A NOTE

ON PACKING PATHS IN PLANAR GRAPHS

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ABSTRACT P.D. Seymour proved that the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem when the union of the supply graph and the demand graph is planar and Eulerian. When only planarity is required, M. Middendorf and F. Pfeiffer proved the problem to be NP-complete. For this case E. Korach and M. Penn proved that the cut criterion is sufficient for the existence of a near-complete packing of paths. Here we generalize this result by showing how a natural strengthening of the cut criterion yields better packings of paths.

Analogously to Seymour's approach, we actually prove a theorem on packing cuts in an arbitrary graph and then the planar edge-disjoint paths case is obtained by planar dualization. The main result is derived from a theorem of A. Sebő on the structure of ± 1 weightings of a bipartite graph with no negative circuits.

1. INTRODUCTION

The edge-disjoint paths problem consists of finding edge-disjoint paths in an undirected graph $G = (V, E)$ that connect specified pairs of terminal nodes. We will say that uv is a **demand edge** if (u, v) is a specified pair of nodes to be connected. Let $H = (V, F)$ denote the graph of demand edges and call H a **demand graph.** If (u, v) is specified k times, that is, k paths are required to connect u and v, then in the demand graph k parallel demand edges occur connecting u and v. The edge-disjoint paths problem is equivalent to finding a family of $|F|$ edge-disjoint circuits in $G + H$ so that each of these circuits contains precisely one demand edge (and then, clearly, each demand edge belongs to a circuit). Such a circuit will be called F**-good** and the family a **complete packing** of F-good cuts. That is, a complete packing is one covering all demand edges. We say that in a concrete instance $G + H$ there is a solution to the edge-disjoint paths problem if such a complete packing exists.

In this paper we will mainly be concerned with the edge-disjoint paths problem in the planar case, that is, when $G + H$ is planar. M. Middendorf and F. Pfeiffer [1993] showed that even in this case the problem is NP-complete.

The following is clearly a necessary condition for the solvability of the edge-disjoint paths problem.

CUT CRITERION $\quad d_G(X) \geq d_H(X)$ for every $X \subseteq V$

where $d_G(X)$ denotes the number of edges of G with precisely one end-node in X. Let $s(X) := s(B) := d_G(X) - d_H(X)$ and we call this quantity the **surplus** of X or of the cut $B := [X, V - X]$ determined by X. Then the cut criterion is equivalent to requiring that the surplus of every cut is non-negative.

The cut criterion is not sufficient in general (not even in the planar case) but there are important special cases when it is. (For a survey, see [Frank, 1990]). For example, P.D. Seymour [1981] proved:

THEOREM 1.1 *When $G + H$ is planar and Eulerian, the edge-disjoint paths problem has a solution if and only if the cut criterion holds.*

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What can one say on the non-Eulerian planar edge-disjoint paths problem? E. Korach and M. Penn proved a theorem stating, roughly, that the cut criterion, though not sufficient, ensures the near-solvability of the edge-disjoint paths problem in the sense that each demand can be met except possibly one from each bounded face:

THEOREM 1.2 [Korach and Penn, 1992] *Suppose that $G+H$ is planar and the cut condition holds. There is at most one demand edge on each bounded face of G so that leaving out these edges from H the problem has a solution.*

Note that in this theorem the infinite face of G plays a special role: no demand edge from it is left out. What if we want to require the same for some or all other faces of G? We should somehow find conditions stronger than the cut criterion.

On the negative side, we show that even in the planar case there is no constant $K \in \mathbf{Z}_+$ such that requiring $d_G(X) \geq d_H(X) + K$ for every subset X of nodes is sufficient for the solvability of the edge-disjoint paths problem however big K is. Even worse, we will see that for any positive ε the requirement $d_G(X) \geq (2-\varepsilon)d_H(X)$ for every $X \subseteq V$ is still not sufficient, in general. On the other hand we will observe that requiring condition $d_G(X) \geq 2d_H(X)$ for every $X \subseteq V$ is already sufficient. We are going to prove a refinement of this statement asserting the sufficiency of a weaker condition. Our main result is a generalization of the theorem of Korach and Penn (but not of Seymour's).

A subset F of edges is called a **join** if $|C \cap F| \leq |C - F|$ holds for every circuit C. We call a ± 1 weighting w of the edge-set **conservative** if the total weight $w(C)$ of every circuit C is non-negative. M. Guan [1962] pointed out that these notions are intimately related: for a conservative w the set N_w of negative edges is a join and conversely, for a join F the weighting w_F is conservative where $w_F(e) := -1$ for $e \in F$ and $:= +1$ for $e \notin F$.

A cut containing precisely one element of F (one negative element) will be called **F-good.** A family of disjoint F-good cuts is called a **complete packing** if every negative edge belongs to one of them.

