Connectivity of orientations of 3-edge-connected graphs

Florian Hörsch, Zoltán Szigeti
Univ. Grenoble Alpes, Grenoble INP, CNRS, G-SCOP, 46 Avenue Félix Viallet, Grenoble, France, 38000.

Abstract
We attempt to generalize a theorem of Nash-Williams stating that a graph has a $k$-arc-connected orientation if and only if it is $2k$-edge-connected. In a strongly directed digraph we call an arc deletable if its deletion leaves a strongly connected digraph. Given a 3-edge-connected graph $G$, we define its Frank number $f(G)$ to be the minimum number $k$ such that there exist $k$ orientations of $G$ with the property that every edge becomes a deletable arc in at least one of these orientations. We are interested in finding a good upper bound for the Frank number. We prove that $f(G) \leq 7$ for every 3-edge-connected graph. On the other hand, we show that a Frank number of 3 is attained by the Petersen graph. Further, we prove better upper bounds for more restricted classes of graphs and establish a connection to the Berge-Fulkerson conjecture. We also show that deciding whether all edges of a given subset can become deletable in one orientation is NP-complete.

1. Introduction
This paper deals with ways of orienting graphs so that the obtained directed graph has certain connectivity properties. Our goal is to generalize classical results of Robbins and Nash-Williams.

Let $G = (V, E)$ be an undirected graph. For some $F \subseteq E$, $G(F)$ denotes $(V, F)$. For a set $X \subseteq V$, the subgraph induced by $X$ is denoted by $G[X]$. We use $\delta G(X)$ to denote the set of edges between $X$ and $V - X$ and $d_G(X)$ for $|\delta G(X)|$. For some vertex $v \in V$, $d_G(\{v\})$ is called the degree of $v$. The graph $G$ is called cubic if $d_G(\{v\}) = 3$ for all $v \in V$. We say that $G$ is $k$-edge-connected if $|\delta G(X)| \geq k$ for all nonempty, proper $X \subset V$. We call $G$ Eulerian if every vertex of $G$ is of even degree. For some $e \in E$, we denote by $G/e$ the graph obtained from $G$ by contracting $e$, that is deleting $e$ and identifying its two endvertices. For some $F = \{e_1, \ldots, e_t\} \subseteq E$, we denote $G/e_1/\ldots/e_t$ by $G/F$. For some subgraph $H$ of $G$, we abbreviate $G/E(H)$ to $G/H$. An orientation of $G$ is a directed graph $D = (V, A)$ such that each edge $uv \in E$ is replaced by exactly one of the arcs $uv$ or $vu$. $G$ or a subset of $V$ is trivial if it contains only one vertex. $G$ is called essentially $(k+1)$-edge-connected if $G$ is $k$-edge-connected and for all edge-cuts of size $k$ one side is trivial. A cycle is a connected graph each vertex of which is of degree 2. A path is a connected graph in which two
vertices are of degree 1 and all other vertices are of degree 2. A cycle packing is a collection of vertex-disjoint cycles of $G$. We say that a vertex or an edge is in the cycle packing if it is contained in one of the cycles of the packing.

An edge set $M$ of $G$ is called matching if each vertex of $G$ is incident to at most 1 edge of $M$. A matching $M$ is perfect if each vertex of $G$ is incident to exactly one edge of $M$. We say that $G$ is $k$-edge-colorable if the edge set of $G$ can be partitioned into $k$ matchings.

Let $D = (V, A)$ be a directed graph. For some $F \subseteq A$, $D(F)$ denotes $(V, F)$. The subgraph induced by some $X \subseteq V$ is denoted by $D[X]$. We use $\delta_D^+(X)$ to denote the set of arcs from $X$ to $V - X$ and $\delta_D^-(X)$ for $\delta_D^+(V - X)$. For a vertex $v \in V$, we call $|\delta_D^+(v)|$ the in-degree and $|\delta_D^-(v)|$ the out-degree of $v$.

The graph that is obtained from $D$ by replacing each arc by an edge between the same two vertices is called the underlying graph of $D$. $D$ is called weakly connected if its underlying graph is connected. $D$ is called strongly connected if $|\delta_D^+(X)| \geq 1$ for every nonempty, proper $X \subset V$. More generally, we say that $D$ is $k$-arc-connected if $|\delta_D^+(X)| \geq k$ for every nonempty, proper $X \subset V$. We call $D$ Eulerian if $|\delta_D^+(v)| = |\delta_D^-(v)|$ for every $v \in V$. For some $a \in A$, we denote by $D/a$ the directed graph obtained from $D$ by contracting $a$, that is deleting $a$ and identifying its head and its tail. For some $F = \{a_1, \ldots, a_t\} \subseteq A$, we denote $D/a_1/\ldots/a_t$ by $D/F$. For some subgraph $H$ of $D$, we abbreviate $D/E(H)$ to $D/H$. $\hat{D}$ denotes the orientation that arises from $D$ by reversing the orientation of all arcs. A circuit is a strongly connected orientation of a cycle. A directed path is an orientation of a path such that at most one arc enters and at most one arc leaves each vertex. Subscripts may be omitted when the graph or directed graph is clear from the context. We also use basic notions of complexity theory which can be found in Chapter 15 of [3].

As one of the first important results in the theory of graph orientations, Robbins proved in 1939 that a graph has a strongly connected orientation if and only if it is 2-edge-connected [10]. This was later generalized by Nash-Williams [7] who proved that for any positive integer $k$, a graph has a $k$-arc-connected orientation if and only if it is $2k$-edge-connected. This naturally raises the question whether odd edge-connectivity also yields distinctive orientability properties. Our approach to this consists in relaxing the goal to obtain exactly one orientation of the graph to allowing several of them. We say that an arc is deletable in a $k$-arc-connected orientation of a $(2k+1)$-edge-connected graph if its deletion leaves it $k$-arc-connected and ask how many orientations are necessary for each edge of the original graph to become a deletable arc in at least one of the orientations. Surprisingly, the number of necessary orientations is bounded by a constant depending only upon $k$. This is a consequence of a theorem of De Vos, Johnson and Seymour [1]. We focus on the case $k = 1$, meaning we want to find orientations of a 3-edge-connected graph such that for every edge of the graph, the deletion of the associated arc leaves a strongly connected graph in at least one of the orientations. In honor of András Frank who proposed this problem and had an immense impact on the development of the theory of graph orientations, we call the minimum number of necessary orientations for a graph $G$ the Frank number $f(G)$. Observe that the Frank number of any 4-edge-connected graph is 1 as it has a 2-arc-connected orientation by the theorem of Nash-Williams. On the other hand, any graph $G$ containing a 3-edge-cut has Frank number at least 2. This follows directly from the fact that in any
strongly connected orientation of $G$, there is one arc of the 3-edge-cut that is oriented differently than the other two arcs. This arc cannot be deletable in this orientation, so at least one more orientation is required. It is an interesting question to find upper bounds for the Frank number of graphs. The following theorem of De Vos, Johnson and Seymour \cite{1} allows to conclude a first constant bound:

**Theorem 1.** Let $G = (V,E)$ be a 3-edge-connected graph. Then there is a partition $\{E_1, \ldots, E_9\}$ of $E$ such that $G - E_i$ is 2-edge-connected for all $i = 1, \ldots, 9$.

This implies the following:

**Corollary 1.** $f(G) \leq 9$ for every 3-edge-connected graph $G$.

Indeed, by Robbins’ Theorem, for all $i = 1, \ldots, 9$, there is a strongly connected orientation of $G - E_i$. Giving an arbitrary orientation to the arcs of $E_i$ yields an orientation in which the arcs of $E_i$ are deletable.

The main contribution of this paper is to further narrow down the values attained by the Frank number. We first show a better upper bound.

**Theorem 2.** $f(G) \leq 7$ for every 3-edge-connected graph $G$.

In attempt to improve on this, we also establish a relationship between our problem and a well-known conjecture about matchings in cubic graphs, the conjecture of Berge-Fulkerson mentioned in Section \ref{2}.

**Theorem 3.** $f(G) \leq 5$ for every 3-edge-connected graph $G$ unless the conjecture of Berge-Fulkerson fails.

Further, we prove a stronger bound for two more restricted classes of 3-edge-connected graphs.

