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NOTE

The Two Ear Theorem on Matching-Covered Graphs

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We give a simple and short proof for the two ear theorem on matching-covered graphs which is a well-known result of Lovász and Plummer. The proof relies only on the classical results of Tutte and Hall on perfect matchings in (bipartite) graphs. © 1998 Academic Press

1. INTRODUCTION

We consider only finite undirected loopless graphs in this note. A set M of edges is called *matching* if no two edges in M have a common end vertex. A matching M of a graph G is *perfect* if M covers all the vertices of G. We shall denote the number of perfect matchings of a graph G by $\Phi(G)$. Let M be a matching of G. A path P is said to be *alternating* if the edges of P are alternately in and not in M. For a subgraph F of G, the subset of M contained in F is denoted by $M(F)$.

A graph G with a perfect matching is called *elementary* if the edges which belong to some perfect matching of G form a connected subgraph. Note that if G is elementary, then after adding some edges to G the resulting graph remains elementary. G is *matching-covered* if it is connected and each edge belongs to a perfect matching of G. Clearly, if G is matching-covered then it is elementary.

Let G be an arbitrary graph. A subgraph H of G is *nice* if $G-V(H)$ has a perfect matching. A sequence of subgraphs of G, $(G_0, G_1, ..., G_m)$ is a *graded ear-decomposition* of G if G_0 is an edge, $G_m=G$, every G_i for $i=0, 1, ..., m$ is a nice matching-covered subgraph of G and G_{i+1} is obtained from G_i by adding at most two disjoint odd paths which are

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openly disjoint from G_i but their end-vertices belong to G_i. Clearly, if G possesses a graded ear-decomposition, then it is matching-covered. Lovász and Plummer [6, 7] proved the following important result on matching-covered graphs.

THEOREM 1. *Every matching-covered graph has a graded ear-decomposition.*

The proof of this theorem relies on the following theorem of Lovász and Plummer [7]. For the sake of completeness we shall show in Section 3 how Theorem 2 implies Theorem 1.

THEOREM 2. *Let G be an elementary graph and let $e_1, ..., e_k$ be edges not in G but having both end-vertices in $V(G)$. Suppose that $\Phi(G+e_1+\cdots+e_k) > \Phi(G)$. Then there exist i and j, $1 \leqslant i \leqslant j \leqslant k$ such that $\Phi(G+e_i+e_j) > \Phi(G)$.*

Theorem 2 can be easily derived from the case when $k=3$. However, the original proof of this case in [7] is involved and it is far from being simple. Here we shall derive Theorem 2 from the following theorem, which was formulated by Cheriyan and Geelen. The main contribution of this note is a new proof of Theorem 3 which relies only on Tutte's theorem and Hall's theorem.

THEOREM 3. *Let G be an elementary graph and let e_1, e_2, e_3 be edges not in G but having both end-vertices in $V(G)$ so that $G+e_1+e_2+e_3$ has a perfect matching M containing e_1, e_2, e_3. Suppose that for each e_i $(1 \leqslant i \leqslant 3)$, no perfect matching of $G+e_i$ contains e_i. Then for each e_i $(1 \leqslant i \leqslant 3)$ there exists an e_j $(1 \leqslant j \leqslant 3)$ $i \neq j$ such that $G+e_i+e_j$ has a perfect matching containing e_i and e_j.*

However, we mention that the obvious generalization of Theorem 3 for more than three edges is not true, here is a counterexample. Let G be the cycle $(1, 2, ..., 8)$ on eight vertices and let $15, 24, 37, 68$ be the four new edges. Then for the edge 15 the generalization of Theorem 3 does not hold.

Little and Rendl [8] have given a shorter proof for Theorem 1 than the original one, but our proof is even shorter and simpler. Recently, Carvalho *et al.* [2] generalized Theorem 1 by showing that a matching-covered graph of maximum degree Δ has at least $\Delta!$ graded ear-decompositions.

2. PRELIMINARIES

Let us recall the two classical and basic results in matching theory due to Hall [3] and Tutte [9].

THEOREM 4 [3]. *A bipartite graph $B=(U, V; E)$ possesses a perfect matching if and only if $|U|=|V|$ and $|\Gamma(X)| \geqslant |X|$ for all $X \subseteq U$, where $\Gamma(X)$ denotes the set of neighbors of X.*

THEOREM 5 [9]. *A graph G has a perfect matching if and only if for every $X \subseteq V(G)$, $c_0(G-X) \leqslant |X|$, where $c_0(G-X)$ denotes the number of odd components of the graph obtained from G by deleting a vertex set X.*

We shall use the following easy corollary of Hall's theorem; see [7].