2. RESULTS

For planar graphs the problem of packing F-good circuits (that is, the edge-disjoint paths problem) and the problem of packing F-good cuts are clearly equivalent via planar dualization. But very often results concerning packing F-good cuts extend to non-planar graphs. For example, Seymour's above-mentioned result was originally proved in a more general form:

THEOREM 2.1 [P.D. Seymour, 1981] *Given a bipartite graph and a subset F of its edges, there is a complete packing of F-good cuts if and only if F is a join.*

(To see the equivalence of Theorems 1.1 and 2.1 in the planar case, notice that a planar graph is Eulerian if and only if its planar dual is bipartite and that the cut condition holds if and only if the edge-set in the dual corresponding to the demand edges is a join.)

For those who are familiar with T-joins and T-cuts, we note that Seymour actually proved his result in an equivalent form:

THEOREM 2.1A *For a bipartite graph the minimum cardinality of a T-join is equal to the maximum number of disjoint T-cuts.*

Korach and Penn also proved their result in the form of packing F-good cuts in an arbitrary graph $\hat{G} = (V, \hat{E})$:

THEOREM 2.2 [E. Korach and M. Penn, 1992] *Let F be a join in $\hat{G}$ and let the components of the subgraph $(V(F), F)$ be K_0, $F_1, F_2, \ldots, F_l$. Then it is possible to contract at most one edge from each F_i so that the resulting graph has a complete packing of F-good cuts.*

When $\hat{G}$ is planar, its planar dual corresponds to $G + H$ in Theorem 1.2. Moreover, the elements of F correspond to the demand edges (that is, the edges of H). It is an easy exercise to see that a tree F_i corresponds to a subset of demand edges in H which lie in one face of G.

In order to generalize Theorem 2.2, assume that the components of the subgraph $(V(F), F)$ are divided into two groups: $\{K_0, K_1, \ldots, K_k\}$ and $\{F_1, \ldots, F_l\}$, $(k \geq 0, l \geq 0)$. For a circuit C let $k_+(C)$ denote the number of those components K_i $(i \geq 1!)$ from which C contains at least one edge.

MAIN THEOREM 2.3 *If*

$$w_F(C) \geq k_+(C) \tag{2.1}$$

for every circuit C of $\hat{G}$, then it is possible to contract at most one edge from each F_i so that the resulting graph has a complete packing of F-good cuts. In particular, if $l = 0$, there is a complete packing of F-good cuts.

In other words, the theorem says that there is a packing of F-good cuts which is nearly complete in the sense that each edge of K_i $(i = 0, \ldots, k)$ is covered and each but at most one edge of F_j $(j = 1, \ldots, l)$ is covered.

In the same way as Theorem 1.2 arises from Theorem 2.2, one can specialize Theorem 2.3 to the planar case. Here we do this only when $l = 0$.

COROLLARY 2.4 *Let $G + H$ be planar. Suppose that the surplus $s(B)$ of every cut B is at least the number of those bounded faces of G which contain a demand edge from B. Then the edge-disjoint paths problem has a solution.*

This immediately implies:

COROLLARY 2.5 *In the planar case the edge-disjoint paths problem has a solution if $d_G(X) \geq 2d_H(X)$ for every $X \subseteq V$.*

On the other hand, we construct an example showing that the coefficient 2 in the corollary cannot be replaced by $2 - \varepsilon$ for any positive ε. Let $K_1, K_2, \ldots, K_t$ be t disjoint copies of the complete graph on four nodes and choose two disjoint edges e_i, f_i in each K_i. Glue together the copies by identifying each e_i into one edge e_0. In the resulting graph let e_0 and each f_i be demand edges. (For $t = 3$, see Figure 1). Clearly, the edge-disjoint paths problem has no solution in this instance. Moreover, $d_G(X) \geq d_H(X)\ 2t/(t + 1)$.

Figure 1

3. PROOF

Seymour's Theorem 2.1 tells us that in a bipartite graph for any join F *there is* a complete packing of F-good cuts. A. Sebő [1990] found an extremely elegant approach to describe a complete packing, "canonical" in a sense. His theorem (Theorem 3.1 below), which might be called the **Distance Theorem,** not only generalizes Seymour's but directly implies Theorem 2.2 of Korach and Penn, as well. To show this, Sebő first constructed a bipartite graph by subdividing each edge by a new node, he applied then his Distance Theorem, and finally, from the obtained cut packing, extracted the family of disjoint cuts required in the theorem of Korach and Penn.

Our proof will follow the similar line: the difference is that the Distance Theorem will be applied to an auxiliary bipartite graph which is constructed in a more sophisticated way.