**Theorem 4.** $f(G) \leq 3$ for every 3-edge-connected 3-edge-colorable graph $G$.

**Theorem 5.** $f(G) \leq 3$ for every essentially 4-edge-connected graph $G$.

For the lower bound, we show that there are graphs whose Frank number is strictly bigger than 2, more precisely:

**Theorem 6.** The Frank number of the Petersen graph is 3.

A drawing of the Petersen graph can be found in Figure \ref{1}.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{petersen_graph.png}
\caption{The Petersen graph}
\end{figure}
One of the main difficulties in improving the upper bound on the Frank number consists in finding a useful class of deletable sets. We consider the problem of testing algorithmically whether a set is deletable. More formally, we define the following problem:

**DELETABILITY**

Instance: A graph $G = (V, E)$ and a set $S \subseteq E$.

Question: Is there an orientation $D$ of $G$ such that $D - s$ is strongly connected for all $s \in S$?

The following result shows that an efficient algorithm for DELETABILITY seems out of reach. This implies that a good characterization of deletable sets is hard to attain.

**Theorem 7.** DELETABILITY is NP-complete for cubic 3-edge-connected graphs.

In Section 2, we present several classical results we will make use of and make some preparatory observations. Also, we introduce an auxiliary graph that will help to reduce the problems to cubic graphs later. In Section 3, we deal with the general case of 3-edge-connected graphs proving Theorems 2 and 3. Section 4 is concerned with essentially 4-edge-connected graphs, in particular the proof of Theorem 5. In Section 5, we prove Theorem 6. Finally, in Section 6, we conclude our work and give directions for further research on this topic.

2. Preliminaries

In the first part of this section, we give classical results we will make use of later. In the second part, we make some easy preparatory observations which will prove useful later. In the third part, we introduce a way to construct a cubic graph from an arbitrary graph of minimum degree at least 3 and give some basic properties of it.

2.1. Previous results

The following result was proven by Nash-Williams [7] and is the starting point of our work as it characterizes the graphs admitting a $k$-arc-connected orientation.

**Theorem 8.** A graph has a $k$-arc-connected orientation if and only if it is $2k$-edge-connected.

In fact, there is an immense strengthening of this theorem. For any two vertices $u, v$ of a graph $G$, let $\lambda_G(u, v)$ be the maximum number of edge-disjoint paths between $u$ and $v$. An orientation $D$ of $G$ is called well-balanced if for any $u, v \in V(G)$, there exist at least $\lfloor \frac{\lambda_G(u, v)}{2} \rfloor$ directed paths from $u$ to $v$ and also from $v$ to $u$ in $D$. An odd-vertex pairing of $G$ is a partition of the set of vertices of odd degree of $G$ into pairs. An odd-vertex pairing $P$ is called admissible if the restriction of any Eulerian orientation of $G + P$ to $E(G)$ yields a well-balanced orientation of $G$. Nash-Williams [7] proved the following:

**Theorem 9.** Every graph has an admissible odd-vertex pairing.
Observe that this implies that every graph has a well-balanced orientation. Also, as well-balanced orientations of $2k$-edge-connected graphs are $k$-arc-connected, Theorem 9 implies Theorem 8.

There are several other well-known results we make use of in this article. The first one is about packing spanning trees and is also due to Nash-Williams [8].

**Theorem 10.** Every $2k$-edge-connected graph has $k$ edge-disjoint spanning trees.

The next one concerns matching theory and was proven by Petersen [9].

**Theorem 11.** Every cubic $2$-edge-connected graph has a perfect matching.

The next one concerns minimum edge cuts and can for example be found as Theorem 7.1.2 in [3].

**Proposition 1.** Let $G$ be a $3$-edge-connected graph and let $\delta(X)$ and $\delta(X')$ be $3$-edge-cuts of $G$. Then $\delta(X)$ and $\delta(X')$ are not crossing, i.e. one of $X-X'$, $X'-X$, $X \cap X'$ or $V(G) - (X \cup X')$ is empty.

Let $G = (V,E)$ be a graph and $T \subseteq V$. A $T$-join is defined to be a set $F \subseteq E$ such that the set of odd degree vertices of $G[F]$ is $T$. We use the following characterization of the existence of $T$-joins that can be found as Proposition 12.7 in [5].

**Proposition 2.** Let $G = (V,E)$ be a graph and $T \subseteq V$. Then $G$ contains a $T$-join if and only if every connected component of $G$ contains an even number of elements of $T$.

The following theorem is due to Menger [6] and is a fundamental characterization of $k$-edge-connected graphs.

**Theorem 12.** A graph $G = (V,E)$ is $k$-edge-connected if and only if $\lambda_G(u,v) \geq k$ for all $u,v \in V$.

Further, we mention an intensively studied conjecture which was proposed independently by Berge and Fulkerson [12].

**Conjecture 1.** Every cubic $2$-edge-connected graph has 6 perfect matchings covering each edge exactly twice.

We also consider the following algorithmic problem which is well-known in the literature:

**Monotone Not-all-equal-3SAT (MNAE3SAT)**

**Instance:** A set $X$ of boolean variables, a formula consisting of a set $C$ of clauses each containing 3 distinct variables none of which are negated.

**Question:** Is there a truth assignment to the variables of $X$ such that every clause in $C$ contains at least one true and one false literal?

This problem will be used in the reduction in Section 6 which is justified by the following result due to Schaefer [11].

**Theorem 13.** MNAE3SAT is NP-complete.
2.2. Preparatory results

The following two results show that connectivity properties are maintained when contracting or blowing up sufficiently connected subgraphs. As they are of basic nature, they are given without proof.

**Proposition 3.** Let $G$ be a graph.

(a) If $G$ is $k$-edge-connected, then so is any contraction of $G$.

(b) If $G$ is essentially $k$-edge-connected, then so is any contraction of $G$.

**Proposition 4.** For a subgraph $Q$ of a directed graph $D$,

(a) if $D$ is strongly connected, then so is $D/Q$.

(b) if $D/Q$ and $Q$ are strongly connected, then so is $D$.

The following observation concerns Eulerian orientations and follows easily from Proposition 5 of [4]. For the sake of completeness we provide an easy proof for it.

**Proposition 5.** Let $G = (V,E)$ be an Eulerian graph and $\{e_v, f_v\}$ two edges incident to $v$ for all $v \in V' \subseteq V$. Then there is an Eulerian orientation of $G$ such that exactly one of $e_v$ and $f_v$ enters $v$ for all $v \in V'$.

**Proof** Let $G'$ be the graph obtained from $G$ by detaching each vertex $v \in V'$ into two vertices $u_v$ and $w_v$ such that in $G'$, $u_v$ is incident to $\{e_v, f_v\}$ and $w_v$ to $\delta_G(\{v\}) - \{e_v, f_v\}$. As $G$ is Eulerian, so is $G'$. Hence there exists an Eulerian orientation $D'$ of $G'$. By identifying $u_v$ and $w_v$ in $D'$ for all $v \in V'$, we obtain the required orientation. ■

We will next introduce a little more notation related to the Frank number. Given a directed graph $D$, we call a set $F \subseteq A(D)$ deletable if $D - f$ is strongly connected for all $f \in F$. Given a graph $G$, we call a set $F \subseteq E(G)$ deletable if there exists an orientation $\vec{G}$ of $G$ such that $\vec{F}$ is deletable in $\vec{G}$.

The following result is a direct consequence of the definition of strongly connected directed graphs.

**Proposition 6.** Given a directed graph $D = (V,A)$, a set $F \subseteq A$ is deletable if and only if $\delta_D(X)$ contains either at least one arc of $A - F$ or at least two arcs for every nonempty, proper $X \subset V$.

Consequently, given a graph $G$, a subset $F \subseteq E(G)$ is deletable if and only if there is an orientation satisfying the above properties.

Finally, we show one more result about strongly connected orientations of 3-edge-connected graphs which we need in the proof of Theorem [5].

**Lemma 1.** Let $D$ be a strongly connected orientation of a 3-edge-connected graph $G$ and $C$ a circuit of $D$. Then $C$ contains an arc $a$ such that $D - a$ is strongly connected.

