Hypergraph connectivity augmentation

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Abstract

The hypergraph augmentation problem is to augment a hypergraph by hyperedges to meet prescribed local connectivity requirements. We provide here a minimax theorem for this problem. The result is derived from the degree constrained version of the problem by a standard method. We shall construct the required hypergraph for the latter problem by a greedy type algorithm. A similar minimax result will be given for the problem of augmenting a hypergraph by weighted edges (hyperedges of size two with weights) to meet prescribed local connectivity requirements. Moreover, a special case of an earlier result of Schrijver on supermodular colourings shall be derived from our theorem.

1. Introduction

Let us be given a hypergraph $\mathcal{G}$ on the finite set V and for any pair s, t of vertices a connectivity requirement $\lambda(s,t)$ between s and t, where $\lambda(s,t)$ is a non-negative integer. We are looking for a hypergraph $\mathcal{H}$ so that adding the hyperedges of $\mathcal{H}$ to $\mathcal{G}$ the new hypergraph satisfies the connectivity requirements.

For a hypergraph $\mathcal{H}$ the degree of a set X denoted by $d_{\mathcal{H}}(X)$ is the number of hyperedges H of $\mathcal{H}$ for which none of $H \cap X$ and $H \cap (V - X)$ is empty.

Using this notation our problem is the following. Find a hypergraph $\mathcal{H}$ so that for all $X \subseteq V$

$$d_{\mathcal{G}+\mathcal{H}}(X) \geq \lambda(s,t) \text{ for all } s \in X, t \notin X.$$

Introducing the set function $R(X) := \max\{\lambda(s,t) : s \in X, t \notin X\}$, the inequalities above can be transformed into

$$d_{\mathcal{H}}(X) \geq R(X) - d_{\mathcal{G}}(X) \text{ for all } X \subseteq V.$$

Finally, let us define $p(X) := R(X) - d_{\mathcal{G}}(X)$. So the hypergraph $\mathcal{H}$ must satisfy

$$d_{\mathcal{H}}(X) \geq p(X) \text{ for all } X \subseteq V.$$

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The set function $p(X)$ has the following properties: it is integer valued, symmetric and skew-supermodular (for definitions see Section 2). We remark here that $p(X)$ is not necessarily non-negative.

The question is what is the minimum value of a hypergraph $\mathcal{H}$ which satisfies a symmetric, skew–supermodular set function p, that is

$$d_{\mathcal{H}}(X) \geq p(X) \text{ for all } X \subseteq V.$$

Here the value of the hypergraph $\mathcal{H}$ is not the number of the hyperedges in $\mathcal{H}$ because in that case the problem is trivial. (Let $\mathcal{H}$ contain the set V k times where $k = \max\{p(X) : X \subseteq V\}$.) Thus it is natural to define val($\mathcal{H}$) to be $\sum_{H \in \mathcal{H}} |H|$.

The following theorem solves the problem.

Theorem 1. *Let p be a symmetric, skew–supermodular, integer valued function on the ground set V. Then*

$$\min\{\text{val}(\mathcal{H}) : d_{\mathcal{H}}(X) \geq p(X) \text{ for all } X \subseteq V\} = \max\{\textstyle\sum p(V_i)\}$$

where the maximum is taken over all subpartitions $\{V_1, ..., V_l\}$ of V.

The edge–connectivity augmentation problem for graphs was solved by A.Frank [1]. He derived the minimax theorem from the degree constrained version of the problem. We shall follow this line.

Theorem 2. *Let p be a skew–supermodular integer valued function on the ground set V. Furthermore, let $m(v)$ be a non–negative integer valued function on V. Then there exists a hypergraph $\mathcal{H}$ such that*

$$d_{\mathcal{H}}(v) = m(v) \text{ for all } v \in V, \tag{1}$$

$$d_{\mathcal{H}}(X) \geq p(X) \text{ for all } X \subseteq V, \tag{2}$$

if and only if for all $X \subseteq V$

$$p(X) \leq \min\{m(X), m(V - X)\}. \tag{3}$$

Furthermore, $\mathcal{H}$ can be chosen so that $|\mathcal{H}| = k := \max\{p(X) : X \subseteq V\}$.