In order to formulate the Distance Theorem, let B be a connected bipartite graph and F' a join of B, that is, $w := w_{F'}$ is conservative. Choose an arbitrary node s of B and let $\lambda(v)$ denote the minimum w-weight of a path between s and v. Clearly, $\lambda(s) = 0$ and it is also evident that $|\lambda(x) - \lambda(y)| = 1$ for every edge xy of B. Let i be any integer between the smallest and the largest distance from s. Define a **level set** $L_i := \{x \in V(B) : \lambda(s, x) = i\}$ and a **down-set** $D_i := \{x \in V(B) : \lambda(x) \leq i\}$.

Let $\mathcal{D}_i$ denote the family of connected components of the subgraph induced by D_i. Note that each edge leaving a member of $\mathcal{D}_i$ connects a node of L_i and a node of L_{i+1}. Now the family $\mathcal{D} := \cup\mathcal{D}_i$ is a laminar family (i.e., any two members are either disjoint or one includes the other) and the cuts defined by the members of $\mathcal{D}$ form a partition of the edge-set of B.

We call a member of $\mathcal{D}$ containing s a **root-component** while the members not containing s are called the **non-root-components.** (There is no root-components in $\mathcal{D}_i$ if $i < 0$, and there is precisely one if $i \geq 0$.)

DISTANCE THEOREM 3.1 [A. Sebő, 1990 and 1994] *Given a bipartite graph B and a join F' of B, every negative edge (that is, every element of F') enters a non-root-member of $\mathcal{D}$ and each non-root-member of $\mathcal{D}$ is entered by exactly one negative edge. Equivalently, the cuts determined by the distance components form a complete packing of F'-good cuts.*

Let us call a node $v \neq s$ **singular** if there are precisely two edges u_1v, u_2v incident to v, both of them negative, and $\lambda(v) - 1 = \lambda(u_1) = \lambda(u_2)$.

LEMMA 3.2 [Sebő, 1990] *Each component T of F' contains at most one singular node. Moreover, if s belongs to T, then T contains no singular node.*

Proof. We claim that

$$\text{there are no two adjacent negative edges } x_1y, x_2y \text{ for which } \lambda(y) + 1 = \lambda(x_1) = \lambda(x_2). \tag{3.1}$$

This can be seen directly by some easy manipulations with the minimum weight paths or we can argue that if (3.1) were not true, then there would be two negative edges entering the component of $\mathcal{D}_{\lambda(y)}$ containing y and this would contradict the Distance Theorem.

Now suppose, indirectly, that there are two singular nodes a, b in T. Let P be the unique path in T connecting a and b and let y be a node of P for which $\lambda(y)$ is minimum. Now the two incident edges x_1y and x_2y of P violate (3.1). •

We will need another simple lemma. For a sub-tree T of a graph $\hat{G} = (V, \hat{E})$ we call a circuit C **continuous in** T if $x, y \in V(C) \cap V(T)$ implies that the unique path in T connecting x and y belongs to C.

LEMMA 3.3 *If F is a sub-forest of a graph so that w_F is not conservative, then there is a negative circuit which is continuous in each component of F.*

Proof. We claim that a negative circuit C containing a minimum number of positive edges will do. If not, then there are two nodes x, y of C belonging to some component T of F so that the unique path P in T connecting x and y is internally disjoint from the two subpaths C_1 and C_2 of C connecting x and y. Now both circuits $P \cup C_1$ and $P \cup C_2$ contain a positive edge since F is a forest, hence both have less positive edges than C has. But one of them is a negative circuit contradicting the choice of C. •

Proof of the main theorem. We define an auxiliary graph B in two steps. For reference purposes let us assume that the nodes of $\hat{G}$ are linearly ordered. First, subdivide each edge $e = uv$ of $\hat{G}$ by a new node called a **subdividing node** and denoted by s_e, that is, replace e by two new edges $e' := us_e$ and $e'' := vs_e$. If, say, $u < v$, then e' will be called the **first half** of e while e'' the **second half.** Then for each node

$v \in \cup(V(K_i) : i = 1, \ldots, k)$ split v into two in the following sense: add a new copy v' of v, called a **split node,** to the graph, along with a new edge $v'v$, called a **split edge,** and for each original edge $e = uv \notin F$ replace the edge $s_e v$ by $s_e v'$. (Intuitively, this operation means that we split apart at v the negative and the positive edges.) (See Figure 2 where solid lines denote the edges belonging to the join).

Figure 2

It is immediately seen that the resulting graph B is bipartite. (Note that B has three types of nodes: original, subdividing and split nodes). In B let F' consist of the split edges plus the halves of the elements of F. Let s be an original node from $V(K_0)$. From the hypothesis of Theorem 2.3 and from the construction of B, Lemma 3.3 implies that F' is a join and thus we can apply Theorem 3.1.

