \in e$. For $\vec{\mathcal{Z}} \subseteq \mathcal{A}$, $\mathcal{Z}$ denotes the set of underlying hyperedges of $\vec{\mathcal{Z}}$. For $\mathcal{Z} \subseteq \mathcal{E}$ and $X \subseteq V$, we denote by $\boldsymbol{V(\mathcal{Z})}$ the set of vertices that belong to at least one hyperedge in $\mathcal{Z}$ and by $\boldsymbol{\mathcal{Z}(X)}$ the set of hyperedges in $\mathcal{Z}$ that are contained in X. A *mixed s-hyperarborescence* is a mixed hypergraph that has an orientation that is an s-hyperarborescence. A mixed s-hyperarborescence is called *spanning* in $\mathcal{F}$ if its vertex set is V. For a family $\mathcal{P}$ of subsets of V, we denote by $\boldsymbol{e_{\mathcal{E} \cup \mathcal{A}}(\mathcal{P})}$ the number of hyperedges in $\mathcal{E}$ and dyperedges in $\mathcal{A}$ that enter some member of $\mathcal{P}$. For $X \subseteq V$, we use $e_{\mathcal{E} \cup \mathcal{A}}(X)$ for $e_{\mathcal{E} \cup \mathcal{A}}(\{X\})$. A *packing* of mixed subhypergraphs in $\mathcal{F}$ is a set of mixed subhypergraphs that are hyperedge- and dyperedge-disjoint. We say that $\mathcal{F}$ has an *(f, g)-bounded/k-regular/(ℓ, ℓ')-limited* packing of mixed hyperarborescences if $\mathcal{E}$ has an orientation $\vec{\mathcal{E}}$ such that the dypergraph $(V, \vec{\mathcal{E}} \cup \mathcal{A})$ has an (f, g)-bounded/k-regular/(ℓ, ℓ')-limited packing of hyperarborescences.

3 Known results

In this section we list the results on packing arborescences that are related to the new results. We start with the fundamental result of Edmonds [5] on packing spanning arborescences with fixed roots.

Theorem 1 (Edmonds [5]) *Let $D = (V, A)$ be a digraph and S a multiset of V. There exists a packing* *of spanning s-arborescences* $(s \in S)$ *in* D *if and only if*

$$d_A^-(X) \geq |S_{V-X}| \text{ for every } \emptyset \neq X \subseteq V.$$

Theorem 1 was extended for the case when the roots of the arborescences are not fixed but the number of arborescences in the packing rooted at any vertex is bounded. For a subpartition $\mathcal{P}$ of V, $\cup\boldsymbol{\mathcal{P}}$ denotes the set of elements of V that belong to some member of $\mathcal{P}$.

Theorem 2 (Frank [7], Cai [3]) *Let* $D = (V, A)$ *be a digraph,* $f, g : V \to \mathbb{Z}_+$ *functions and* $k \in \mathbb{Z}_+$. *There exists an* (f, g)*-bounded packing of* k *spanning arborescences in* D *if and only if*

$$g(v) \geq f(v) \quad \text{for every } v \in V, \tag{1}$$

$$e_A(\mathcal{P}) \geq k|\mathcal{P}| - \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{2}$$

If S is a multiset of V and $f(v) = g(v) = |S_v|$ for every $v \in V$, then Theorem 2 reduces to Theorem 1.

Theorem 2 can be generalized for the case when the arborescences are not necessarily spanning but every vertex must belong to the same number of arborescences in the packing. For $g : V \to \mathbb{Z}_+$ and $k \in \mathbb{Z}_+$, let $\boldsymbol{g_k}(v) = \min\{k, g(v)\}$ for every $v \in V$. For convenience, we present not the original version of the result of [2] which is about packing branchings but one that fits better to our framework.

Theorem 3 (Bérczi, Frank [2]) *Let* $D = (V, A)$ *be a digraph,* $f, g : V \to \mathbb{Z}_+$ *functions and* $k, \ell, \ell' \in \mathbb{Z}_+$. *There exists an* (f, g)*-bounded* k*-regular* (ℓ, ℓ')*-limited packing of arborescences in* D *if and only if*

$$g_k(v) \geq f(v) \quad \text{for every } v \in V, \tag{3}$$

$$\min\{g_k(V), \ell'\} \geq \ell \tag{4}$$

$$e_A(\mathcal{P}) \geq k|\mathcal{P}| - \min\{\ell' - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{5}$$

For $k = \ell = \ell'$, Theorem 3 reduces to Theorem 2.

An elegant extention of Theorem 1 for packing reachability arborescences was provided in [16].

Theorem 4 (Kamiyama, Katoh, Takizawa [16]) *Let* $D = (V, A)$ *be a digraph and* S *a multiset of* V. *There exists a packing of reachability s-arborescences* $(s \in S)$ *in* D *if and only if*

$$d_A^-(X) \geq |S_{P_D^X - X}| \text{ for every } X \subseteq V.$$

When each vertex is reachable from every vertex of S, Theorem 4 reduces to Theorem 1. Theorem 4 can be proved by induction and using Edmonds' result on packing branchings, see Hörsch, Szigeti [15].

Another type of generalizations of Theorem 1 was obtained by adding a matroid constraint.

Theorem 5 (Durand de Gevigney, Nguyen, Szigeti [4]) *Let* $D = (V, A)$ *be a digraph,* S *a multiset of* V *and* $\mathsf{M} = (S, r_\mathsf{M})$ *a matroid. There exists a* M*-based packing of arborescences in* D *if and only if*

$$r_\mathsf{M}(S_X) + d_A^-(X) \geq r_\mathsf{M}(S) \text{ for every } \emptyset \neq X \subseteq V.$$

For the free matroid M, Theorem 5 reduces to Theorem 1.

A common generalization of Theorems 4 and 5 was found by Cs. Király [17].

Theorem 6 (Cs. Király [17]) *Let* $D = (V, A)$ *be a digraph,* S *a multiset of* V *and* $\mathsf{M} = (S, r_\mathsf{M})$ *a matroid. There exists a matroid-reachability-based packing of arborescences in* D *if and only if*

$$r_\mathsf{M}(S_X) + d_A^-(X) \geq r_\mathsf{M}(S_{P_D^X}) \quad \text{for every } X \subseteq V. \tag{6}$$

For the free matroid M, Theorem 6 reduces to Theorem 4. When each vertex is reachable from a basis of M, Theorem 6 reduces to Theorem 5.

Gao, Yang [11] provided another characterization of the existence of a matroid-reachability-based packing of arborescences.

Theorem 7 (Gao, Yang [11]) *Let* $D = (V, A)$ *be a digraph,* S *a multiset of* V *and* $\mathsf{M} = (S, r_\mathsf{M})$ *a matroid. There exists a matroid-reachability-based packing of arborescences in* D *if and only if for every strongly connected component* C *of* D *and every* $X \subseteq P_D^C$ *such that* $X \cap C \neq \emptyset$ *and* $d_A^-(X - C) = 0$,

$$r_\mathsf{M}(S_X) + d_A^-(X) \geq r_\mathsf{M}(S_{P_D^C}). \tag{7}$$

Let us show that Theorems 6 and 7 are equivalent.

