

Simultaneous well-balanced orientations of graphs

Zoltán Király^{a,1,2}, Zoltán Szigeti^{b,2}

^a *Department of Computer Science, and CNL (Communication Networks Lab), Eötvös University,
Pázmány Péter sétány 1/C, Budapest, H-1117, Hungary*

^b *Equipe Combinatoire et Optimisation, Université Paris 6, 75252 Paris cedex 05, France*

Received 30 September 2004

Available online 8 February 2006

Abstract

Nash-Williams' well-balanced orientation theorem [C.St.J.A. Nash-Williams, On orientations, connectivity and odd-vertex-pairings in finite graphs, *Canad. J. Math.* 12 (1960) 555–567] is extended for orienting several graphs simultaneously.

We prove that if $G_1, \dots, G_k$ are pairwise edge-disjoint subgraphs of a graph G , then G has a well-balanced orientation $\vec{G}$ such that the inherited orientations $\vec{G}_i$ of G_i are well-balanced for all $1 \leq i \leq k$. We also have new results about simultaneous well-balanced orientations of non-disjoint subgraphs of an Eulerian graph as well as those of different contractions of a graph.

© 2006 Elsevier Inc. All rights reserved.

Keywords: Orientation; Connectivity

1. Introduction

This paper concerns undirected and directed graphs, more precisely we consider orientations of undirected graphs. Multiple edges are allowed, but loops are forbidden. The starting point is Robbins' theorem [13] which states that an undirected graph G has a strongly connected orientation if and only if G is 2-edge-connected. The following generalization was proved by Nash-Williams in [11]: An undirected graph G has a k -arc-connected orientation if and only if G is $2k$ -edge-connected (see Theorem 2 below). This result about global edge-connectivity can be easily proved by applying Lovász' splitting off theorem [8]. Nash-Williams [11] also provided

E-mail addresses: kiraly@cs.elte.hu (Z. Király), szigeti@math.jussieu.fr (Z. Szigeti).

¹ Research supported by OTKA grants T 037547 and T046234.

² Research supported by EGRES group (MTA-ELTE).

the following extension on local edge-connectivity (for a stronger statement see Theorem 3): Any undirected graph G has a *well-balanced* orientation $\vec{G}$, that is for every ordered pair of vertices (u, v) , if the maximum number of edge disjoint (u, v) -paths was $\lambda_G(u, v)$ in G , then the maximum number of arc disjoint directed (u, v) -paths is at least $\lfloor \lambda_G(u, v)/2 \rfloor$ in the resulting directed graph $\vec{G}$. The well-balanced orientation may also be required to be *smooth*, that is the in-degree and the out-degree of every vertex differ by at most one. A smooth well-balanced orientation is called *best-balanced*. In fact, Nash-Williams proved an even stronger result in [11] the so-called odd vertex pairing theorem (see Theorem 5).

Nash-Williams [12] formulated an extension of his orientation theorem: For an arbitrary subgraph H of an undirected graph G there exists a best-balanced orientation of H that can be extended to a best-balanced orientation of G (see Theorem 4). He mentioned that

“Given Theorem 5, the proof of Theorem 4 is not unreasonably difficult. At a certain stage in the proof of Theorem 3 ... we had occasion to select an arbitrary di-Eulerian orientation Δ of the finite Eulerian graph $G + P$ the proof of Theorem 4 depends essentially on the idea of modifying this step ... by choosing Δ to be, not just any di-Eulerian orientation of $G + P$, but one which satisfies certain additional restrictions.”

The main contribution of the present paper is to provide a simple proof for a generalization of this result, namely if $G_1, \dots, G_k$ are pairwise edge-disjoint subgraphs of a graph G , then G has a best-balanced orientation $\vec{G}$ such that the inherited orientations $\vec{G}_i$ are best-balanced orientations of G_i for all $1 \leq i \leq k$. We also have a new result about simultaneous best-balanced orientations of contractions of G .

For an Eulerian graph G we can prove more: There exist simultaneous best-balanced orientations of $G - v$ for all $v \in V$. This solves a conjecture of Frank [2], a special case of an interesting conjecture about k -vertex-connected orientations (see Conjecture 1), and generalizes a theorem of Berg and Jordán [1]. We also provide a couple of consequences of the theorems.