CLAIM 1. *If for a bipartite graph $B=(U, V; E)$, $|U|=|V|$ and $|\Gamma(X)| \geqslant |X|+1$ for all $\emptyset \neq X \subset U$, then B is matching-covered.*

For a graph G let $\operatorname{def}(G) := \max\{c_0(G-X)-|X| : X \subseteq V(G)\}$. A vertex set X of G is called *barrier* if X attains this maximum, that is if $G-X$ has exactly $|X|+\operatorname{def}(G)$ odd components. By a *maximal* barrier we mean one that is inclusionwise maximal. A graph G is called *factor-critical* if for each vertex v of G there exists a perfect matching in $G-v$. A barrier X is called *strong* if each odd component of $G-X$ is factor-critical. For more results on strong barriers see Király [4].

The following well-known corollary of Tutte's Theorem can be found for example in [1].

CLAIM 2. *Let G be a graph so that it has an even number of vertices and it has no perfect matching. Let X be a maximal barrier of G. Then $c_0(G-X) \geqslant |X|+2$ and X is a strong barrier.*

The following claim is obvious.

CLAIM 3. *Let G be an elementary graph. Then for any barrier $X \neq \emptyset$ of G, $G-X$ has no even components.*

In fact, elementary graphs can be characterized this way. A graph with a perfect matching is elementary if and only if for any barrier $X \neq \emptyset$ of G, $G-X$ has no even components (see [7]), but we shall not use this characterization. We mention that by Claim 3 the notion of maximal barriers and strong barriers coincide for elementary graphs.

Lovász [5] proved that for elementary graphs (i) the maximal barriers form a partition of the vertex set and (ii) an edge belongs to a perfect matching if and only if its end-vertices lie in different maximal barriers. We do not want to rely on these results, instead we prove the following claim. This claim will be applied frequently in our proof.

CLAIM 4. *Let X be a strong barrier of an elementary graph G. Then each edge leaving X belongs to some perfect matching of G.*

Proof. Since all the components of $G-X$ are factor-critical by Claim 3 it suffices to prove that each edge e of the bipartite graph B, obtained from G by deleting the edges spanned by X and contracting each component of $G-X$ into one vertex, belongs to a perfect matching of B, that is B is matching-covered. Let us denote the colour class of B different from X by Y. Clearly, $|X|=|Y|$. Furthermore, for any set $\emptyset \neq Z \subset Y$, $|\Gamma(Z)| \geqslant |Z|+1$, otherwise $\Gamma(Z)$ would violate in G either the Tutte's condition or Claim 3, both cases lead to contradiction. Then, by Claim 1, B is matching-covered which was to be proved. ∎

3. THE PROOF

Proof of Theorem 3. Let us assume that there is no perfect matching of $G' := G+e_1+e_2$ containing e_1 and e_2. We shall prove that there is a perfect matching of $G+e_1+e_3$ containing e_1 and e_3. Let us denote the vertices of e_i by x_i, y_i.

(1) *There exists a strong barrier P in G' containing x_1 and y_1.* $G'-x_1-y_1$ has no perfect matching by assumption; thus by Claim 2 there exists a barrier of G' containing x_1 and y_1. Let P be a maximal barrier of G' containing x_1 and y_1. Then, by Claim 2 P is a strong barrier; that is, by Claim 3 each component F_i of $G-P$ $(1 \leqslant i \leqslant |P|)$ is factor-critical.

(2) *e_2 is in one of the factor-critical components (say in F_1) of $G'-P$.* Indeed, by Claim 4, e_2 does not enter P. Moreover, x_2 and y_2 cannot be contained in P; otherwise $P-x_1-y_1-x_2-y_2$ violates the Tutte's condition in $G+e_3-x_1-y_1-x_2-y_2$, contradicting the assumption that $G'' := G+e_1+e_2+e_3$ has the perfect matching M containing e_1, e_2, and e_3.

(3) *x_3 and y_3 are in different factor-critical components of $G'-P$.* This follows from the fact that $G'-x_1-y_1+e_3$ contains the perfect matching $M-e_1$. It also follows that

(4) *for each F_i $(1 \leqslant i \leqslant |P|)$ exactly one edge m_i of M leaves F_i in G'.*

Now, suppose that m_1 enters P. P is a strong barrier in $G+e_2$; thus, by Claim 4 m_1 belongs to a perfect matching M_1 of $G+e_2$. Then $(M_1-M_1(F_1)) \cup M(F_1)$ is a perfect matching of $G+e_2$ containing e_2. This contradiction shows that

(5) *in $G''e_3$ leaves the factor-critical component of $G'-P$ that contains e_2; that is $m_1=e_3$.*

Assume without loss of generality that x_3 is in F_1. We know that $H := F_1-x_3$ has a perfect matching, for example $M(H)$.