We shall prove that the hyperedges of the required hypergraph can be constructed step by step by a greedy type algorithm.

In Section 5 we shall derive from Theorem 2 a result due to Schrijver [4] on supermodular colourings.

In the remaining part of the Introduction we show a simple application of Theorem 1. Let us consider the following problem. We want to augment a hypergraph $\mathcal{G}$ by a set F of edges (hyperedges of size two) with suitable rational weights on the edges so that the resulting hypergraph satisfies prescribed local edge-connectivity requirements and the total weight of the new edges is minimum. As above, we can formulate this problem as follows.

Minimize $1c_F$ so that $d_{c_F}(X) \geq p(X)$ for all $X \subset V$, where c_F is the weighting of F, $d_{c_F}(X)$ denotes the sum of the weights of the edges of F leaving X and p is the above defined set function.

Claim $\min\{1c_F : d_{c_F}(X) \geq p(X) \text{ for all } X \subseteq V\} = \max\{1/2 \sum p(V_i)\}$,

where the maximum is taken over all subpartitions $\{V_1, ..., V_l\}$ *of* V.

Proof. min≥max: $1c_F \geq 1/2 \sum d_{c_F}(V_i) \geq 1/2 \sum p(V_i)$.

min≤max: Let $\mathcal{H}$ be a hypergraph and $\{V_1, ..., V_l\}$ be a subpartition of V satisfying the minimax formula in Theorem 1. Using $\mathcal{H}$ we define the required edge set and weighting. Let us replace each hyperedge H of $\mathcal{H}$ by a circuit C_H on the vertex set of H with weight 1/2 on each edge of C_H. Then $d_{c_F}(X) \geq d_{\mathcal{H}}(X) \geq p(X)$ and $\sum p(V_i) = \sum_{H \in \mathcal{H}} |H| = 2(1c_F)$, and we are done. □

Remarks

1. Clearly, this claim is true for any integer valued, symmetric, skew–supermodular set function p.
2. Note that the above constructed weighting is always half integral.
3. In the special case, when the starting hypergraph is empty, the same result was obtained by Gomory and Hu [3], and when the starting hypergraph is a graph, this was proved later by A. Frank in [1].

2. Definitions, Preliminaries

In this paper all (set) functions are integer valued. A set function p is called **skew–supermodular** if at least one of the following two inequalities holds

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \qquad (p1)$$

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X), \qquad (p2)$$

whenever X and Y are two subsets of V. We call p **symmetric** if $p(X) = p(V - X)$ for all $X \subseteq V$. Let $\mathcal{H}$ be a hypergraph. Then the **value** of $\mathcal{H}$ is defined to be $\text{val}(\mathcal{H}) = \sum\{|H| : H \in \mathcal{H}\}$. $d_{\mathcal{H}}(X)$ denotes the number of hyperedges H of $\mathcal{H}$ for which none of $H \cap X$ and $H \cap (V - X)$ is empty.

A.Frank showed in [2] that $R(X)$ defined in the Introduction is skew-supermodular. It is well-known that the degree function $d_{\mathcal{G}}(X)$ satisfies both $(p1)$ and $(p2)$ so $p(X)$ defined in the Introduction is skew-supermodular, indeed. In this section we present some simple propositions we shall need later. It is easy to see that the following holds.

Proposition 3. *Let p be a skew–supermodular function on V. Then*

$$p'(X) := \max\{p(X), p(V - X)\}$$

is a symmetric, skew–supermodular function. □

Proposition 4. *Let p be a skew–supermodular function on V. Let Z be a subset of V. Then $p'(X) := \max\{p(X \cup X') : X' \subseteq Z\}$ is skew–supermodular on $V - Z$.*

Proof. Let $X, Y \subseteq V - Z$. Let X' (respectively, Y') be the subset of Z which defines $p'(X)$ (resp. $p'(Y)$). p is skew–supermodular, thus either ($p1$) (Case I.) or ($p2$) (Case II.) holds for $X \cup X'$ and $Y \cup Y'$.