PROOF: We have to prove that (6) and (7) are equivalent.

(6) $\Longrightarrow$ (7): If (6) holds, then let C be a strongly connected component of D and $X \subseteq P_D^C$ such that $X \cap C \neq \emptyset$ and $d_A^-(X - C) = 0$. Then, we have $P_D^X = P_D^C$ and (6) implies (7).

(7) $\Longrightarrow$ (6): Now if (7) holds, then let X be a subset of V. Let $C_1, \ldots, C_k$ be the strongly connected components of D in a topological ordering that is if there exists an arc from C_i to C_j then $i < j$. Let

$$\begin{aligned} \boldsymbol{J} &= \{1 \leq j \leq k : X \cap C_j \neq \emptyset\}, \\ \boldsymbol{X_j} &= (X \cap C_j) \cup \bigcup_{\substack{i \in J - \{j\} \\ C_i \subseteq P_D^{C_j}}} P_D^{C_i} \qquad \text{for every } j \in J. \end{aligned}$$

Note that $X_j \subseteq P_D^{C_j}$, $X_j \cap C_j \neq \emptyset$ and $d_A^-(X_j - C_j) = 0$ for every $j \in J$.

Claim 8 $d_A^-(X) \geq \sum_{j \in J} d_A^-(X_j)$.

PROOF: If uv enters X_j, then $v \in X \cap C_j \subseteq X$ and $u \notin X_j$. If $u \in X$, then $u \in X \cap C_{j'}$ for some $j' \in J$. Since $C_{j'}$ is strongly connected, $u \in C_{j'}$ and $v \in X \cap C_j$, we have $C_{j'} \subseteq P_D^{C_j}$, so $u \in X_j$ which is a contradiction. It follows that $u \notin X$, so uv enters X. Since $(X \cap C_j) \cap (X \cap C_{j'}) = \emptyset$ for distinct $j, j' \in J$, the claim follows. $\square$

Claim 9 $\sum_{j \in J} (r_{\mathsf{M}}(S_{P_D^{C_j}}) - r_{\mathsf{M}}(S_{X_j})) \geq r_{\mathsf{M}}(S_{P_D^X}) - r_{\mathsf{M}}(S_X)$.

PROOF: We prove it by induction on $|J|$. For $|J| = 1$, say $J = \{j\}$, the claim follows from $P_D^{C_j} = P_D^{X_j}$. Suppose that the inequality holds for $|J| - 1$. Let ℓ be the largest value in J. Note that we have

$$\begin{aligned} P_D^{X_\ell} \cap (X_\ell \cup P_D^{X - X_\ell}) &\supseteq X_\ell, & P_D^{X - X_\ell} \cap X &\supseteq X - X_\ell, \\ P_D^{X_\ell} \cup (X_\ell \cup P_D^{X - X_\ell}) &\supseteq P_D^X, & P_D^{X - X_\ell} \cup X &\supseteq X_\ell \cup P_D^{X - X_\ell}. \end{aligned}$$

Then, by induction, submodularity and monotonicity of r_{M}, we have

$$\begin{aligned} \sum_{j \in J} (r_{\mathsf{M}}(S_{P_D^{C_j}}) - r_{\mathsf{M}}(S_{X_j})) &\geq (r_{\mathsf{M}}(S_{P_D^{X_\ell}}) - r_{\mathsf{M}}(S_{X_\ell})) + (r_{\mathsf{M}}(S_{P_D^{X - X_\ell}}) - r_{\mathsf{M}}(S_{X - X_\ell})) \\ &\geq (r_{\mathsf{M}}(S_{P_D^X}) - r_{\mathsf{M}}(S_{X_\ell \cup P_D^{X - X_\ell}})) + (r_{\mathsf{M}}(S_{X_\ell \cup P_D^{X - X_\ell}}) - r_{\mathsf{M}}(S_X)) \\ &= r_{\mathsf{M}}(S_{P_D^X}) - r_{\mathsf{M}}(S_X), \end{aligned}$$

and the claim follows. $\square$

By Claims 8, 9 and (7), we get that (6) holds. $\square$

Theorem 1 was generalized for dypergraphs as follows.

Theorem 10 (Frank, T. Király, Z. Király [10]) *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, $s \in V$ and $k \in \mathbb{Z}_+$. There exists a packing of k spanning s-hyperarborescences in $\mathcal{D}$ if and only if*

$$d_{\mathcal{A}}^-(X) \geq k \qquad \textit{for every } \emptyset \neq X \subseteq V - s.$$

Theorem 10 easily implies the following corollary.

Corollary 11 *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph and S a multiset of V. There exists a k-regular packing of s-hyperarborescences ($s \in S$) in $\mathcal{D}$ if and only if*

$$|S_v| \leq k \qquad \textit{for every } v \in V, \tag{8}$$

$$d_{\mathcal{A}}^-(X) \geq k - |S_X| \qquad \textit{for every } \emptyset \neq X \subseteq V. \tag{9}$$

Theorem 2 was generalized for mixed graphs as follows.

Theorem 12 (Gao, Yang [12]) *Let $F=(V,E\cup A)$ be a mixed graph, $f,g:V\to\mathbb{Z}_+$ functions, and $k\in\mathbb{Z}_+$. There exists an (f,g)-bounded packing of k spanning mixed arborescences in F if and only if* (1) *holds and*

$$e_{E\cup A}(\mathcal{P})\geq k|\mathcal{P}|-\min\{k-f(\overline{\cup\mathcal{P}}),g(\cup\mathcal{P})\}\text{ for every subpartition }\mathcal{P}\text{ of }V.$$

If F is a digraph, then Theorem 12 reduces to Theorem 2.

Theorem 12 can be generalized for mixed hypergraphs.

Theorem 13 (Hörsch, Szigeti [14]) *Let $\mathcal{F}=(V,\mathcal{E}\cup\mathcal{A})$ be a mixed hypergraph, $f,g:V\to\mathbb{Z}_+$ functions, and $k\in\mathbb{Z}_+$. There exists an (f,g)-bounded packing of k spanning mixed hyperarborescences in $\mathcal{F}$ if and only if* (1) *holds and*

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P})\quad\geq\quad k|\mathcal{P}|-\min\{k-f(\overline{\cup\mathcal{P}}),g(\cup\mathcal{P})\}\qquad\text{for every subpartition }\mathcal{P}\text{ of }V.$$

If $\mathcal{F}$ is a mixed graph then Theorem 13 reduces to Theorem 12. Theorem 13 is derived from matroid intersection in [14]. One of the main contribution of this paper is to provide a generalization of Theorem 13 in Subsection 4.2. The new result will be obtained from the theory of generalized polymatroids.