2. Notation, definitions

We denote a directed graph by $\vec{G} = (V, A)$ and an undirected graph by $G = (V, E)$. For a directed graph $\vec{G}$, a set $X \subseteq V$ and $u, v \in V$, let $\delta_{\vec{G}}(X) := |\{uv \in A : u \in X, v \notin X\}|$, $\varrho_{\vec{G}}(X) := \delta_{\vec{G}}(V - X)$, $f_{\vec{G}}(X) := \varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X)$, $\lambda_{\vec{G}}(u, v) := \min\{\delta_{\vec{G}}(Y) : u \in Y, v \in V - Y\}$, and $\vec{G} := (V, \{vu : uv \in A\})$.

For an undirected graph G , a set $X \subseteq V$ and $u, v \in V$, let $d_G(X) := |\{uv \in E : u \in X, v \notin X\}|$, $T_G := \{v \in V : d_G(v) \text{ is odd}\}$, $\lambda_G(u, v) := \min\{d_G(Y) : u \in Y, v \in V - Y\}$, $R_G(X) := \max\{\lambda_G(x, y) : x \in X, y \in V - X\}$ (let $R_G(\emptyset) = R_G(V) = 0$), $b_G(X) := d_G(X) - 2 \cdot \lfloor R_G(X)/2 \rfloor$, $G[X] := G - (V - X)$.

Observe that for all $X \subseteq V$, we have $0 \leq b_G(X) \leq d_G(X)$ and

$$f_{\vec{G}}(X) = \sum_{v \in X} f_{\vec{G}}(v). \quad (1)$$

Let $G = (V, E)$ be an undirected graph. G is called *k-edge-connected* if $G - F$ is connected for all $F \subseteq E$ with $|F| \leq k - 1$. In this paper, if it is not explicitly stated, graphs may be disconnected, and we use the notion *Eulerian graph* for a possibly disconnected graph with all degrees even. Similarly, a directed graph is called Eulerian if at every vertex the in-degree equals the out-degree. Let $D = (V, A)$ be a directed graph. D is *strongly connected* if for every ordered pair

$(u, v) \in V \times V$ of vertices there is a directed (u, v) -path in D . D is called *k-arc-connected* if $G - F$ is strongly connected for all $F \subseteq A$ with $|F| \leq k - 1$. D is called *k-vertex-connected* if $|V| > k$ and $G - X$ is strongly connected for all $X \subseteq V$ with $|X| \leq k - 1$. An orientation $\vec{G}$ of G is called *well-balanced* if

$$\lambda_{\vec{G}}(x, y) \geq \left\lfloor \frac{\lambda_G(x, y)}{2} \right\rfloor \quad \text{for all } (x, y) \in V \times V, \quad (2)$$

and $\vec{G}$ is called *smooth* if

$$|f_{\vec{G}}(v)| \leq 1 \quad \text{for all } v \in V. \quad (3)$$

A smooth well-balanced orientation is called *best-balanced*. Note that if $\vec{G}$ is best-balanced then so is $\overleftarrow{G}$.

A *pairing* M of G is a new graph on vertex set T_G in which each vertex has degree one. Let M be a pairing of G . An orientation $\vec{M}$ of M is called *good* if

$$f_{\vec{M}}(X) \leq b_G(X) \quad \text{for all } X \subseteq V. \quad (4)$$

M is *well-orientable* if there exists a good orientation of M , M is *strong* if every orientation of M is good and M is *feasible* if

$$d_M(X) \leq b_G(X) \quad \text{for all } X \subseteq V. \quad (5)$$

Clearly an oriented pairing $\vec{M}$ is good if and only if $\overleftarrow{M}$ is good. We say that two arc disjoint directed graphs $\vec{G}$ and $\vec{H}$ on the same vertex set V are *compatible* if

$$f_{\vec{G}}(v) = f_{\vec{H}}(v) \quad \text{for all } v \in V \quad (6)$$

or equivalently if $\vec{G} + \overleftarrow{H}$ is Eulerian.

3. Eulerian graphs

The following statements are well known and/or are easy exercises.

Proposition 1. Every undirected Eulerian graph G has an Eulerian orientation and every Eulerian orientation of G is best-balanced.

Proposition 2. If $\vec{G}^1$ and $\vec{G}^2$ are Eulerian orientations of a graph G , then $\vec{G}^2$ can be obtained from $\vec{G}^1$ by reversing the orientation of some directed cycles.