Case I.

$$\begin{aligned} p'(X) + p'(Y) &= p(X \cup X') + p(Y \cup Y') \\ &\le p((X \cup X') \cap (Y \cup Y')) + p((X \cup X') \cup (Y \cup Y')) \\ &= p((X \cap Y) \cup (X' \cap Y')) + p((X \cup Y) \cup (X' \cup Y')) \\ &\le p'(X \cap Y) + p'(X \cup Y). \end{aligned}$$

Case II.

$$\begin{aligned} p'(X) + p'(Y) &= p(X \cup X') + p(Y \cup Y') \\ &\le p((X \cup X') - (Y \cup Y')) + p((Y \cup Y') - (X \cup X')) \\ &= p((X - Y) \cup (X' - Y')) + p((Y - X) \cup (Y' - X')) \\ &\le p'(X - Y) + p'(Y - X). \end{aligned}$$

□

Proposition 5. *Let p be a skew–supermodular function on V. Let V_1 be a subset of V. Let*

$$p'(X) := \begin{cases} p(X) - 1 & \text{if } X \cap V_1 \neq \emptyset, \\ p(X) & \text{otherwise.} \end{cases}$$

Then p' is skew–supermodular on V.

Proof. It is straightforward. □

Let $m : V \longrightarrow Z_+$ be a non–negative integer valued function on V. Let p be a skew–supermodular set function on V. Assume that $p(X) \le m(X)$ for all $X \subseteq V$. We call a set $X \subseteq V$ **tight** if $m(X) = p(X)$. Then we have the following.

Proposition 6. *Let X and Y be two tight sets. Then either $X \cap Y$ and $X \cup Y$ or $X - Y$, $Y - X$ are tight sets. Furthermore, in the latter case $m(X \cap Y) = 0$.*

Proof. Since p is skew–supermodular, either ($p1$) or ($p2$) holds for X and Y.

Case I. Assume first that ($p1$) holds. Then

$$\begin{aligned} m(X) + m(Y) &= p(X) + p(Y) \\ &\le p(X \cap Y) + p(X \cup Y) \\ &\le m(X \cap Y) + m(X \cup Y) \\ &= m(X) + m(Y). \end{aligned}$$

Thus equality holds everywhere, implying that $X \cap Y$ and $X \cup Y$ are tight.

Case II. Assume that $(p2)$ holds. Then

$$\begin{aligned} m(X) + m(Y) &= p(X) + p(Y) \\ &\leq p(X - Y) + p(Y - X) \\ &\leq m(X - Y) + m(Y - X) \\ &= m(X) + m(Y) - 2m(X \cap Y). \end{aligned}$$

This implies that $X - Y$ and $Y - X$ are tight. Furthermore, $m(X \cap Y) = 0$. □

3. The proof of Theorem 2

In this section we prove Theorem 2. Moreover, we shall characterize the hyperedges which can be contained in a hypergraph satisfying the requirements of Theorem 2.

Proof. The only if part is trivial, so we prove the other direction. Let us consider a minimal counter–example, minimal with respect to $|V| + k$.

By Proposition 3, we may assume that p is symmetric.

Lemma 7. *Let $H \subseteq V$. Then there exists a hypergraph $\mathcal{H}$ with*

i.) $H \in \mathcal{H}$,
ii.) $\mathcal{H}$ *satisfies (1) and (2),*
iii.) $\mathcal{H}$ *contains exactly* $k := \max\{p(X) : X \subseteq V\}$ *hyperedges*

if and only if

a.) p *and* m *satisfy (3),*
b.) $X \cap H \neq \emptyset$ *for all set* $X \subseteq V$ *with* $p(X) = k$,
c.) $|X \cap H| \leq m(X) - p(X) + 1$ *for all* $X \subseteq V$,
$|H| \leq m(X) - p(X)$ *for all* $V \supseteq X \supseteq H$,
d.) $m(v) \geq 1$ *for all* $v \in H$.

Remark. Note that in fact b.) and c.) imply a.).