Now a generalization of Theorem 4 for mixed graphs follows. For convenience, we present not the original version of the result but one due to Gao, Yang [11] that fits better to our framework.

Theorem 14 (Matsuoka, Tanigawa [20]) *Let $F=(V,E\cup A)$ be a mixed graph and S a multiset of V. There exists a packing of reachability mixed s-arborescences ($s\in S$) in F if and only if for every strongly connected component C of F and every set $\mathcal{P}$ of subsets of P_F^C such that $Z\cap C\neq\emptyset$ and $e_{E\cup A}(Z-C)=0$ for every $Z\in\mathcal{P}$ and $Z\cap Z'\cap C=\emptyset$ for every $Z,Z'\in\mathcal{P}$,*

$$e_{E\cup A}(\mathcal{P})\geq|S_{P_F^C}||\mathcal{P}|-\sum_{Z\in\mathcal{P}}|S_Z|.$$

If F is a digraph, then Theorem 14 reduces to Theorem 4.

A common generalization of Theorems 7 and 14 was provided by Gao, Yang [11].

Theorem 15 (Gao, Yang [11]) *Let $F=(V,E\cup A)$ be a mixed graph, S a multiset of V and $\mathsf{M}=(S,r_{\mathsf{M}})$ a matroid. There exists a matroid-reachability-based packing of mixed arborescences in F if and only if for every strongly connected component C of F and every set $\mathcal{P}$ of subsets of P_F^C such that $Z\cap C\neq\emptyset$ and $e_{E\cup A}(Z-C)=0$ for every $Z\in\mathcal{P}$ and $Z\cap Z'\cap C=\emptyset$ for every $Z,Z'\in\mathcal{P}$,*

$$e_{E\cup A}(\mathcal{P})\quad\geq\quad r_{\mathsf{M}}(S_{P_F^C})|\mathcal{P}|-\sum_{Z\in\mathcal{P}}r_{\mathsf{M}}(S_Z).\tag{10}$$

For $E=\emptyset$, Theorem 15 reduces to Theorem 7. For the free matroid M, Theorem 15 reduces to Theorem 14. Hörsch, Szigeti [15] pointed out that Theorem 15 holds for mixed hypergraphs. That more general result was proved in [15] by induction using a result on matroid-based packing of mixed hyperbranchings in mixed hypergraphs from [6]. Here we propose another approach to prove Theorem 15. It will be derived from Theorem 18, a new orientation result.

We need a matroid construction for hypergraphs and one for mixed hypergraphs. Given a hypergraph $\mathcal{H}=(V,\mathcal{E})$, let $\boldsymbol{\mathcal{I}_{\mathcal{H}}}=\{\mathcal{Z}\subseteq\mathcal{E}:|V(\mathcal{Z}')|>|\mathcal{Z}'|\text{ for all }\emptyset\neq\mathcal{Z}'\subseteq\mathcal{Z}\}$. Lorea [19] showed that $\mathcal{I}_{\mathcal{H}}$ is the set of independent sets of a matroid $\boldsymbol{\mathsf{M}_{\mathcal{H}}}$ on $\mathcal{E}$, called the *hypergraphic matroid* of the hypergraph $\mathcal{H}$. We also need the *k-hypergraphic matroid* $\boldsymbol{\mathsf{M}^k_{\mathcal{H}}}$ of $\mathcal{H}$ which is the k-sum matroid of $\mathsf{M}_{\mathcal{H}}$, that is the matroid on ground set $\mathcal{E}$ in which a subset of $\mathcal{E}$ is independent if it can be partitioned into k independent sets of $\mathsf{M}_{\mathcal{H}}$. Hörsch, Szigeti [14] extended the previous construction for mixed hypergraphs as follows. Let $\mathcal{F}=(V,\mathcal{A}\cup\mathcal{E})$ be a mixed hypergraph. For a subpartition $\mathcal{P}$ of V, $\boldsymbol{\mathcal{A}(\mathcal{P})}$ and $\boldsymbol{\mathcal{E}(\mathcal{P})}$ denote the set of dyperedges and the set of hyperedges that enter some member of $\mathcal{P}$. Let $\boldsymbol{\mathcal{H}_{\mathcal{F}}}=(V,\boldsymbol{\mathcal{E}_{\mathcal{A}}}\cup\mathcal{E})$ the underlying hypergraph of $\mathcal{F}$ and $\boldsymbol{\mathcal{D}_{\mathcal{F}}}=(V,\mathcal{A}\cup\mathcal{A}_{\mathcal{E}})$ the *directed extension* of $\mathcal{F}$ where $\boldsymbol{\mathcal{A}_{\mathcal{E}}}=\bigcup_{e\in\mathcal{E}}\mathcal{A}_e$ and for $e\in\mathcal{E}$, $\boldsymbol{\mathcal{A}_e}=\{(e-x,x):x\in e\}$. The *extended k-hypergraphic matroid* $\boldsymbol{\mathsf{M}^k_{\mathcal{F}}}$ of $\mathcal{F}$ on $\mathcal{A}\cup\mathcal{A}_{\mathcal{E}}$ is obtained from $\mathsf{M}^k_{\mathcal{H}_{\mathcal{F}}}$ by replacing every $e\in\mathcal{E}$ by $|e|$ parallel copies of itself, associating these elements to the dyperedges in $\mathcal{A}_e$ and associating every hyperedge of $\mathcal{E}_{\mathcal{A}}$ to the corresponding dyperedge in $\mathcal{A}$. It is shown in [14] that the rank function of the extended k-hypergraphic matroid $\mathsf{M}^k_{\mathcal{F}}$ satisfies for all $\mathcal{Z}\subseteq\mathcal{A}\cup\mathcal{A}_{\mathcal{E}}$,

$$r_{\mathsf{M}^k_{\mathcal{F}}}(\mathcal{Z})=\min\{|\mathcal{Z}\cap\mathcal{A}(\mathcal{P})|+|\{e\in\mathcal{E}(\mathcal{P}):\overline{\mathcal{Z}}\cap\mathcal{A}_e\neq\emptyset\}|+k(|V|-|\mathcal{P}|):\mathcal{P}\text{ partition of }V\}.\tag{11}$$

Generalized polymatroids were introduced by Hassin [13] and independently by Frank [8]. A set function p (resp. b) on S is called *supermodular* (resp. *submodular*) if (12) (resp. (12)) holds. If the supermodular (resp. submodular) inequality holds for intersecting sets then p (resp. b) is called *intersecting supermodular* (resp. *intersecting submodular*).

$$\begin{aligned} p(X)+p(Y) &\leq p(X\cap Y)+p(X\cup Y) \text{ for all } X,Y\subseteq S,\\ b(X)+b(Y) &\geq b(X\cap Y)+b(X\cup Y) \text{ for all } X,Y\subseteq S.\end{aligned}$$

For a pair (p,b) of set functions on S and $\alpha,\beta\in\mathbb{R}$, let us introduce the polyhedra

$$\begin{aligned} \boldsymbol{Q(p,b)} &= \{x\in\mathbb{R}^S : p(Z)\leq x(Z)\leq b(Z) \text{ for all } Z\subseteq S\},\\ \boldsymbol{K(\alpha,\beta)} &= \{x\in\mathbb{R}^S : \alpha\leq x(S)\leq\beta\}.\end{aligned}$$

If $p(\emptyset)=b(\emptyset)=0$, $-p$ and b are submodular and $b(X)-p(Y)\geq b(X-Y)-p(Y-X)$ for all $X,Y\subseteq S$, the polyhedron $Q(p,b)$ is called a *generalized-polymatroid*, shortly *g-polymatroid*. The polyhedron $K(\alpha,\beta)$ is called a *plank*. The Minkowski sum of the n g-polymatroids $Q(p_i,b_i)$ is denoted by $\boldsymbol{\sum_1^n Q(p_i,b_i)}$. We will need the following results on g-polymatroids, for more details see [9].