**Proof** Let $(G, D, C)$ be a counterexample that minimizes the number of vertices of $D$. Let $e$ be an edge of $G$ that is incident to a vertex of $C$ and that does not belong to $C$. By the 3-edge-connectivity of $G$, $e$ exists. Since $D$ is strongly
connected, \( \vec{e} \) belongs to a directed path \( P \) whose end-vertices belong to \( C \) but whose internal vertices do not. Then \( P \) can be extended by a possibly trivial directed subpath of \( C \) to form a circuit \( C^* \). Let \( (G', D', C') \) be obtained from \((G, D, C)\) by contracting \( C^* \). Then, by Propositions \( \text{II(a)} \) and \( \text{II(a)} \), the assumptions of the lemma are satisfied for \((G', D', C')\). By the minimality of \((G, D, C)\), \( C' \) contains an arc \( a' \) such that \( D' - a' \) is strongly connected. Let \( a \) be the arc of \( C \) in \( D \) that corresponds to \( a' \). Since \( D' - a' \) and \( C^* \) are strongly connected, by Proposition \( \text{II(b)} \), so is \( D - a \). ■

2.3. Cubic extensions

We introduce for any graph \( G = (V, E) \) of minimum degree at least 3 an auxiliary graph \( H_G \) that is cubic. For each vertex \( v \in V \) of degree at least 4, \( H_G \) contains a set \( S_v \) of \( d_G\{\{v\}\} \) vertices, for each vertex of degree 3, let \( S_v = \{v\} \). Next, we add on \( S_v \) a cycle \( C_v \) for each \( v \in V \) of degree at least 4.

Finally, for each edge \( uv \in E \), we add an edge between \( S_u \) and \( S_v \) to \( H_G \). We do this in a way so that \( H_G \) becomes cubic. We call \( H_G \) a cubic extension of \( G \).

**Proposition 7.** Let \( G = (V, E) \) be a graph of minimum degree at least 3 and \( H_G \) be a cubic extension of \( G \).

(a) If \( G \) is 3-edge-connected and \( G - v \) is connected for all \( v \in V \), then \( H_G \) is 3-edge-connected.

(b) If \( G \) is essentially 4-edge-connected and \( G - v \) is 2-edge-connected for all \( v \in V \), then \( H_G \) is essentially 4-edge-connected.

**Proof** (a) Assume for a contradiction that \( d_{H_G}(X) \leq 2 \) for some nonempty, proper \( X \subset V(H_G) \). Since \( G \) is 3-edge-connected, there is at least one \( v \in V \) such that \( S_v \cap X \) and \( S_v - X \) are nonempty. It follows that \( 2 \leq d_{C_v}(X) \leq d_{H_G}(X) \leq 2 \).

This yields that for every \( u \in V - v \) we have \( S_u \subseteq X \) or \( S_u \subseteq V(H_G) - X \) and for all \( uw \in E \) with \( v \notin \{u, w\} \) we have \( S_u \cup S_w \subseteq X \) or \( S_u \cup S_w \subseteq V(H_G) - X \).

If there are vertices \( u, w \in V \) such that \( S_u \subseteq X \) and \( S_w \subseteq V(H_G) - X \), it follows that \( G - v \) is not connected, contradicting the assumption. Therefore, by symmetry we may assume that \( X \subseteq S_v \) for some \( v \in V \). As \( X = S_v \), there are at least 3 edges leaving \( X \) because \( v \) is of degree at least 3 in \( G \), a contradiction.

If \( X \) is a nonempty, proper subset of \( S_v \), we have \( d_{C_v}(X) \geq 2 \) and there is at least one edge between \( X \) and \( V(H_G) - S_v \), a contradiction.

(b) By (a), \( H_G \) is 3-edge-connected. For the sake of a contradiction, suppose that there is some non-trivial \( X \subset V(H_G) \) such that \( d_{H_G}(X) = 3 \). If there are two vertices \( u, v \in V \) such that \( S_u \cap X, S_u - X, S_v \cap X \) and \( S_v - X \) are nonempty, we have \( 2 + 2 \leq d_{C_u}(X) + d_{C_v}(X) \leq d_{H_G}(X) \leq 3 \), a contradiction.

Now consider the case that there is exactly one \( v \in V \) such that \( S_v \cap X \) and \( S_v - X \) are nonempty. We have that \( d_{H_G}(X) - d_{C_v}(X) \leq 1 \). It follows that in \( H_G \) there is at most one edge between \( X - S_v \) and \( V(H_G) - X - S_v \). If \( X - S_v \) and \( V(H_G) - X - S_v \) are nonempty, then \( G - v \) is not 2-edge-connected, a contradiction to the assumption. By symmetry, we may therefore assume that \( X \subseteq S_v \). We have that \( d_{C_v}(X) = 2 \) and there are \( |X| \) edges between \( X \) and \( V(H_G) - S_v \). It follows that \( |X| = 1 \) which is a contradiction.

Finally assume that \( S_v \subseteq X \) or \( S_v \cap X = \emptyset \) for all \( v \in V \). Let \( X' = \{v \in V : S_v \subseteq X\} \). As \( G \) is essentially 4-edge-connected, we may assume by symmetry that \( X' = \{v\} \) for some vertex \( v \) of degree 3. This yields that \( |X| = |S_v| = 1 \) which is a contradiction. ■
3. 3-edge-connected graphs

This section is dedicated to proving Theorems 2, 3 and 4. In the first part, we show that a certain class of edge sets is deletable. After, we show how to cover cubic 3-edge-connected graphs with such sets. Next, we use this to conclude cubic versions of Theorems 2 and 3 and to prove Theorem 4. Finally, we extend this to obtain the general versions of Theorems 2 and 3.

3.1. A class of deletable edge sets

Given a packing \( C \) of cycles in a 3-edge-connected graph \( G \), the special set of \( C \) is defined to be the set of edges in \( E(G) - E(C) \) that belong to no 3-edge-cut of \( G/C \).

Lemma 2. Let \( M \) be the special set of a cycle packing \( C \) of a 3-edge-connected graph \( G \). Then \( M \) is deletable.

**Proof** Let \( G' = G/C \). Since \( G \) is 3-edge-connected, so is \( G' \) by Proposition 3(a). Consider a well-balanced orientation \( D' \) of \( G' \) which exists by Theorem 9. Then \( D' \) is strongly connected. Let \( D \) be the orientation of \( G \) obtained from \( D' \) by orienting all cycles of \( C \) as a circuit.

We have to show that \( D - f \) is strongly connected for all \( f \in M \). By Proposition 1(b), it is enough to show that \( D' - f \) is strongly connected for all \( f \in M \). Let \( f = uv \) for some \( f \in M \) and suppose that there exists some non-empty, proper \( X \subset V(D') \) with \( |\delta_{D'-f}(X)| = 0 \). Obviously \( u \in X \) and \( v \in V(D') - X \). Since \( G' \) is 3-edge-connected and \( f \) belongs to no 3-edge-cut in \( G' \), Theorem 12 guarantees that \( \lambda_{G'}(u,v) \geq 4 \). As \( D' \) is well-balanced, it follows that \( 0 = |\delta_{D'-f}^+(X)| = |\delta_{D'}^+(X)| - 1 \geq \lfloor \frac{\lambda_{G'}(u,v)}{2} \rfloor - 1 \geq 2 - 1 = 1 \), a contradiction. \( \blacksquare \)

3.2. Covering cubic graphs with special sets

In the following we show that any cubic 3-edge-connected graph can be covered by 7 special sets. For technical reasons, we will need the following slight strengthening.

Lemma 3. For every cubic 3-edge-connected graph, there exist 7 cycle packings satisfying the following conditions:

(a) Every edge is in the special set of at least one cycle packing.

(b) Every edge is in exactly 4 of the cycle packings.

**Proof** For the sake of a contradiction, let \( G \) be a counterexample to the lemma that minimizes \( |V(G)| \).

Claim 1. \( G \) is essentially 4-edge-connected.