Proof. First we show the necessity of the conditions. Assume the hypergraph $\mathcal{H}$ satisfies the requirement of the lemma. It is easy to see that a.), b.) and d.) hold. To see c.) assume that $H \in \mathcal{H}$. Let $\mathcal{H}' := \mathcal{H} - H$,

$$p'(X) := \begin{cases} p(X) - 1 & \text{if } X \cap H \neq \emptyset, \\ p(X) & \text{otherwise.} \end{cases}$$

$$m'(v) := \begin{cases} m(v) - 1 & \text{if } v \in H, \\ m(v) & \text{otherwise.} \end{cases}$$

Then p' is skew–supermodular by Proposition 5, m' is non-negative by d.) and clearly $\mathcal{H}'$ satisfies (1) and (2) for p', m'. Thus (3) holds for p' and m'.

Let $X \subseteq V$. Then $p(X) - 1 \leq p'(X) \leq m'(X) = m(X) - |X \cap H|$.
Finally, let $H \subseteq X \subseteq V$. Then $p(X) = p(V - X) = p'(V - X) \leq m'(X) = m(X) - |H|$.

Secondly, we show the sufficiency of the conditions. Let H be a subset of V satisfying a.), b.), c.) and d.).
Let p' and m' be defined as above.

Claim 8. *p' and m' satisfy the conditions of Theorem 2.*

Proof. By Proposition 5, p' is skew–supermodular and by d.) m' is non-negative.
To prove (3), let X be a subset of V.

If $X \cap H = \emptyset$, then $p'(X) = p(X) \leq m(X) = m'(X)$ by a.).
If $X \cap H \neq \emptyset$, then by c.), $p'(X) = p(X) - 1 \leq m(X) - |X \cap H| = m'(X)$.
Thus $p'(X) \leq m'(X)$ for all $X \subseteq V$. $\quad (*)$

If $X \cap H = \emptyset$, then by c.), $p'(X) = p(X) = p(V - X) \leq m(V - X) - |H| = m'(V - X)$.
If $X \cap H \neq \emptyset$, then using the previous inequality $(*)$ and the symmetry of p, $p'(X) = p(X) - 1 = p(V - X) - 1 \leq p'(V - X) \leq m'(V - X)$.
Thus $p'(X) \leq m'(V - X)$ for all $X \subseteq V$. □

By b.) $\max\{p'(X) : X \subseteq V\} = k - 1$. By the minimality of the counter–example, there exists a hypergraph $\mathcal{H}'$ satisfying (1) and (2) for p' and m' and containing $k - 1$ hyperedges.

Claim 9. *$\mathcal{H} := \mathcal{H}' \cup \{H\}$ satisfies* (1) *and* (2) *for p and m.*

Proof. First we prove that (1) holds.
If $v \notin H$ then $d_{\mathcal{H}}(v) = d_{\mathcal{H}'}(v) = m'(v) = m(v)$.
If $v \in H$ then $d_{\mathcal{H}}(v) = d_{\mathcal{H}'}(v) + 1 = m'(v) + 1 = m(v)$.

Secondly we prove that (2) holds. Let $X \subseteq V$.
If $X \cap H = \emptyset$, then $d_{\mathcal{H}}(X) = d_{\mathcal{H}'}(X) \geq p'(X) = p(X)$.
If $X \supseteq H$, then $d_{\mathcal{H}}(X) = d_{\mathcal{H}'}(X) = d_{\mathcal{H}'}(V - X) \geq p'(V - X) = p(V - X) = p(X)$.
If $X \cap H \neq \emptyset$ and $H - X \neq \emptyset$, then $d_{\mathcal{H}}(X) = d_{\mathcal{H}'}(X) + 1 \geq p'(X) + 1 = p(X)$. □

By Claim 9 the proof of Lemma 7 is complete. □

To prove Theorem 2 we have to show that there exists a subset H of V satisfying b.), c.) and d.) in Lemma 7. We shall need the following lemma.