Theorem 16 (Frank [9]) *The following hold:*

1. *Let $Q(p,b)$ be a g-polymatroid, $K(\alpha,\beta)$ a plank and $M=Q(p,b)\cap K(\alpha,\beta)$.*
   - *(i) $M\neq\emptyset$ if and only if $p\leq b$, $\alpha\leq\beta$, $\beta\geq p(S)$ and $\alpha\leq b(S)$.*
   - *(ii) M is a g-polymatroid.*
   - *(iii) If $M\neq\emptyset$, then $M=Q(p^\alpha_\beta,b^\alpha_\beta)$ with*

$$\boldsymbol{p^\alpha_\beta(Z)}=\max\{p(Z),\alpha-b(S-Z)\},\quad \boldsymbol{b^\alpha_\beta(Z)}=\min\{b(Z),\beta-p(S-Z)\}. \tag{12}$$

2. *Let $Q(p_1,b_1)$ and $Q(p_2,b_2)$ be two non-empty g-polymatroids and $M=Q(p_1,b_1)\cap Q(p_2,b_2)$.*
   - *(i) $M\neq\emptyset$ if and only if $p_1\leq b_2$ and $p_2\leq b_1$.*
   - *(ii) If p_1,b_1,p_2,b_2 are integral and $M\neq\emptyset$, then M contains an integral element.*
3. *Let $Q(p_i,b_i)$ be n non-empty g-polymatroids. Then $\sum_1^n Q(p_i,b_i)=Q(\sum_1^n p_i,\sum_1^n b_i)$.*

4 Main results

4.1 A new orientation result

To prove the new orientation result, Theorem 18, we need a result of Frank, see Theorem 15.4.13 in [9].

Theorem 17 (Frank [9]) *Let $G=(V,E)$ be a graph and h an integer-valued, intersecting supermodular function such that $h(V)=0$. There exists an orientation $\vec{G}=(V,\vec{E})$ of G such that*

$$d^-_{\vec{E}}(X) \geq h(X) \quad \text{for every } X\subseteq V$$

if and only if

$$e_E(\mathcal{P}) \geq \sum_{X\in\mathcal{P}} h(X) \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{13}$$

We can now extend an orientation theorem which is implicitly contained in Gao, Yang [11] as follows.

Theorem 18 *Let $F=(V,E\cup A)$ be a mixed graph and h an integer-valued, intersecting supermodular function on V. There exists an orientation $\vec{E}$ of E such that*

$$d^-_{\vec{E}\cup A}(X) \geq h(X)-h(P^X_F) \quad \text{for every } X\subseteq V \tag{14}$$

if and only if for every strongly connected component C of F and every set $\mathcal{P}$ of subsets of P^C_F such that $Z\cap C\neq\emptyset, e_{E\cup A}(Z-C)=0$ for every $Z\in\mathcal{P}$; and $Z\cap Z'\cap C=\emptyset$ for every $Z,Z'\in\mathcal{P}$,

$$e_{E\cup A}(\mathcal{P}) \geq \sum_{Z\in\mathcal{P}}(h(Z)-h(P^C_F)). \tag{15}$$

PROOF: Let $(\boldsymbol{F} = (\boldsymbol{V},\ \boldsymbol{E} \cup \boldsymbol{A}),\ \boldsymbol{h})$ be a minimum counterexample for Theorem 18. Let $\boldsymbol{C}$ be a strongly connected component of F such that $e_{E\cup A}(\overline{C}) = 0$. Let $(\boldsymbol{F_1} = (\boldsymbol{V_1},\ \boldsymbol{E_1} \cup \boldsymbol{A_1}),\ \boldsymbol{h_1})$ be obtained from (F,h) by deleting the elements in C. As $e_{E\cup A}(\overline{C}) = 0$, we have $e_{E_1\cup A_1}(X) = e_{E\cup A}(X)$, $P^X_{F_1} = P^X_F$ and $h_1(X) = h(X)$ for every $X \subseteq V_1$. Then, since (F,h) satisfies (15), so does (F_1,h_1). Hence, by the minimality of (F,h), there exists an orientation $\vec{\boldsymbol{E}}_{\boldsymbol{1}}$ of E_1 such that

$$d^-_{\vec{E}_1\cup A_1}(X) \quad \geq \quad h(X) - h(P^X_F) \qquad \text{for every } X \subseteq V_1. \tag{16}$$

Let us now consider the subgraph $\boldsymbol{F_2} = (\boldsymbol{C},\ \boldsymbol{E_2} \cup \boldsymbol{A_2})$ of F induced by C. Moreover, let us define $\boldsymbol{h_2}(X) = \max\{h(Y) - d^-_A(Y) : Y \subseteq P^C_F, Y \cap C = X, e_{E\cup A}(Y - C) = 0\}$ for every $\emptyset \neq X \subseteq C$. For any non-empty set X_i in C, let $\boldsymbol{Y_i}$ be a set that provides $h_2(X_i)$. Gao, Yang [11] proved that