**Proof** For the sake of a contradiction, let \( \{A_1, A_2\} \) be a partition of \( V(G) \) such that \( |A_i| \geq 2 \) and a 3-edge-cut \( F := \{e_1, e_2, e_3\} \) exists between \( A_1 \) and \( A_2 \). Construct the graphs \( G_i \) from \( G \) by contracting \( A_{3-i} \) to \( v_i \). As \( G_i \) is cubic, 3-edge-connected by Proposition 5(a) and smaller than \( G \), there exists a set of cycle packings \( C_i = \{C_{1i}, \ldots, C_{7i}\} \) of \( G_i \) satisfying (a) and (b).
Observe that since \( G \) is cubic, \((b)\) implies that for \( j \in \{1, 2, 3\} \), there are exactly two cycle packings in \( \mathcal{C} \) that contain \( \{e_1, e_2, e_3\} - \{e_j\} \). It follows that \( e_i \) is in exactly 6 cycle packings of \( \mathcal{C}' \). By relabeling if needed, we may assume that \( \mathcal{C}_1 \) is the cycle packing that does not contain \( e_i \) and \( \{e_1, e_2, e_3\} - \{e_j\} \) is contained in \( \mathcal{C}'_{2j} \) and \( \mathcal{C}'_{2j+1} \). We may also assume, by \((a)\), that \( e_j \) is in the special set of \( \mathcal{C}'_{2j} \).

We construct \( \mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_7\} \) so that \( E(\mathcal{C}_k) = E(\mathcal{C}'_k) \cup E(\mathcal{C}'_{k+1}) \) for \( k = 1, \ldots, 7 \). Observe that this is a set of seven cycle packings. We finish the proof by showing that \( \mathcal{C} \) satisfies \((a)\) and \((b)\).

First observe that \((b)\) follows directly from the construction and the fact that an edge is in \( \mathcal{C}_k \) if and only if it is in \( \mathcal{C}'_k \) or \( \mathcal{C}'_{k+1} \). For \((a)\), let first \( e \) be an edge in \( G[A_i] \). By \((a)\), there exists a \( k \in \{1, \ldots, 7\} \) such that \( e \) is in the special set of \( \mathcal{C}_k \). First observe that \( e \) is in \( E(G) - E(\mathcal{C}_k) \). If \( e \) is in a 3-edge-cut \( F' \) of \( G(\mathcal{C}_k) \), since \( F' \) is not a 3-edge-cut of \( G/\mathcal{C}_k \), \( F' \) contains an edge of \( G[A_{k-1}] \). This yields that \( F \) and \( F' \) are crossing 3-edge-cuts of \( G \), a contradiction to Proposition \( \text{11} \).

Now consider the edge \( e_j \) for some \( j \in \{1, 2, 3\} \). As \( e_j \in E(G_i) - E(\mathcal{C}'_{2j}) \), we have \( e_j \in E(G) - E(\mathcal{C}_{2j}) \). Again, assume that \( e_j \) is in a 3-edge-cut \( F' \) of \( G(\mathcal{C}_{2j}) \). As \( F' \) is not a 3-edge-cut in \( G/\mathcal{C}_1 \) and \( G/\mathcal{C}_{2j} \), \( F' \) and \( F \) are crossing in \( G \) contradicting Proposition \( \text{11} \). This finishes the proof of the claim. ■

By Theorem \( \text{11} \) \( G \) contains a perfect matching \( M \). Since \( G \) is cubic, the connected components of \( G - M \) form a cycle packing \( \mathcal{C}_1 \). Now consider the graph \( G' := G/\mathcal{C}_1 \) (including arising loops) and let \( T \) be its set of odd-degree vertices.

**Claim 2.** The edge set of \( G' \) can be partitioned into three \( T \)-joins \( F_1, F_2 \) and \( F_3 \).

**Proof** As \( G \) is essentially 4-edge-connected by Claim \( \text{11} \) \( G' \) is 4-edge-connected by Proposition \( \text{11} \). By Theorem \( \text{11} \) there exist two edge-disjoint spanning trees \( F'_1, F'_2 \) of \( G' \). By Proposition \( \text{11} \), each of them contains a \( T \)-join \( F_i, i = 1, 2 \). As \( F_1 \cup F_2 \) is Eulerian, \( F_3 = E(G') - F_1 - F_2 \) is also a \( T \)-join. ■

**Claim 3.** For \( i = 1, 2, 3 \), there exist \( V \)-joins \( S_{2i} \) and \( S_{2i+1} \) of \( G \) such that \( S_{2i} \cap S_{2i+1} = F_i \) and \( S_{2i} \cup S_{2i+1} = (E - M) \cup F_i \).

**Proof** For \( i = 1, 2, 3 \), let \( T_i \) be the set of vertices in \( V \) not incident to an edge in \( F_i \). Since \( G \) is cubic, \( F_i \subseteq M \) is a matching and \( d_G(V(C)) \) and \( d_{F_i}(V(C)) \) are of the same parity, \( |T_i \cup V(C)| \) is even for every \( C \in \mathcal{C}_1 \) and hence, by Proposition \( \text{11} \) \( G - M \) contains a \( T_i \)-join \( N_i \). Let \( S_{2i} := F_i \cup N_i \) and \( S_{2i+1} := F_i \cup (E - M - N_i) \). By construction, \( S_{2i} \) and \( S_{2i+1} \) are \( V \)-joins in \( G \) such that \( S_{2i} \cap S_{2i+1} = F_i \) and \( S_{2i} \cup S_{2i+1} = (E - M) \cup F_i \). ■

For \( j = 2, \ldots, 7 \), we define \( \mathcal{C}_j \) to be the set of nontrivial connected components of \( G - S_j \). Observe that all of them are cycles as \( S_j \) is a \( V \)-join and \( G \) is cubic.

**Claim 4.** \( \mathcal{C}_1, \ldots, \mathcal{C}_7 \) satisfy \((a)\) and \((b)\).

**Proof** (a) For \( e \in M \), since \( G \) is essentially 4-edge-connected, \( e \) is in the special set of \( \mathcal{C}_1 \).
For $e \in E - M$, let $f$ and $g$ be the two edges of $M$ adjacent to $e$. Since $F_1, F_2$ and $F_3$ are disjoint, there is an $F_i$ that contains neither $f$ nor $g$. Then, since $G$ is cubic and by Claim 3, one of the $V$-joins $S_{2i}$ and $S_{2i+1}$, say $S_i$, contains $e$ but none of the edges adjacent to $e$. It follows that both endvertices of $e$ in $G$ are in cycles of $C_j$. As $G$ is essentially 4-connected, it follows that both endvertices of $e$ in $G/C_j$ are of degree at least 4. As $G$ is essentially 4-connected, so is $G/C_j$ by Proposition 2(b). This yields that $e$ is in no 3-edge-cut of $G/C_j$ and so $e$ is in the special set of $C_j$.

(b) For $e \in M$, by Claim 2, $e$ is in exactly one $F_i$, say $F_1$. Then, by Claim 3, $e$ is in $C_4, \ldots, C_7$ and not in $C_1, C_2, C_3$.

For $e \in E - M$, $e$ is in $C_1$ and, by Claim 3, in exactly one of $C_{2i}$ and $C_{2i+1}$ for $i = 1, 2, 3$. ■

Claim 4 finishes the proof of Lemma 3. ■

3.3. Cubic case

We first show how to conclude a cubic version of Theorem 2.

**Theorem 14.** Let $G$ be a cubic 3-edge-connected graph. Then $f(G) \leq 7$.

**Proof** By Lemma 3, $E(G)$ can be covered by 7 special sets $S_1, \ldots, S_7$. By Lemma 2, there exist orientations $D_1, \ldots, D_7$ of $G$ such that $S_i$ is deletable in $D_i$ for $i = 1, \ldots, 7$. It follows that the Frank number of $G$ is at most 7. ■

Next, we use Lemma 2 to show that perfect matchings with a certain additional property are deletable. As corollaries, we obtain Theorem 4 and a cubic version of Theorem 3.

**Lemma 4.** Let $M$ be a perfect matching of a cubic 3-edge-connected graph $G$ intersecting every 3-edge-cut of $G$ in exactly one edge. Then $M$ is deletable.

**Proof** As $G$ is cubic and $M$ is a perfect matching of $G$, the connected components of $G - M$ form a packing $C$ of cycles. We show that $G/C$ is 4-edge-connected. By Proposition 3(a) and since $G$ is 3-edge-connected, so is $G/C$. A 3-edge-cut of $G/C$ would provide a 3-edge-cut of $G$ intersecting $M$ in 3 edges contradicting the assumption. It follows that $M$ is the special set of $C$ and therefore deletable by Lemma 2. ■

We first show how to conclude Theorem 4 from Lemma 4.