(6) *$H-e_2$ has a perfect matching M_2.* Otherwise, for a maximal barrier X of $H-e_2$, we have by Claim 2 $c_0(H-e_2-X) \geqslant |X|+2$. Then, by Claim 2 $P' := P \cup X \cup x_3$ is a strong barrier in $G+e_3$, and e_3 enters P';

thus by Claim 4 $G+e_3$ contains a perfect matching containing e_3, a contradiction.

(7) *$M(G''-H)\cup M_2$ is a perfect matching of $G+e_1+e_3$ containing e_1 and e_3*, as we claimed. ∎

THEOREM 3 $\Rightarrow$ THEOREM 2.

Proof. We may suppose that $(*)$ no proper subset of $\{e_1, ..., e_k\}$ satisfies the conditions of the theorem. Then we claim that $k\leqslant 3$. Assume that $k\geqslant 4$ and let $G' := G+e_4+\cdots+e_k$. Then, by $(*)$, $\Phi(G')=\Phi(G)$ and $\Phi(G'+e_i)=\Phi(G')$ $i=1, 2, 3$, but $\Phi(G'+e_1+e_2+e_3)>\Phi(G)=\Phi(G')$. Theorem 3 implies that for some $1\leqslant i<j\leqslant 3$, $\Phi(G'+e_i+e_j)>\Phi(G')$; that is, $\Phi(G'+e_i+e_j+e_4+\cdots+e_k)>\Phi(G)$, contradicting $(*)$. If $k\leqslant 3$, then Theorem 3 directly implies Theorem 2. ∎

THEOREM 2 $\Rightarrow$ THEOREM 1.

Proof. Assume that for some i the nice matching-covered subgraph G_i has already been contructed. (G_0 can be chosen as an arbitrary edge of G.) If G_i does not span $V(G)$ then let e be an edge connecting $V(G_i)$ and $V(G)-V(G_i)$. Let M_i be a perfect matching of $G-V(G_i)$ and M_e a perfect matching of G containing e. The symmetric difference of M_i and M_e consists of vertex disjoint cycles and a set $(P_1, ..., P_k)$ of alternating paths connecting vertices in $V(G_i)$. If G_i spans $V(G)$ but does not contain all the edges of G then the edges in $E(G)-E(G_i)$ are denoted by $(P_1, ..., P_k)$. Clearly, after adding all these paths to G_i, the resulting graph is a nice matching-covered subgraph of G. We have to show that G_{i+1} can be constructed by adding at most two of these paths to G_i. We define for $i=1, ..., k$ e_i to be the edge connecting the two end-vertices of P_i. Clearly, for a subset $(P_{i_1}, ..., P_{i_r})$ of $(P_1, ..., P_k)$, $G_i+P_{i_1}+\cdots+P_{i_r}$ is matching-covered if and only if $G_i+e_{i_1}+\cdots+e_{i_r}$ is matching-covered. Thus Theorem 2 implies the theorem. ∎

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REFERENCES

1. I. Anderson, Perfect matchings of a graph, *J. Combin. Theory Ser. B* **10** (1971), 183–186.
2. M. H. Carvalho, C. L. Lucchesi and U. S. R. Murty, Ear decompositions of matching covered graphs, submitted.
3. P. Hall, On representatives of subsets, *J. London Math. Soc.* **10** (1935), 26–30.
4. Z. Király, The calculus of barriers, manuscript.

5. L. Lovász, On the structure of factorizable graphs, *Acta Math. Acad. Sci. Hungar.* **23** (1972), 179–195.
6. L. Lovász and M. D. Plummer, On bicritical graphs, *in* “Infinite and Finite Sets, II,” Colloq. Math. Soc. J. Bolyai, Vol. 10, pp. 1051–1079, Amsterdam, North-Holland, 1975.
7. L. Lovász and M. D. Plummer, “Matching Theory,” North-Holland, Amsterdam, 1986.
8. C. H. C. Little and F. Rendl, An algorithm for the ear decomposition of a 1-factor covered graph, *J. Austral. Math. Soc. Ser. A* **46** (1989), 296–301.
9. W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.* **22** (1947), 107–111.