Claim 19 *h_2 is an intersecting supermodular function on C.*

PROOF: For intersecting sets X_1 and X_2 in C, let $\boldsymbol{X_3} = X_1 \cap X_2$, $\boldsymbol{X_4} = X_1 \cup X_2$, $\boldsymbol{Y'_3} = Y_1 \cap Y_2$ and $\boldsymbol{Y'_4} = Y_1 \cup Y_2$. Note that, for $i = 3,4$, we have $Y'_i \subseteq P^C_F, Y'_i \cap C = X_i$ and $e_{E\cup A}(Y'_i - C) = 0$, and hence $h(Y'_i) - d^-_A(Y'_i) \leq h_2(X_i)$ Then, by the intersecting supermodularity of h and $-d^-_A$, we get that

$$\begin{aligned} h_2(X_1) + h_2(X_2) &= h(Y_1) - d^-_A(Y_1) + h(Y_2) - d^-_A(Y_2) \\ &\leq h(Y'_3) - d^-_A(Y'_3) + h(Y'_4) - d^-_A(Y'_4) \\ &\leq h_2(X_3) + h_2(X_4) \\ &= h_2(X_1 \cap X_2) + h_2(X_1 \cup X_2), \end{aligned}$$

so h_2 is intersecting supermodular. □

Let $\boldsymbol{h'}$ be defined by $h'(X) = h_2(X) - h(P^C_F)$ for every $\emptyset \neq X \subseteq C$ and $h(\emptyset) = 0$. By the Claim 19, h' is intersecting supermodular on C. Let $\boldsymbol{\mathcal{P}} = \{X_1, \ldots, X_t\}$ be a subpartition of C and $\boldsymbol{\mathcal{P}'} = \{Y_i : X_i \in \mathcal{P}\}$. Then $\mathcal{P}'$ is a set of subsets of P^C_F such that $Y_i \cap C \neq \emptyset$ and $e_{E\cup A}(Y_i - C) = 0$ for $1 \leq i \leq t$ and $Y_i \cap Y_j \cap C = \emptyset$ for $1 \leq i < j \leq t$. It follows, by (15), that

$$\begin{aligned} e_{E_2}(\mathcal{P}) &= e_{E\cup A}(\mathcal{P}') - e_A(\mathcal{P}') \\ &\geq \sum_{Y_i\in\mathcal{P}'} (h(Y_i) - h(P^C_F) - d^-_A(Y_i)) \\ &= \sum_{Y_i\in\mathcal{P}'} (h_2(X_i) - h(P^C_F)) \\ &= \sum_{X_i\in\mathcal{P}} h'(X_i). \end{aligned}$$

Thus the graph (C, E_2) satisfies (13). In particular, we get that $0 = e_{E_2}(C) \geq h'(C)$. Moreover, $h'(C) = h_2(C) - h(P^C_F) \geq h(P^C_F) - h(P^C_F) = 0$. Hence $h'(C) = 0$. Then, by Theorem 17, there exists an orientation $\vec{\boldsymbol{E}}_{\boldsymbol{2}}$ of E_2 such that $d^-_{\vec{E}_2}(X) \geq h'(X) = h_2(X) - h(P^C_F)$ for every $X \subseteq C$. It follows that for every $Y \subseteq P^C_F$ with $Y \cap C \neq \emptyset$ and $e_{E\cup A}(Y - C) = 0$, we have

$$d^-_{\vec{E}_2}(Y) = d^-_{\vec{E}_2}(Y \cap C) \geq h(Y) - h(P^C_F) - d^-_A(Y). \tag{17}$$

Let $\vec{\boldsymbol{F}} = (V, \vec{E} \cup A)$, where $\vec{E} = \vec{E}_1 \cup \vec{E}_2$. To finish the proof we show that $\vec{F}$ satisfies (14). If $X \subseteq V_1$, then, by (16), (14) holds. Otherwise, $X \cap C \neq \emptyset$. If $X \subseteq C$, then, by (17) applied for X, (14) holds. We suppose from now on that $X \cap C \neq \emptyset \neq X - C$. Let $\boldsymbol{Z} = P^{X-C}_F$, $\boldsymbol{Y} = Z \cap P^C_F$ and $\boldsymbol{W} = Y \cup (X \cap C)$. Then $X \cap Z = X - C, P^C_F \cap (X \cup Z) = W$ and $P^C_F \cup (X \cup Z) = P^X_F$, $e_{E\cup A}(Y) = 0$. Thus, by (16) for $X - C$, (17) for W and the intersecting supermodularity of h, we have

$$\begin{aligned} d^-_{\vec{E}\cup A}(X) &\geq d^-_{\vec{E}_1\cup A}(X - C) + d^-_{\vec{E}_2\cup A}(W) \\ &\geq (h(X - C) - h(Z)) + (h(W) - h(P^C_F)) \\ &\geq (h(X) - h(X \cup Z)) + (h(X \cup Z) - h(P^X_F)) \\ &= h(X) - h(P^X_F), \end{aligned}$$

so (14) holds. □

Theorem 15 now easily follows from Theorems 6 and 18.

PROOF: Let (F, S, M) be an instance of Theorem 15 that satisfies (10). Then, for $h(X) = -r_{\mathsf{M}}(S_X)$ for all $X \subseteq V$, (15) holds, so by Theorems 18 applied for (F, h), there exists an orientation $\vec{E}$ of E such that in $\vec{F} = (V, \vec{E} \cup A)$ (14) holds. Let $X \subseteq V$. Since $P^X_{\vec{F}} \subseteq P^X_F$ and r_{M} is non-decreasing, we have $r_{\mathsf{M}}(S_{P^X_{\vec{F}}}) \leq r_{\mathsf{M}}(S_{P^X_F})$. By (14) applied for $P^X_{\vec{F}}$, we have $r_{\mathsf{M}}(S_{P^X_{\vec{F}}}) \geq r_{\mathsf{M}}(S_{P^X_F})$. Hence $r_{\mathsf{M}}(S_{P^X_{\vec{F}}}) = r_{\mathsf{M}}(S_{P^X_F})$. Thus (14) implies that (6) holds in $(\vec{F}, S, \mathsf{M})$. Then, by Theorems 6, there exists a matroid-reachability-based packing of arborescences in $(\vec{F}, S, \mathsf{M})$. Since $r_{\mathsf{M}}(S_{P^X_{\vec{F}}}) = r_{\mathsf{M}}(S_{P^X_F})$, by replacing the arcs in $\vec{E}$ by the edges in E, we obtain a matroid-reachability-based packing of mixed arborescences in (F, S, M). $\Box$

We mention that Theorem 17 and hence Theorem 18 also works for mixed hypergraphs. This shows that the result of Hörsch, Szigeti [15] can also be obtained from a theorem of Fortier et al. [6] on matroid-reachability-based packing of hyperarborescences.

4.2 A new result on packing mixed hyperarborescences

The main contribution of the present paper is a common generalization of Theorems 3 and 13.

Theorem 20 *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}_+$ functions, and $k, \ell, \ell' \in \mathbb{Z}_+ - \{0\}$. There exists an (f, g)-bounded k-regular (ℓ, ℓ')-limited packing of mixed hyperarborescences in $\mathcal{F}$ if and only if* (3) *and* (4) *hold and*

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) \quad \geq \quad k|\mathcal{P}| - \min\{\ell' - f(\overline{\cup\mathcal{P}}), g_k(\cup\mathcal{P})\} \qquad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \tag{18}$$

If $\mathcal{F}$ is a digraph, then Theorem 20 reduces to Theorem 3. If $k = \ell = \ell'$, then Theorem 20 reduces to Theorem 13. Theorem 20 will follow from Theorem 21.