**Proof** (of Theorem 4) Let $G$ be a 3-edge-colorable 3-edge-connected graph. Then $G$ is cubic and has 3 disjoint perfect matchings $M_1, M_2, M_3$ covering the edge set of $G$. Let $\delta(X)$ be a 3-edge-cut of $G$. Since $G$ is cubic and $d(X) = 3$, we obtain that $|X|$ is odd. Then, since $M_i$ is a perfect matching, $\delta(X)$ intersects each $M_i$. As $d(X) = 3$ and the matchings are disjoint, $\delta(X)$ intersects each of $M_1, M_2, M_3$ exactly once. It follows by Lemma 2 that each of $M_1, M_2, M_3$ is deletable, so $f(G) \leq 3$. ■

Next, we prove in a similar way the following cubic version of Theorem 3.

**Theorem 15.** Let $G$ be a cubic 3-edge-connected graph that satisfies Conjecture 1. Then $f(G) \leq 5$.
Proof By assumption, there exist 6 perfect matchings $M_1, \ldots, M_6$ of $G$ covering each edge of $G$ exactly twice.

Let $\delta(X)$ be a 3-edge-cut of $G$. Since $G$ is cubic and $d(X) = 3$, we obtain that $|X|$ is odd. Then, since $M_i$ is a perfect matching, $\delta(X)$ intersects each $M_i$. Since each of the 3 edges of $\delta(X)$ belongs to exactly 2 $M_i$'s, $\delta(X)$ intersects each of $M_1, \ldots, M_6$ exactly once. It follows by Lemma 4 that each of $M_1, \ldots, M_6$ is deletable. As every edge of $G$ is covered by at least one of $M_1, \ldots, M_5$, it follows that $f(G) \leq 5$.

3.4. Non-cubic case

We first show how to prove the general case of Theorem 2.

Proof (of Theorem 2) Let $G$ be a counterexample minimizing $|V(G)|$.

Claim 5. $G$ is 2-vertex-connected.

Proof For the sake of a contradiction, assume that $G$ has a cut vertex $v$. So $G$ has two non-trivial subgraphs $G_1$ and $G_2$ such that $G_1 = G/G_2$ and $G_2 = G/G_1$. As $G$ is 3-edge-connected, so is $G_i$ by Proposition 3(a). Since $G_i$ is smaller than $G$, $G_i$ has Frank number at most 7. So there exist 7 orientations $D_i$ of $G_i$ such that for each edge $e$ of $G_i$, one of $D_i - \vec{e}$ is strongly connected. We can now construct the 7 orientations $D_j$ of $G$ by giving each edge in $G_i$ its orientation in $D_j$ also in $D_j$. Now consider an edge $e$ of $G_i$ and let $D_j - \vec{e}$ be strongly connected. Since $D_j - \vec{e} = (D_j - \vec{e})/D_j^{3-i}$ and $D_j^{3-i}$ are strongly connected, Proposition 4(b) implies that so is $D_j - \vec{e}$. It follows that $G$ has Frank number at most 7, a contradiction.

Let $H_G$ be a cubic extension of $G$ as defined in Section 2.3. By Claim 5 and Proposition 7(a), $H_G$ is 3-edge-connected. Then, by Theorem 14, the Frank number of $H_G$ is at most 7, that is there exist 7 orientations $D'_i$ of $H_G$ such that for each edge $e$ of $H_G$, one of $D'_i - \vec{e}$ is strongly connected. Let $D_i$ be the orientation of $G$ obtained from $D'_i$ by contracting the subgraphs $C_v$ for all $v \in V$. For any $e \in E(G) \subset E(H_G)$, one of $D_i - \vec{e}$ is strongly connected, therefore, by Proposition 4(a), so is $D_i - \vec{e}$. It follows that the Frank number of $G$ is at most 7, a contradiction.

The same reduction and Theorem 15 show Theorem 3.

4. Essentially 4-edge-connected graphs

This section is dedicated to proving Theorem 5. Again, first we prove the result for cubic graphs and then we show how it implies the non-cubic case.

4.1. Cubic case

In the case of essentially 4-edge-connected graphs Lemma 2 can be generalized. More precisely, every matching of $G$ is deletable. We prove the following slightly stronger statement.
Lemma 5. Let $G$ be an essentially 4-edge-connected graph, $M$ a matching of $G$ and $\mathcal{C}$ a cycle packing of $G - M$. Then there exists an orientation of $G$ in which $\overline{M}$ is deletable and each cycle of $\mathcal{C}$ is oriented as a circuit.

Proof Let $\mathcal{F}$ be the set of maximal 2-edge-connected subgraphs of $G - M$. Let $G' = (V', E' \cup M)$ be the graph obtained from $G$ by contracting each graph of $\mathcal{F}$. Note that $G' - M$ is a forest. Since $G$ is essentially 4-edge-connected, by Proposition 5(b), so is $G'$ and every vertex of degree 3 in $G'$ is an original vertex of $G$. Then, since $M$ is a matching of $G$, every vertex $v$ of degree 3 in $G'$ is incident to at least 2 edges $e_1^v, e_2^v$ in $E'$.

By Theorem 8 and Lemma 5, there exists a set of maximal 2-edge-connected subgraphs of $G - M$ such that each cycle of $\mathcal{C}$ contained in $F$ is oriented as a circuit.

For all $F \in \mathcal{F}$, by Proposition 5(a), Theorem 8 and Proposition 5(b), there exists a strongly connected orientation $\tilde{G}$ of $F$ such that each cycle of $\mathcal{C}$ contained in $F$ is oriented as a circuit in $\tilde{G}$.

We will finish the proof by showing that $\overline{G} - \overline{e}$ is strongly connected for all $e \in M$. Since $\overline{F}$ is strongly connected for all $F \in \mathcal{F}$ and $\bigcup_{F \in \mathcal{F}} E(F)$ contains no edge in $M$, it suffices to prove, by Proposition 4(b), that $\overline{G} - \overline{e}$ is strongly connected for all $e \in M$. Let $X$ be a subset of $V'$. By Proposition 4(b), it is enough to prove that either at least two arcs or at least one arc of $\overline{E'}$ leave $X$.

If there are $x \in X$ and $y \in V' - X$ of degree at least 4, then, since $G'$ is essentially 4-edge-connected, there is no 3-edge-cut separating $x$ and $y$ in $G'$ and therefore, as $\overline{\mathcal{F}}$ is well-balanced, there are 2 arcs leaving $X$, and we are done.

Hence, by considering $V' - X$ and $\overline{G} - \overline{e}$ if necessary, we may assume without loss of generality that $X$ only contains vertices of degree 3 and there is no arc of $\overline{E'}$ leaving $X$. By construction, every vertex $v$ of $X$ has at least one arc $e_1^v$ or $e_2^v$ of $\overline{E'}$ leaving $v$. As there is no arc of $\overline{E'}$ leaving $X$, $\overline{G}[X]$ contains a circuit $C$ of arcs in $\overline{E'}$. This cycle $C$ provides a contradiction since $G' - M$ is a forest.

We are now ready to prove a cubic version of Theorem 5.

Theorem 16. Let $G$ be a cubic essentially 4-edge-connected graph. Then $f(G) \leq 3$.

Proof Since $G$ is cubic and 2-edge-connected, by Theorem 11, $G$ has a perfect matching $M_1$ and the connected components of $G - M_1$ form a packing $\mathcal{C}$ of cycles. By Lemma 5, there exists an orientation $D_1$ of $G$ such that each cycle of $\mathcal{C}$ is oriented as a circuit and $M_1$ is deletable in $D_1$. By Lemma 1, each $C_i \in \mathcal{C}$ contains a deletable arc $e_i$ in $D_1$. Note that the connected components of $G - M_1 - \cup\{e_i : C_i \in \mathcal{C}\}$ form a packing of paths which is the union of two matchings $M_2$ and $M_3$. By Lemma 5, there exist orientations $D_2$ and $D_3$ of $G$ such that $M_2$ is deletable in $D_2$ and $M_3$ is deletable in $D_3$. Since $E(G) = M_1 \cup M_2 \cup M_3 \cup \cup\{e_i : C_i \in \mathcal{C}\}$, Theorem 16 follows.
4.2. Non-cubic case

We now generalize the results of the previous part to arbitrary essentially 4-edge-connected graphs.