Lemma 10. *$m(v) \geq 1$ for all $v \in V$.*

Proof. Let $Z := \{v \in V : m(v) = 0\}$ and suppose Z is not empty. Let us define a set function p' on $V - Z$ as follows.

$$p'(X) := \max\{p(X \cup X') : X' \subseteq Z\}.$$

Then by Proposition 4, p' is skew–supermodular and it is symmetric since

$$p'(X) = \max\{p(X \cup X') : X' \subseteq Z\} = \max\{p(V - X - X') : X' \subseteq Z\} = p'((V - Z) - X).$$

Let $m'(v) := m(v)$ for $v \in V - Z$. Since $m(X') = 0$ for all $X' \subseteq Z$, we have for all $X \subseteq V - Z$

$$p'(X) = \max\{p(X \cup X') : X' \subseteq Z\} \leq \max\{m(X \cup X') : X' \subseteq Z\} = m(X) = m'(X).$$

Thus p' and m' satisfy the conditions of Theorem 2. Now the ground set is smaller, thus there exists a hypergraph $\mathcal{H}$ which satisfies (1) and (2) for p' and m'. Since p' is defined by as a maximum, this hypergraph is good for p and m. This contradiction proves the assertion. □

Let V_1 be a minimal (for inclusion) set intersecting each set Y for which $p(Y) = k$. We show that V_1 satisfies c.) and thus by Lemma 7 there exists a hypergraph satisfying (1) and (2) and containing this hyperedge.

Lemma 11. *$p(X) \leq m(X) + 1 - |V_1 \cap X|$ for all $X \subseteq V$. Furthermore, if $X \supseteq V_1$ then $p(X) \leq m(X) - |V_1|$.*

Proof. We prove the lemma by induction on $|X|$. For $X = \emptyset$ it is true by (3). Let X be an arbitrary subset of V and assume that the lemma holds for each smaller set. We may assume that there exists a vertex $y \in V_1 \cap X$, for otherwise the lemma holds by (3). By the minimality of V_1, there exists a set $Y \subseteq (V - V_1) \cup \{y\}$ for which $p(Y) = k$. Then

$$|V_1 \cap (X - Y)| = |V_1 \cap X| - 1. \tag{4}$$

Note that by Lemma 10 and (4) we have the following inequalities.

$$m(X - Y) \geq m((X - Y) \cap V_1) \geq |(X - Y) \cap V_1| = |V_1 \cap X| - 1. \tag{5}$$

$$m(X \cap Y) \geq 1. \tag{6}$$

Since $y \in X \cap Y$, $|X - Y| < |X|$ and the lemma is true for $X - Y$, that is

$$p(X - Y) \leq m(X - Y) + 1 - |V_1 \cap (X - Y)|. \tag{7}$$

By the skew–supermodularity of p, $(p1)$ (Case I) or $(p2)$ (Case II.) holds for X and Y.

Case I.

$$\begin{aligned} p(X) + p(Y) &\leq p(X \cap Y) + p(X \cup Y) && \text{by (p1)} \\ &\leq m(X \cap Y) + p(X \cup Y) && \text{by (3)} \\ &= m(X) - m(X - Y) + p(X \cup Y) \\ &\leq m(X) + 1 - |V_1 \cap X| + p(X \cup Y). && \text{by (5)} \end{aligned}$$

This implies the desired inequalities, since $p(Y) = k, p(X \cup Y) \leq k$ and if $X \supseteq V_1$ then $V - (X \cup Y)$ is disjoint from V_1 thus $p(X \cup Y) = p(V - (X \cup Y)) \leq k - 1$.

Case II.

$$\begin{aligned} p(X) + p(Y) &\leq p(X - Y) + p(Y - X) && \text{by (p2)} \\ &\leq m(X - Y) + 1 - |V_1 \cap (X - Y)| + p(Y - X) && \text{by (7)} \\ &\leq m(X) - m(X \cap Y) + 1 - |V_1 \cap X| + 1 + p(Y - X) && \text{by (4)} \\ &\leq m(X) - |V_1 \cap X| + 1 + p(Y - X) && \text{by (6)} \\ &\leq m(X) - |V_1 \cap X| + k, && \text{since } (Y - X) \cap V_1 = \emptyset. \end{aligned}$$

□

Observe that V_1 can be constructed easily since the minimal sets Y with $p(Y) = \max\{p(X) : X \subset V\}$ are pairwise disjoint by the skew-supermodularity of p.