Theorem 21 *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}_+$ functions, and $k, \ell, \ell' \in \mathbb{Z}_+ - \{0\}$. Let $\mathbf{M}_{\boldsymbol{v}} = (\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v), r_v)$ be the free matroid for all $v \in V$ and $\mathbf{M}^{\boldsymbol{k}}_{\boldsymbol{\mathcal{F}}}$ the extended k-hypergraphic matroid of $\mathcal{F}$ on $\mathcal{A} \cup \mathcal{A}_\mathcal{E}$. Let us define the following polyhedron*

$$\boldsymbol{T} = (\sum_{v\in V}(Q(0, r_v) \cap K(k - g_k(v), k - f(v)))) \cap K(k|V| - \ell', k|V| - \ell) \cap Q(0, r_{\mathsf{M}^k_\mathcal{F}}).$$

(a) The characteristic vectors of the dyperedge sets of the (f, g)-bounded k-regular (ℓ, ℓ')-limited packings of hyperarborescences in orientations of $\mathcal{F}$ are exactly the integer points of T.

(b) $T \neq \emptyset$ if and only if (3) *and* (4) *hold and for every $\mathcal{Z} \subseteq \mathcal{A} \cup \mathcal{A}_\mathcal{E}$,*

$$\sum_{v\in V} \max\{0, k - g_k(v) - d^-_{\mathcal{Z}}(v)\} \quad \leq \quad r_{\mathsf{M}^k_\mathcal{F}}(\overline{\mathcal{Z}}), \tag{19}$$

$$k|V| - \ell' - \sum_{v\in V} \min\{d^-_{\mathcal{Z}}(v), k - f(v)\} \quad \leq \quad r_{\mathsf{M}^k_\mathcal{F}}(\overline{\mathcal{Z}}). \tag{20}$$

(c) (19) *and* (20) *are equivalent to* (18).

PROOF: (a) To prove the **necessity**, let $\boldsymbol{\mathcal{B}_1, \dots, \mathcal{B}_{\ell^*}}$ be an (f, g)-bounded k-regular packing of hyperarborescences in an orientation $\vec{\boldsymbol{\mathcal{F}}}$ of $\mathcal{F}$, where $\ell \leq \ell^* \leq \ell'$. Let $\boldsymbol{S}$ be the root set of the hyperarborescences in the packing. Note that $|S| = \ell^*$. Let $\vec{\boldsymbol{\mathcal{Z}}}$ be the dyperedge set of the packing. Since the packing is k-regular, we have $k = d^-_{\vec{\mathcal{Z}}}(v) + |S_v|$ for all $v \in V$. Then $k|V| = |\vec{\mathcal{Z}}| + |S|$. Since the packing is (f, g)-bounded, we have $f(v) \leq |S_v| \leq g_k(v)$ for all $v \in V$. Let $\boldsymbol{m}$ be the characteristic vector of $\vec{\mathcal{Z}}$ and $\boldsymbol{m_v}$ the restriction of m on $\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)$ for all $v \in V$. Then m_v is a characteristic vector, so $m_v \in Q(0, r_v)$ for all $v \in V$. Since for all $v \in V$, $d^-_{\vec{\mathcal{Z}}}(v) = m_v(\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v))$, we have $m_v \in K(k - g_k(v), k - f(v))$. It follows that $m \in \sum_{v\in V}(Q(0, r_v) \cap K(k - g_k(v), k - f(v)))$. Since $\ell \leq |S| \leq \ell'$, $k|V| = |\vec{\mathcal{Z}}| + |S|$ and $|\vec{\mathcal{Z}}| = m(\mathcal{A} \cup \mathcal{A}_\mathcal{E})$, we have $m \in K(k|V| - \ell', k|V| - \ell)$. By Theorem 10, since $\vec{\mathcal{Z}}$ is the dyperedge set of a k-regular packing of hyperarborescences in $\vec{\mathcal{F}}$, $\vec{\mathcal{Z}}$ is the dyperedge set of a packing of k spanning hyperbranchings in $\vec{\mathcal{F}}$. Note that the underlying hypergraph of each hyperbranching is independent in $\mathsf{M}_{\mathcal{H}_\mathcal{F}}$. It follows that $\mathcal{Z}$ is independent in $\mathsf{M}^k_{\mathcal{H}_\mathcal{F}}$. Then, since $\vec{\mathcal{Z}}$ is in the orientation $\vec{\mathcal{F}}$ of $\mathcal{F}$, $\vec{\mathcal{Z}}$ is independent in $\mathsf{M}^k_\mathcal{F}$. Thus, since m is the characteristic vector of $\vec{\mathcal{Z}}$, $m \in Q(0, r_{\mathsf{M}^k_\mathcal{F}})$. By consequence, m is an integer points of T.

To prove the **sufficiency**, let $\boldsymbol{m} = (m_v)_{v\in V}$ be an integer point of T, that is $m_v \in Q(0, r_v) \cap K(k - g_k(v), k - f(v))$ for all $v \in V$ and $m \in K(k|V| - \ell', k|V| - \ell) \cap Q(0, r_{\mathsf{M}^k_{\mathcal{F}}})$. Since m_v is an integer point in $Q(0, r_v)$, m_v is the characteristic vector of a subset $\vec{\boldsymbol{\mathcal{Z}}}_{\boldsymbol{v}}$ of $\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)$. Since $m_v \in K(k - g_k(v), k - f(v))$, we have

$$k - g_k(v) \le m_v(\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)) = |\vec{\mathcal{Z}}_v| = m_v(\rho_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)) \le k - f(v). \tag{21}$$

Let $\vec{\boldsymbol{\mathcal{Z}}} = \bigcup_{v\in V} \vec{\mathcal{Z}}_v$. Note that $d^-_{\vec{\mathcal{Z}}}(v) = |\vec{\mathcal{Z}}_v|$ for all $v \in V$. Then, by $f \ge 0$, we have $k - d^-_{\vec{\mathcal{Z}}}(v) \ge f(v) \ge 0$ for all $v \in V$. Since $m \in K(k|V| - \ell', k|V| - \ell)$, we have

$$k|V| - \ell' \le m(\mathcal{A} \cup \mathcal{A}_\mathcal{E}) = |\vec{\mathcal{Z}}| = m(\mathcal{A} \cup \mathcal{A}_\mathcal{E}) \le k|V| - \ell. \tag{22}$$

Since $m \in Q(0, r_{\mathsf{M}^k_{\mathcal{F}}})$, we get that $\vec{\mathcal{Z}}$ is independent in $\mathsf{M}^k_{\mathcal{F}}$. It follows that $\vec{\mathcal{Z}}$ is a subset of the dyperedge set of an orientation $\vec{\boldsymbol{\mathcal{F}}}$ of $\mathcal{F}$ and in the hypergraph $\boldsymbol{\mathcal{H}_\mathcal{F}} = (V, \mathcal{E}_\mathcal{A} \cup \mathcal{E})$ we have for all $X \subseteq V$,

$$|\mathcal{Z}(X)| \le r_{\mathsf{M}^k_{\mathcal{H}_\mathcal{F}}}(\mathcal{Z}(X)) \le k(|X| - 1). \tag{23}$$