Proof (of Theorem 5). Let $G = (V, E)$ be a counterexample minimizing $|V|$. 

Claim 6. $G - v$ is 2-edge-connected for all $v \in V$.

Proof For the sake of a contradiction, assume that $G - v$ is not 2-edge-connected for some $v \in V$. If $G - v$ is disconnected, we obtain a contradiction using the same argument as in the proof of Claim 5. We therefore have a partition $A_1 \cup A_2$ of $V - \{v\}$ such that $A_1$ and $A_2$ are only connected by a single edge $e_0$ in $G - v$. Let us denote the end-vertices of $e_0$ by $u_i \in A_i$. Consider the graph $G_i$ that arises from $G$ by contracting $A_{3-i} \cup \{v\}$ into a vertex $v_i$. Note that $E(G_1) \cap E(G_2) = \{e_0\}$. Since $G$ is essentially 4-edge-connected, so is $G_i$. Moreover, $G_i$ is smaller than $G$. It follows that there exist 3 orientations $D_j^i$ of $G_i$ such that one of $D_j^i - \vec{e}$ is strongly connected for all $e \in E(G_i)$. We may suppose that $D_1^1 - \vec{e}_0$ and $D_2^2 - \vec{e}_0$ are strongly connected. Reversing the arcs in $D_j^i$ if needed, we may assume that $e_0$ has the same orientation in $D_1^1$ and $D_2^2$. We can construct the 3 orientations $D_j$ of $G$ by merging $D_1^1$ and $D_2^2$. We will finish the proof by showing that for all $e \in E$, there exists a $j$ such that $D_j - \vec{e}$ is strongly connected. Let $e \in E$ and $j \in \{1, 2, 3\}$ such that both $D_j^1 - \vec{e}$ and $D_j^2 - \vec{e}$ are strongly connected. Assume that there is a set $X \subset V$ that has no arc leaving in $D_j - \vec{e}$. Without loss of generality, we may assume that $v \in X$. As $(X \cap A_j) \cup \{v_i\}$ has an arc leaving in $D_j^i - \vec{e}_0$, $e_0$ must be directed away from $v_i$ in $D_j^i$ for $i = 1, 2$. This is a contradiction as $D_j^1$ and $D_j^2$ were chosen to both have the same orientation of $e_0$. ■

Let $H_G$ be a cubic extension of $G$ as defined in Section 2.3. By Claim 5 and Proposition 7(b), $H_G$ is a cubic essentially 4-edge-connected graph. Then, by Theorem 16, the Frank number of $H_G$ is at most 3. There exist therefore 3 orientations $D_j^i$ of $H_G$ such that for each edge $e \in E(H_G)$, there is some $j \in \{1, 2, 3\}$ such that $D_j^i - \vec{e}$ is strongly connected. Consider now the 3 orientations $D_j$ of $G$ which arise from $D_j^i$ by contracting the subgraphs $C_e$ for all $e \in V$. By Proposition 4(a), if $D_j^i - \vec{e}$ is strongly connected for an edge $e \in E$, so is $D_j - \vec{e}$. It follows that the Frank number of $G$ is at least 3, a contradiction. ■

5. The Petersen graph

In this section, we show that there are graphs of Frank number higher than two, more precisely we prove Theorem 6.

Proof (of Theorem 6). Let $G = (V, E)$ be the Petersen graph, see Figure 1. We frequently make use of the symmetry properties of $G$. By Theorem 5 and since $G$ is essentially 4-edge-connected, but not 4-edge-connected, it suffices to prove that its Frank number is different from 2. Suppose that $G$ has Frank number 2 and let $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ be two orientations of $G$ such that $D_1 - \vec{e}$ or $D_2 - \vec{e}$ is strongly connected for each edge $e$ of $G$. (•)
We say that an arc of $D_1$ is stable if the same arc exists in $D_2$, otherwise it is changing. Let $S$ and $C$ be the set of stable and changing arcs, respectively. Note that $D_1$ and $D_2$ also satisfy and stable and changing arcs are exchanged. Hence, whatever is proved for stable arcs is also true for changing arcs.

We first show that $S$ and $C$ induce a 2-edge-coloring of $G$ with certain properties and then that no such 2-edge-coloring exists. Observe that none of the considered colorings are required to be proper. For a 2-edge-coloring $R, B$ of $G$, we define an auxiliary graph $H^{R,B} := (V, F)$ where $uv \in F$ if there exists a 3-path $tuvw$ in $G(R)$ or in $G(B)$ or there exists a $(u, v)$-path that is a connected component of $G(R)$ or of $G(B)$.

**Lemma 6.** $G$ has a 2-edge-coloring $R, B$ such that

1. no monochromatic 3-star exists, \hspace{1cm} (1)
2. $H^{R,B}$ is bipartite. \hspace{1cm} (2)

**Proof** We show that the 2-edge-coloring induced by $S$ and $C$ satisfies \[1\] and \[2\]. To show \[1\] we need the following claim.

**Claim 7.** Each vertex is incident to at least one stable arc.

**Proof** Suppose that a vertex $v$ is incident only to changing arcs. Since $G$ is cubic and $D_1$ is strongly connected, either the in-degree or the out-degree of $v$ is 1, say $\vec{e}$ is the only arc entering $v$. Then $\vec{e}$ is the only arc leaving $v$ in $D_2$. Then, $D_1 - \vec{e}$ and $D_2 - \vec{e}$ are not strongly connected which is a contradiction.

To show \[2\] we need the following claims.

**Claim 8.** The weakly connected components of $D_1(S)$ are directed paths or circuits.

**Proof** By Claim \[7\] applied for stable arcs and then for changing arcs, the connected components of $D_1(S)$ are paths or cycles. If two stable arcs are incident to a vertex $v$ then one of them enters and the other one leaves $v$. Otherwise, let $e$ be the third arc incident to $v$. Then $\vec{e}$ is the only arc leaving $v$ in $D_2$. Then, $D_1 - \vec{e}$ and $D_2 - \vec{e}$ are not strongly connected which is a contradiction. Now the claim follows.

**Claim 9.** Let $P$ be a weakly connected component of $D_1(S)$ that is a directed $(u, v)$-path. Then the in-degrees of $u$ and $v$ in $D_1$ are of different parity.

**Proof** Since $G$ is cubic and $u$ and $v$ are incident to exactly one stable arc in $D_1$, $u$ and $v$ are incident to exactly two changing arcs in $D_1$. Then, by Claim \[8\] applied for $D_1(C)$, exactly one changing arc enters both $u$ and $v$ in $D_1$. Since $P$ is a directed path between $u$ and $v$, the claim follows.

**Claim 10.** Let $tuvw$ be a 3-path in $D_1(S)$. Then the in-degrees of $u$ and $v$ are of different parity in $D_1$.

**Proof** By Claim \[8\] exactly one stable arc enters both $u$ and $v$ in $D_1$. By Claim \[7\] the two other arcs incident to $u$ and $v$ are changing. If both are entering or leaving then $D_1 - uv$ and $D_2 - uv$ are not strongly connected which is a contradiction. Now the claim follows.
Claim 11. $H^{S,C}$ is a bipartite graph.

Proof Since $G$ is cubic and $D_1$ and $D_2$ are strongly connected, each vertex is of in-degree 1 or 2. ByClaims 9 and 10, each edge of $H^{S,C}$ is between a vertex of in-degree 1 and a vertex of in-degree 2, so $H^{S,C}$ is bipartite. ■

By Claim 7 applied for $R := S$ and $B := C$ and by Claim 11, Lemma 6 follows.

We show that $G$ does not admit any 2-edge-coloring satisfying (1) and (2) and obtain a contradiction to Lemma 6.

The following result yields a strong property such a coloring would have to satisfy.

Lemma 7. Let $R, B$ be a 2-edge-coloring satisfying (1) and (2). Then $G$ has a 5-cycle that contains a monochromatic 4-path whose end-vertices are incident to 2 edges of the other color.

Proof We first show two weaker statements which are useful in the proof later on.

Claim 12. $G$ has a 5-cycle that contains a monochromatic 3-path.