4. Hypergraph connectivity augmentation

Now we are in a position to prove Theorem 1.

Proof. The min $\geq$ max being trivial we prove only the other direction. We shall prove it with a standard method using Theorem 2. Let $m : V \longrightarrow Z_+$ be a degree constraint on V so that there exists a hypergraph $\mathcal{H}$ satisfying (1) and (2) in Theorem 2 for p and m and $m(V)$ is minimal. (Such an m exists since $m(v) = \max\{p(X) : X \subseteq V\}$ for all $v \in V$ satisfies (3) in Theorem 2.)

Recall that a set X is tight if $p(X) = m(X)$. Let $Z := \{v \in V : m(v) = 0\}$ and $V'; = V - Z$. By the minimality of $m(V)$ and the symmetry of p each $v \in V'$ is contained in a tight set. Thus there exists a set system $\{X_1, ...X_l\}$ satisfying the following.

i.) X_i is tight for all $i = 1, ..., l$,
ii.) $\bigcup_1^l X_i \supseteq V'$,
iii.) $\sum |X_i|$ is minimal.

Claim. $X_i \cap X_j = \emptyset$.

Proof. Assume that $X_i \cap X_j \neq \emptyset$. By Proposition 6, either $X_i \cup X_j$ is tight or $X_i - X_j$, $X_j - X_i$ are tight sets and in the latter case $m(X \cap Y) = 0$. Observe that this implies that $X \cap Y \subseteq Z$. $\{X_i, X_j\}$ will be replaced by $X_i \cup X_j$ in the former case, and by $X_i - X_j$, $X_j - X_i$ in the latter case. In both cases we have a contradiction with iii.) □

This subpartition shows the other direction.

min $\leq \mathrm{val}(\mathcal{H}) = \sum_{H \in \mathcal{H}} |H| = \sum_{v \in V} d_{\mathcal{H}}(v) = \sum_{v \in V} m(v) = m(V) = \sum_1^l m(X_i) = \sum_1^l p(X_i) \leq$ max. □□□

5. Supermodular colourings

In this section we show that Theorem 2 implies a special case of an earlier result of Schrijver [4] on supermodular colourings. Let p be a supermodular function. (A set function p is **supermodular** if ($p1$) holds for all sets X and Y.) A colouring of V with k colours is called **good k–colouring with respect to** p if for all subsets X of V the number of different colour classes intersecting X is at least $p(X)$. Schrijver proved in [4] that the obvious necessary condition for the existence of a good k–colouring is also sufficient, that is the following holds.

Theorem 12. *Let p be an integer valued supermodular function and let $k := \max\{p(X) : X \subseteq V\}$ be non-negative. Then there exists a good k–colouring if and only if $p(X) \leq |X|$ for all $X \subseteq V$.*

Proof. Let

$$V' := V \cup v,$$

$$p'(X) := \begin{cases} p(X) & \text{if } v \notin X \\ p(V - X) & \text{if } v \in X. \end{cases}$$

$$m'(u) := \begin{cases} 1 & \text{if } u \in V, \\ \max\{p(X) : X \subseteq V\} & \text{if } u = v. \end{cases}$$

It is easy to see that p' is skew–supermodular.

Lemma 13. *p' and m' satisfy* (3) *in Theorem 2.*

Proof. First of all observe that p' is symmetric. Let $X \subseteq V'$.
If $v \notin X$, then $p'(X) = p(X) \leq |X| = m'(X)$.
If $v \in X$, then $p'(X) = p(V - X) \leq m'(v) + |X \cap V| = m'(X)$.
Thus $p'(X) \leq m'(X)$ for all $X \subseteq V'$. This was to be proved since p' is symmetric. □

By Theorem 2, there exists a hypergraph $\mathcal{H}$ so that

$$d_{\mathcal{H}}(u) = m'(u) \text{ for all } u \in V', \tag{8}$$

$$d_{\mathcal{H}}(X) \geq p'(X) \text{ for all } X \subseteq V', \tag{9}$$

The hypergraph $\mathcal{H}$ defines a partition of V by (8) and this partition is a good k–colouring by (9) and by the definition of $m'(v)$. □

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