Let $\boldsymbol{S}$ be the multiset of V such that $|S_v| = k - d^-_{\vec{\mathcal{Z}}}(v)$ for all $v \in V$. Since $d^-_{\vec{\mathcal{Z}}} \ge 0$, (8) holds. Since for all $X \subseteq V$, by (23), we have

$$d^-_{\vec{\mathcal{Z}}}(X) = \sum_{v\in X} d^-_{\vec{\mathcal{Z}}}(v) - |\vec{\mathcal{Z}}(X)| = \sum_{v\in X}(k - |S_v|) - |\mathcal{Z}(X)| \ge k|X| - |S_X| - k(|X| - 1) = k - |S_X|,$$

(9) holds for $\vec{\boldsymbol{\mathcal{F}}}' = (V, \vec{\mathcal{Z}})$. Then, by Corollary 11, there exists a k-regular packing of s-hyperarborescences $(s \in S)$ in $\vec{\mathcal{F}}'$ and hence in $\vec{\mathcal{F}}$. Since the number of dyperedges in the packing is $k|V| - |S| = \sum_{v\in V}(k - |S_v|) = \sum_{v\in V} d^-_{\vec{\mathcal{Z}}}(v) = |\vec{\mathcal{Z}}|$, the dyperedge set of the packing is $\vec{\mathcal{Z}}$. Since for all $v \in V$, by (21), we have

$$f(v) \le k - |\vec{\mathcal{Z}}_v| = k - d^-_{\vec{\mathcal{Z}}}(v) = |S_v| = k - d^-_{\vec{\mathcal{Z}}}(v) = k - |\vec{\mathcal{Z}}_v| \le g_k(v) \le g(v),$$

the packing is (f, g)-bounded. Since, by (22), we have

$$\ell \le k|V| - |\vec{\mathcal{Z}}| = |S| = k|V| - |\vec{\mathcal{Z}}| \le \ell',$$

the packing is (ℓ, ℓ')-limited. Finally, since $\vec{\mathcal{F}}$ is an orientation of $\mathcal{F}$, the proof is complete.

(b) By Theorem 16.1, for all $v \in V$, $Q(0, r_v) \cap K(k - g_k(v), k - f(v)) \ne \emptyset$ if and only if $k - g_k(v) \le k - f(v)$ that is (3) holds and $0 \le k - f(v)$ (that holds by the previous inequality) and $k - g_k(v) \le d^-_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)$. Then $Q(0, r_v) \cap K(k - g_k(v), k - f(v)) = Q(p_v, b_v)$ where, by (12), we have for all $\mathcal{Z} \subseteq \mathcal{A} \cup \mathcal{A}_\mathcal{E}$,

$$\boldsymbol{p_v}(\mathcal{Z}_v) = \max\{0, k - g_k(v) - d^-_{\overline{\mathcal{Z}}_v}(v)\}, \quad \boldsymbol{b_v}(\mathcal{Z}_v) = \min\{d^-_{\mathcal{Z}_v}(v), k - f(v)\}. \tag{24}$$

By Theorem 16.3, $\sum_{v\in V} Q(p_v, b_v) = Q(p_\Sigma, b_\Sigma)$ where $\boldsymbol{p_\Sigma} = \sum_{v\in V} p_v$, $\boldsymbol{b_\Sigma} = \sum_{v\in V} b_v$. By Theorem 16.1, $Q(p_\Sigma, b_\Sigma) \cap K(k|V| - \ell', k|V| - \ell) \ne \emptyset$ if and only if $Q(p_v, b_v) \ne \emptyset$ for all $v \in V$, $k|V| - \ell' \le k|V| - \ell$ (which is equivalent to one of the conditions in (4)), $p_\Sigma(\mathcal{A} \cup \mathcal{A}_\mathcal{E}) \le k|V| - \ell$ (which is equivalent to the other condition in (4)) and $b_\Sigma(\mathcal{A} \cup \mathcal{A}_\mathcal{E}) \ge k|V| - \ell'$. Then the intersection is equal to $Q(p, b)$ where, by (12), (24), $p_\Sigma = \sum_{v\in V} p_v$, and $b_\Sigma = \sum_{v\in V} b_v$, we have for all $\mathcal{Z} \subseteq \mathcal{A} \cup \mathcal{A}_\mathcal{E}$,

$$\boldsymbol{p}(\mathcal{Z}) = \max\left\{\sum_{v\in V} \max\{0, k - g_k(v) - d^-_{\overline{\mathcal{Z}}_v}(v)\}, k|V| - \ell' - \sum_{v\in V} \min\{d^-_{\overline{\mathcal{Z}}_v}(v), k - f(v)\}\right\}, \tag{25}$$

$$\boldsymbol{b}(\mathcal{Z}) = \min\left\{\sum_{v\in V} \min\{d^-_{\mathcal{Z}_v}(v), k - f(v)\}, k|V| - \ell - \sum_{v\in V} \max\{0, k - g_k(v) - d^-_{\mathcal{Z}_v}(v)\}\right\}. \tag{26}$$

By Theorem 16.2, $Q(p, b) \cap Q(0, r_{\mathsf{M}^k_{\mathcal{F}}}) \ne \emptyset$ if and only if $Q(p, b) \ne \emptyset$, $p \le r_{\mathsf{M}^k_{\mathcal{F}}}$ which, by (25), is equivalent to (19) and (20), and $b \ge 0$ (which holds by $b \ge p \ge 0$). Note that $k - g_k(v) \le d^-_{\mathcal{A}\cup\mathcal{A}_\mathcal{E}}(v)$ for all $v \in V$ and $b_\Sigma(\mathcal{A} \cup \mathcal{A}_\mathcal{E}) \ge k|V| - \ell'$ follow from $p \le r_{\mathsf{M}^k_{\mathcal{F}}}$ applied for $\mathcal{Z} = \emptyset$ and the proof is complete.

(c) We note that (19) is equivalent to

$$k|V| - g_k(V) - \sum_{v\in V} \min\{d^-_{\mathcal{Z}}(v), k - g_k(v)\} \le r_{\mathsf{M}^k_{\mathcal{F}}}(\overline{\mathcal{Z}}). \tag{27}$$

First we show that (19) and (20) imply (18). Let $\mathcal{P}$ be a subpartition of V. Let $\mathcal{Z} = \bigcup_{v\in\overline{\cup\mathcal{P}}} \rho_{\mathcal{A}}(v) \cup \bigcup_{e\in\mathcal{E}(\mathcal{F}(\overline{\cup\mathcal{P}}))} \mathcal{A}_e$ and $\mathcal{P'} = \mathcal{P} \cup \{v\}_{v\in\overline{\cup\mathcal{P}}}$. Note that $d^-_{\mathcal{Z}}(v) = 0$ for all $v \in \cup\mathcal{P}$,

$$\sum_{v\in V} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} \le k|\overline{\cup\mathcal{P}}| - h(\overline{\cup\mathcal{P}}) \text{ for } h \in \{g_k, f\}, \tag{28}$$