Proof Suppose not. Since $G$ is cubic, there are two adjacent edges of the same color, without loss of generality $ab, ae \in R$. Then, by the assumption for $abcde$ and $abihe$, $bc, bi, ch \in B$. Thus, by the assumption for $bcdji$, $hi, bi \in B$. Thus $cbihg$ forms a monochromatic 4-path in the 5-cycle $cbihg$, contradicting the assumption. See Figure 2a. ■

This result is helpful in proving a strengthening of itself.

Claim 13. $G$ has a 5-cycle that contains a monochromatic 4-path.

Proof Suppose not. By Claim 12 without loss of generality $bc, cd, de \in B$. Then, by the assumption for $ab, ac, af \in R$ and $cg, dj \in R$. By (1) for $f$, one of $fg$ and $fj$ is in $R$. By symmetry, without loss of generality $fg \in R$. Then, by (1) for $g, gh \in B$. So, by the assumption for $cdehg$, $eh \in R$. Then, by the assumption for $abihe$, $hi, bi \in B$. Thus $cbihg$ forms a monochromatic 4-path in the 5-cycle $cbihg$, that contradicts the assumption. See Figure 2c. ■

By Claim 13 without loss of generality $ab, bc, de, ea \in R$. Then, by (2) for $ab, ae \in R$. Then, by the assumption for $deabc$ and by $ehib$, $H^{R,B}$ contains the 3-cycle $abe$ that contradicts (2). See Figure 2c. Hence, $hi \in R$. If $cg, dj \in R$, then, by the 3-paths of $jdeabc$ and by $cd$, $H^{R,B}$ contains the 5-cycle $abe$ that contradicts (2). See Figure 2d. Hence, by symmetry, without loss of generality $dj \in B$.

Now suppose for the sake of a contradiction that $G$ does not contain a 5-cycle that contains a monochromatic 4-path whose end-vertices are incident to 2 edges of the other color. If $cg \in B$, then $ab, ac, af \in R$. Then, by the assumption for $e, deabc$ and by $ehib$, $H^{R,B}$ contains the 3-cycle $abe$ that contradicts (2). See Figure 2c. Hence, $hi \in R$. If $ij \in R$, then, by the 3-paths of $abcghi$ and by $bi$, $H^{R,B}$ contains the 5-cycle $beghi$ that contradicts (2). See Figure 2e. Hence
Figure 2
hg ∈ B. If fg ∈ B, then afghde contradicts the assumption. See Figure 2|
Hence fg ∈ R. Then, by the 3-paths of deabcfg and by chg, H_{R,B} contains
the 5-cycle abcgef that contradicts 2. See Figure 2}. This finishes the proof of
Lemma 7. ■

By Lemma 7, G has a 5-cycle, without loss of generality abcede, that contains
a monochromatic 4-path whose end-vertices are incident to 2 edges of the other
color. By similar arguments as before, we obtain the partial coloring of Figure
2c By 1 for f, one of fg and fj is in R. By symmetry, without loss of
generality fj ∈ R.

Suppose that fg ∈ B. Then, by 1 for g, gh ∈ R. If ij ∈ R, then, by the
3-paths of deabc and for ghij, and by eh and ib, H_{R,B} contains the 5-cycle cabih
contradicting 2. See Figure 2 If ij ∈ B, then, by the 3-paths of afgcdji and by
fj, H_{R,B} contains the 5-cycle fgcdj contradicting 2. See Figure 2f

Hence fg ∈ R. Then, by 2 for fghij, one of hg and ij is in B. By symmetry, without loss of generality ij ∈ B. If hg ∈ R, then, by the 3-paths of jfghij and for deab, and by eh and af, H_{R,B} contains the 5-cycle fgheia
contradicting 2. See Figure 2g If hg ∈ B, then, by the 3-paths of deabc and by bijdecgh, H_{R,B} contains the 3-cycle cab contradicting 2. See Figure 2h

In all cases we obtain a contradiction which implies that G has Frank number
different from 2. This finishes the proof of Theorem 6. ■

6. Algorithmic aspects

This chapter is dedicated to proving Theorem 7.

Our reduction is from a slightly stronger variation of MNAE3SAT. In the
first part, we introduce this problem and show that it is NP-complete by a
reduction from MNAE3SAT. Next, we introduce our construction and show
that the constructed graph is cubic and 3-edge-connected. The last two parts
are dedicated to showing that the reduction works indeed.

6.1. Boolean formulas

Given a MNAE3SAT formula F = (X, C), we call a truth assignment to the
variables of X feasible if every clause of C contains at least one true and at least
one false literal. We define the formula graph G_F by V(G_F) = X ∪ C and there
is an edge between the vertices corresponding to a variable x_i and a clause C_j
if x_i is contained in C_j. We call a formula F connected if G_F is connected. We
show that MNAE3SAT stays NP-complete with this additional assumption.

Connected Monotone Not-all-equal-3SAT(CMNAE3SAT)

Instance: A set X of boolean variables, a connected formula consisting of a set
C of clauses each containing 3 distinct variables none of which are negated.
Question: Is there a feasible truth assignment to the variables of X?

Lemma 8. CMNAE3SAT is NP-complete.

Proof We show a reduction from MNAE3SAT. Recall that MNAE3SAT is NP-
complete by Theorem 13. Let F be a MNAE3SAT formula. Let G_1, . . . , G_t be
the connected components of G_F. For i = 1, . . . , t, consider the MNAE3SAT
formula $F_i$ that consists of the variables and clauses corresponding to vertices in $G_i$. Observe that $G_{F_i} = G_i$ and so every $F_i$ is an instance of CMNAE3SAT. We will show that $F$ is a positive instance of MNAE3SAT if and only if all of the $F_i$ are positive instances of CMNAE3SAT. First assume that there is a feasible truth assignment for $F$. The restriction of this assignment to the variables of $F_i$ yields a feasible truth assignment for $F_i$ for all $i = 1, \ldots, t$. Now assume that there is a feasible truth assignment for $F_i$ for $i = 1, \ldots, t$. As every vertex corresponding to a variable is contained in exactly one component, every variable is contained in exactly one of the $F_i$ and so we obtain a unique assignment of boolean values to all variables. As every clause of $C$ is contained in some $F_i$, this assignment is feasible for $F$. This finishes the proof.

6.2. The construction

Let $F = (X, C)$ be a CMNAE3SAT formula with $X = \{x_1, \ldots, x_m\}$. Clearly, we may assume that every $x_i \in X$ is contained in at least 2 clauses. For $i = 1, \ldots, m$, we define $p_i$ to be the number of clauses $x_i$ is contained in.

We now construct an instance $(G = (V, E), S)$ of DELETABILITY. For $i = 1, \ldots, m$, $G$ contains a cycle $K_i$ of length $2p_i$. We abbreviate $V(K_i)$ to $V_i$ and $E(K_i)$ to $E_i$. Observe that $V_i$ can be partitioned into two stable sets in a unique way. We call one of these sets $A_i$ and the other one $B_i$. Note that $|A_i| = |B_i| = p_i$. For every clause $C$, $G$ contains a vertex $v_C$. We denote $\{v_C : C \in C\}$ by $V_C$. Further, $G$ contains a cycle $K$ of length $3|C|$. We abbreviate $V(K)$ to $V_K$ and $E(K)$ to $E_K$. We add a perfect matching between $\{v_C : x_i \in C\}$ and $A_i$ for every $i = 1, \ldots, m$ and between $\bigcup_{i=1}^m B_i$ and $V_K$. Observe that this is possible because $|A_i| = p_i$ and $|\bigcup_{i=1}^m B_i| = \sum_{i=1}^m p_i = 3|C| = |V_K|$. Finally, we define $S = \bigcup_{i=1}^m E_i$. Note that $|V| = 10|C|$ and $|E| = 15|C|$, so the construction is polynomial indeed.

![Figure 3](image)

Figure 3 shows the constructed graph for the formula consisting of the variables $x_1, \ldots, x_4$ and the clauses $C_1 = \{x_1, x_2, x_3\}, C_2 = \{x_1, x_2, x_4\}$ and $C_3 = \{x_1, x_3, x_4\}$. The edges of $S$ are marked in red.

Observe that $G$ is cubic as every clause contains exactly 3 variables and by construction. We show that it also satisfies the other desired structural property.