Let us consider the level sets $L_i := \{x \in V(B) : \lambda(s, x) = i\}$. The original nodes of K_0 and $F_1, \ldots, F_l$ and all the split nodes lie in even levels while the original nodes of $K_1, \ldots, K_k$ lie in odd levels. Furthermore,

$(*)$ a subdividing node s_e is in an even level precisely if edge e belongs to $K_1 \cup \ldots \cup K_k$.

Recall the definition of singular nodes. We call an (original) edge in F **singular** if its subdividing node s_e is singular. It follows from Lemma 3.2 that K_0 contains no singular edge and each other component of F contains at most one.

By the Distance Theorem every non-root-component C' is entered by exactly one element of F' and the cuts of B determined by these components are edge-disjoint. In order to get edge-disjoint F-good cuts of $\hat{G}$ let us define a subset $\mathcal{D}'$ of $\mathcal{D}$, as follows.

Let $\mathcal{D}_{odd} := \cup\{\mathcal{D}_i : i \text{ odd }\}$. Discard from $\mathcal{D}_{odd}$ the root-components along with those members which are entered by either a split edge or the second half of a singular edge. Let $\mathcal{D}'$ denote the resulting family.

For any $X \in \mathcal{D}'$ let $\Delta(X)$ denote the cut of $\hat{G}$ determined by the set of original nodes in X. Such a cut is clearly F-good. Let $\hat{\mathcal{B}} := \{\Delta(X) : X \in \mathcal{D}'\}$.

We claim that $\hat{\mathcal{B}}$ consists of disjoint cuts. To see that, let $X, Y \in \mathcal{D}'$. If $X \in \mathcal{D}_i, Y \in \mathcal{D}_j (i \neq j)$, then $|i - j| \geq 2$ and hence $\Delta(X) \cap \Delta(Y) = \emptyset$ follows. If $X, Y \in \mathcal{D}_i$, and, indirectly, there is an edge $e \in \Delta(X) \cap \Delta(Y)$, then the subdividing node s_e is in L_{i-1} or in L_{i+1}. By $(*)$ e belongs to $K_1 \cup \ldots \cup K_k$. (3.1) implies that s_e must belong to L_{i+1} and hence one of X and Y is entered by the second half of e. This contradicts the definition of $\mathcal{D}'$.

Moreover, we claim that, apart from the singular edges occuring (possibly) in F_i's, each other edge e in F belongs to a cut in $\hat{\mathcal{B}}$. Indeed, if e is a non-singular edge in F, then (exactly) one of its two halves in B connects some level sets L_j and L_{j+1} where j is odd. Then this half enters a member of $\mathcal{D}'$ and therefore e belongs to a cut in $\hat{\mathcal{B}}$. If e is a singular edge of some K_i, then $i > 0$ and, by $(*)$, each halves of e connects some level sets L_j and L_{j+1} where j is odd. We conclude that such an e also belongs to a cut in $\hat{\mathcal{B}}$.

We obtained that, after contracting the singular edges belonging to some F_i, the family $\hat{\mathcal{B}}$ is a complete packing of F-good cuts. Since each F_i contains at most one singular edge, the theorem follows. •

4. REMARKS

Recall that in the case $k = 0$ the main theorem specializes to Theorem 2.2. We note that also the proof above specializes to that of Theorem 2.2 given by A. Sebő in [1990].

One can ask how sharp the main theorem is. Let us consider only the following:

COROLLARY *Suppose that for a join F*

$$w_F(C) \geq k(C) \tag{3.1}$$

for every circuit C where $k(C)$ denotes the number of components of F sharing an edge with C. Then there is a complete packing of F-good cuts.

In (3.1) $k(C)$ cannot be replaced by $k(C)-2$ as is shown by K_4, the complete graph on four nodes, when F consists of two disjoint edges. In this case $w_F(C) \geq k(C)-2$ clearly holds for every circuit C but there is no complete packing of F-good cuts since there are no two disjoint cuts at all. (K_4 is not a single bad example: it is not difficult to construct graphs with a join F of arbitrarily many components so that $w_F(C) \geq k(C)-2$ holds for every circuit and there is no complete packing of F-good-cuts.)

We are left with the following:

OPEN PROBLEM Can $k(C)$ in (3.1) be replaced by $k(C)-1$?

Finally, we briefly remark that, given an arbitrary conservative weighting, the distance of two specified nodes can be computed with the help of matchings. Therefore the level sets in the Distance Theorem can be computed in polynomial time for arbitrary conservative weightings. (For further details, see [Sebő, 1994].) Because the reduction we used in the proof of the main theorem is algorithmic, we conclude that there is a polynomial time algorithm for finding the desired packing of F-good cuts. The resulting algorithm is polynomial even in the capacitated version of the main theorem.

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