$\mathcal{P}'$ is a partition of V, and, by (11), we have

$$r_{\mathsf{M}^k_{\mathcal{F}}}(\overline{\mathcal{Z}}) \le |\overline{\mathcal{Z}} \cap \mathcal{A}(\mathcal{P}')| + |\{e \in \mathcal{E}(\mathcal{P}') : \overline{\mathcal{Z}} \cap \mathcal{A}_e \ne \emptyset\}| + k(|V| - |\mathcal{P}'|) = e_{\mathcal{A}\cup\mathcal{A}_{\mathcal{E}}}(\mathcal{P}) + k(|V| - |\mathcal{P}| - |\overline{\cup\mathcal{P}}|). \tag{29}$$

Then (27), (28) applied for $h = g_k$ and (29) imply $e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) \ge k|\mathcal{P}| - g_k(\cup\mathcal{P})$. Similarly, (20), (28) applied for $h = f$ and (29) imply $e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) \ge k|\mathcal{P}| - \ell' + f(\overline{\cup\mathcal{P}})$. Hence (18) follows.

We now show that (18) implies (20) and (27) and hence (19). Let $\mathcal{Z} \subseteq \mathcal{A} \cup \mathcal{A}_{\mathcal{E}}$. By (11), there exists a partition $\mathcal{P}$ of V such that for $\mathcal{K} = \{e \in \mathcal{E}(\mathcal{P}) : \overline{\mathcal{Z}} \cap \mathcal{A}_e \ne \emptyset\}$, we have

$$r_{\mathsf{M}^k_{\mathcal{F}}}(\overline{\mathcal{Z}}) = |\overline{\mathcal{Z}} \cap \mathcal{A}(\mathcal{P})| + |\mathcal{K}| + k(|V| - |\mathcal{P}|). \tag{30}$$

For $h \in \{g_k, f\}$, let $\mathcal{P}_h = \{X \in \mathcal{P} : d^-_{\mathcal{Z}}(v) \le k - h(v) \text{ for all } v \in X\}$. Note that $\mathcal{P}_h$ is a subpartition of V and for every $X \in \mathcal{P} - \mathcal{P}_h$, there exists a vertex $v_X \in X$ such that $d^-_{\mathcal{Z}}(v_X) > k - h(v_X)$. By the definition of $\mathcal{K}$, we have

$$\mathcal{A}_{\mathcal{E}(\mathcal{P}_h)-\mathcal{K}} \subseteq \mathcal{Z} \cap \mathcal{A}_{\mathcal{E}(\mathcal{P}_h)}. \tag{31}$$

Claim 22 $r_{\mathsf{M}^k_{\mathcal{F}}}(\overline{\mathcal{Z}}) + \sum_{v\in V} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} \ge e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_h) - k|\mathcal{P}_h| - h(\overline{\cup\mathcal{P}_h}) + k|V|$.

Proof: By (30), the definitions of $\mathcal{P}_h$ and v_X, $d^-_{\mathcal{Z}} \ge 0, k - h \ge 0$, (31), and $h \ge 0$, we have

$$\begin{aligned}
& r_{\mathsf{M}^k_{\mathcal{F}}}(\overline{\mathcal{Z}}) + \sum_{v\in V} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} \\
= \ & |\overline{\mathcal{Z}} \cap \mathcal{A}(\mathcal{P})| + |\mathcal{K}| + k(|V| - |\mathcal{P}|) + \sum_{v\in\cup\mathcal{P}_h} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} + \sum_{v\in\overline{\cup\mathcal{P}_h}} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} \\
\ge \ & |\overline{\mathcal{Z}} \cap \mathcal{A}(\mathcal{P}_h)| + \sum_{v\in\cup\mathcal{P}_h} d^-_{\mathcal{Z}}(v) + \sum_{X\in\mathcal{P}-\mathcal{P}_h} \sum_{v\in X} \min\{d^-_{\mathcal{Z}}(v), k - h(v)\} + |\mathcal{K}| + k(|V| - |\mathcal{P}|) \\
\ge \ & |\overline{\mathcal{Z}} \cap \mathcal{A}(\mathcal{P}_h)| + |\mathcal{Z} \cap \mathcal{A}(\mathcal{P}_h)| + |\mathcal{Z} \cap \mathcal{A}_{\mathcal{E}(\mathcal{P}_h)}| + \sum_{X\in\mathcal{P}-\mathcal{P}_h} (k - h(v_X)) + |\mathcal{K}| + k(|V| - |\mathcal{P}|) \\
\ge \ & |\mathcal{A}(\mathcal{P}_h)| + |\mathcal{A}_{\mathcal{E}(\mathcal{P}_h)-\mathcal{K}}| + \sum_{X\in\mathcal{P}-\mathcal{P}_h} (k - h(X)) + |\mathcal{K}| + k(|V| - |\mathcal{P}|) \\
\ge \ & e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_h) - |\mathcal{K}| + k(|\mathcal{P}| - |\mathcal{P}_h|) - h(\overline{\cup\mathcal{P}_h}) + |\mathcal{K}| + k(|V| - |\mathcal{P}|) \\
\ge \ & e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_h) - k|\mathcal{P}_h| - h(\overline{\cup\mathcal{P}_h}) + k|V|,
\end{aligned}$$

and the claim follows. □

Claim 22, applied for $h = f$, and (18) provide that $r_{\mathsf{M}^k_{\mathcal{F}}}(\overline{\mathcal{Z}}) + \sum_{v\in V} \min\{d^-_{\mathcal{Z}}(v), k - f(v)\} \ge k|V| - \ell'$, so (20) holds. Similarly, Claim 22, applied for $h = g_k$, and (18) provide that $r_{\mathsf{M}^k_{\mathcal{F}}}(\overline{\mathcal{Z}}) + \sum_{v\in V} \min\{d^-_{\mathcal{Z}}(v), k - g_k(v)\} \ge k|V| - g_k(V)$, so (27) holds. The proof of the theorem is complete. □

We finish the paper by showing that Theorems 16 and 21 imply Theorem 20.

Proof: Let $(\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A}), f, g, k, \ell, \ell')$ be an instance of Theorem 20 that satisfies (3), (4) and (18). Since (18) holds, by Theorem 21(c), (19) and (20) hold. Since (3) and (4) also hold, by Theorem 21(b), the polyhedron T, defined in Theorem 21, is not empty. Then, by Theorem 16.2(ii), T contains an integral element x. By Theorem 21(b), x is the characteristic vector of the dyperedge set of an (f, g)-bounded k-regular (ℓ, ℓ')-limited packing of hyperarborescences in an orientation $\vec{F} = (V, \vec{\mathcal{E}} \cup \mathcal{A})$ of $\mathcal{F}$. By replacing the dyperedges in $\vec{\mathcal{E}}$ by the underlying hyperedges in $\mathcal{E}$, we obtain the required packing. □

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