**Lemma 9.** $G$ is 3-edge-connected.
Proof Assume for the sake of a contradiction that $G$ contains some cut $\delta(Z)$ which consists of at most 2 edges. Without loss of generality, we may assume that $V_K \cap Z$ is nonempty.

Claim 14. $V_K \subseteq Z$.

Proof Assume that there is a vertex $w \in V_K - Z$. As $G - V_K$ arises from $G_F$ by replacing vertices by cycles and $G_F$ is connected by assumption, $G - V_K$ is connected. Then, since a perfect matching exists between $\bigcup_{i=1}^{m} B_i$ and $V_K$, $G - E_K$ is also connected. As $K$ is 2-edge-connected, it follows that $2 \geq d_G(Z) = d_K(Z) + d_{G - E_K}(Z) \geq 2 + 1 = 3$, a contradiction. ■

Claim 15. $V_C \subseteq Z$.

Proof Consider a vertex $v_C$ where $C$ contains the variables $x_i, x_j, x_\ell$. By construction, both $v_C$ and $K$ have a neighbor in each of $V_i, V_j$ and $V_\ell$ and $K_i, K_j$ and $K_\ell$ are connected. Since, by Claim 14 $V_K \subseteq Z$, this yields 3 edge-disjoint paths from $v_C$ to $Z$. It follows, by $d_G(Z) \leq 2$, that $v_C \in Z$. ■

By Claims 14 and 15 there exists a vertex $v \in V_i - Z$ for some $i = 1, \ldots, m$ and $v$ is connected to $K \cup V_C \subseteq Z$ by a path of length 1 and two paths of length 2 and all of these are edge-disjoint. This is a contradiction to $Z$ being separated from $v$ by a cut of at most 2 edges. This finishes the proof of Lemma 9. ■

The remaining part of this section is dedicated to showing that our construction is indeed correct, i.e. $F$ is a positive instance of CMNAE3SAT if and only if $(G, S)$ is a positive instance of DELETABILITY.

6.3. From orientation to truth assignment

Suppose that $(G, S)$ is a positive instance of DELETABILITY, so there is an orientation $D$ of $G$ such that $D - \vec{s}$ is strongly connected for all $s \in S$. Before finding a feasible truth assignment of the formula, we need the following result about the orientation.

Claim 16. Let $i \in \{1, \ldots, m\}$. Then all the arcs between $A_i$ and $B_i$ are directed in the same way.

Proof Let $v$ be any vertex of $K_i$ and $e, f$ the two edges of $K_i$ incident to $v$. Since $e, f \in S$, $D - e$ and $D - f$ are strongly connected. Then, as $G$ is cubic, both of $e$ and $f$ are either entering or leaving $v$. Since $K_i$ is connected, the claim follows. ■

Using Claim 16, we now define a truth assignment of $X$ in the following way: a variable $x_i$ is assigned the value true if the arcs between $A_i$ and $B_i$ are directed from $B_i$ to $A_i$ and false if the arcs between $A_i$ and $B_i$ are directed from $A_i$ to $B_i$.

Consider a clause $C = \{x_i, x_j, x_\ell\}$. The vertex $v_C$ has one neighbor in each of $A_i, A_j$ and $A_\ell$ in $G$. As $D$ is strongly connected and $G$ is cubic, $v_C$ has one in-neighbor $w$, say in $A_\ell$ and $w$ has an in-neighbor in $D[V_\ell]$. It follows by construction that $x_\ell$ is set to true in the truth assignment. Similarly, one of $x_i, x_j, x_\ell$ is set to false. It follows that the assignment is feasible.
6.4. From truth assignment to orientation

Assume that there is a feasible truth assignment for an instance $F$ of CM-NAE3SAT consisting of a variable set $X = \{x_1, \ldots, x_m\}$ and a clause set $C$. Relabeling variables, we may assume that there is some $t \in \{0, \ldots, m\}$ such that $x_i$ is set to true for $i = 1, \ldots, t$ and $x_i$ is set to false for $i = t + 1, \ldots, m$.

Let $A_1 = \bigcup_{i=1}^{t} A_i$, $A_2 = \bigcup_{i=t+1}^{m} A_i$, $B_1 = \bigcup_{i=1}^{t} B_i$ and $B_2 = \bigcup_{i=t+1}^{m} B_i$.

We define an orientation $D$ of $G$ as follows. We orient all edges from $P$ to $R$ where $P$ and $R$ are two consecutive sets in $A_1, V, C, A_2, V, K, B_1, A_1$. Finally, we orient the edges of $K$ as a circuit.

![Diagram 4](image)

Figure 4 shows the obtained orientation for the formula consisting of the variables $x_1, \ldots, x_4$ and the clauses $C_1 = \{x_1, x_2, x_3\}, C_2 = \{x_1, x_2, x_4\}$ and $C_3 = \{x_1, x_3, x_4\}$ when $x_1$ and $x_2$ are set to true and $x_3$ and $x_4$ are set to false.

The following is the orientation’s decisive property:

**Claim 17.** In $D$, every vertex $v_C \in V_C$ has an in-neighbor in $A_1$ and an out-neighbor in $A_2$.

**Proof** Let $C$ contain the 3 variables $x_i, x_j$ and $x_\ell$. As the truth assignment is feasible, one of $x_i, x_j, x_\ell$, say $x_i$, is set to true and a different one, say $x_j$, is set false. Then, by construction, $D$ contains an arc from $A_i \subseteq A_1$ to $v_C$ an arc from $v_C$ to $A_j \subseteq A_2$. 

The following result will finish the proof:

**Claim 18.** Let $s \in S$. Then $D - \vec{s}$ is strongly connected.

**Proof** Since $K$ is oriented as a circuit, all vertices of $K$ are in the same strongly connected component $Q$. By construction, all vertices in $B_1$ have an in-neighbor in $V_K \subseteq Q$ and all vertices in $A_1$ have 2 in-neighbors in $B_1$ in $D$, so at least one in $D - \vec{s}$. It follows, by Claim 17 that all vertices in $A_1 \cup B_1 \cup V_C$ are reachable from $Q$. By similar arguments, $Q$ is reachable from all vertices in $A_2 \cup B_2 \cup V_C$. This yields that $V_C \subseteq Q$. Finally, from every vertex in $A_1 \cup B_1$ there exists a directed path of length 1 or 2 to a vertex $v_C \in V_C$. Similarly, to every vertex in $A_2 \cup B_2$ there exists a directed path of length 1 or 2 from a vertex $v_C \in V_C$. It follows that $D - \vec{s}$ is strongly connected.

This reduction proves Theorem 7.
7. Conclusion

Our work shows that \( f(G) \leq 7 \) for every 3-edge-connected graph \( G \) and that \( f(G) = 3 \) if \( G \) is the Petersen graph. Also, we show a better bound for the more restricted classes of essentially 4-edge-connected graphs and 3-edge-colorable, 3-edge-connected graphs. Further, we show that a graph of Frank number bigger than 5 would imply the failure of Conjecture \( \mathbb{I} \). Moreover, the decision problem whether all edges of a given subset can become deletable in one orientation is proven to be NP-complete.

The most obvious remaining problem is to improve these bounds on the Frank number in the general case. Considering the indications found during our work, we propose the following conjecture:

**Conjecture 2.** \( f(G) \leq 3 \) for any 3-edge-connected graph.

A possible way to make progress towards Conjecture 2 would be the following generalization of Lemma 5. Using the fact that cubic graphs are 4-edge-colorable [1] and similar arguments as before, it implies that \( f(G) \leq 4 \) for any 3-edge-connected graph.

**Conjecture 3.** Let \( M \) be a matching of a 3-edge-connected graph \( G \) intersecting each 3-edge-cut of \( G \) in at most one edge. Then \( M \) is deletable.

It would also be interesting to generalize Frank numbers to arbitrary odd connectivity:

**Open Problem 1.** Given a \((2k+1)\)-edge-connected graph \( G \), what is the minimum number of \( k \)-arc-connected orientations such that each edge becomes an arc whose deletion does not destroy \( k \)-arc-connectivity in at least one of these orientations?

It follows from a theorem in [1] that this number is bounded by a constant depending only upon \( k \). We are particularly interested if this number can be bounded by a constant not depending upon \( k \).

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References