Proposition 3. The edge-set of an undirected graph G can be partitioned into some cycles and $|T_G|/2$ paths. Hence, every graph G has a pairing M such that $d_M(X) \leq d_G(X)$ for all $X \subseteq V$.

Proposition 4. $\vec{G}$ is Eulerian if and only if $f_{\vec{G}}(X) = 0$ for all $X \subseteq V$. Hence if $\vec{G}$ is Eulerian then the contracted graph $\vec{G}/X$ is also Eulerian for all $X \subseteq V$.

For an Eulerian graph G , an *edge-pairing* at vertex v is an arbitrary partition of the edges incident to v into pairs. Suppose that we are given an edge-pairing at each vertex. An Eulerian orientation is called *admissible* if at each vertex every edge-pair becomes a directed path.

Proposition 5. *We are given an Eulerian graph G and an edge-pairing at every vertex. Then there exists an admissible Eulerian orientation of G .*

A nice theorem of Ford and Fulkerson [3] about Eulerian orientations of mixed graphs implies easily the following theorem that plays an important role in this paper.

Theorem 1. *Let M be a pairing of an undirected graph G and $\vec{M}$ be a good orientation of M . Then G has an orientation $\vec{G}$ compatible with $\vec{M}$.*

4. Equivalent forms

Claim 1. *An orientation $\vec{G}$ of an undirected graph G is well-balanced if and only if*

$$f_{\vec{G}}(X) \leq b_G(X) \quad \text{for all } X \subseteq V. \quad (7)$$

Proof. Note that $b_G(X) - f_{\vec{G}}(X) = (d_G(X) - 2\lfloor R_G(X)/2 \rfloor) - (\varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X)) = 2(\delta_{\vec{G}}(X) - \lfloor R_G(X)/2 \rfloor)$. If $\vec{G}$ is well-balanced then clearly $\delta_{\vec{G}}(X) \geq \lfloor R_G(X)/2 \rfloor$, i.e. $b_G(X) \geq f_{\vec{G}}(X)$ for all X . If $\delta_{\vec{G}}(X) \geq \lfloor R_G(X)/2 \rfloor$ for all X , then, by Menger's theorem [10] and the definition of R , $\vec{G}$ is well-balanced. $\square$

Claim 2. *A pairing M of G is strong if and only if M is feasible.*

Proof. If M is feasible, then for each orientation $\vec{M}$, $f_{\vec{M}}(X) \leq d_M(X) \leq b_G(X)$ for all X by (5), that is $\vec{M}$ is good so M is strong. If M is not feasible, then let $X \subseteq V$ with $d_M(X) > b_G(X)$. Let $\vec{M}$ be an orientation of M with $\delta_{\vec{M}}(X) = 0$. Then $f_{\vec{M}}(X) = d_M(X) > b_G(X)$, that is $\vec{M}$ is not good so M is not strong. $\square$

Claim 3. *An undirected graph G has a best-balanced orientation if and only if there exists a well-orientable pairing M of G . If $\vec{M}$ is a good orientation of pairing M then there exists an orientation $\vec{G}$ compatible with $\vec{M}$, and every such orientation is best-balanced.*

Proof. We start by proving the second statement, in which the first part is the repetition of Theorem 1. As $\vec{G}$ is compatible with the oriented pairing $\vec{M}$, it is clearly smooth. By (1), (6) and (4), $f_{\vec{G}}(X) = f_{\vec{M}}(X) \leq b_G(X)$ so $\vec{G}$ is best-balanced by Claim 1.

To prove the first statement, suppose that $\vec{G}$ is best-balanced. Let $u_1, \dots, u_t$ be the vertices with $\delta_{\vec{G}}(u_i) = \varrho_{\vec{G}}(u_i) + 1$ and $v_1, \dots, v_t$ the vertices with $\delta_{\vec{G}}(v_i) = \varrho_{\vec{G}}(v_i) - 1$. Then the oriented pairing consisting of arcs $u_i v_i$ ($1 \leq i \leq t$) is compatible with $\vec{G}$ and is good by (1) and by Claim 1. The other direction follows from the second statement. $\square$

5. Theorems

The following four theorems are due to Nash-Williams [11,12].

Theorem 2 (Nash-Williams). *A graph G has a k -arc-connected orientation if and only if G is $2k$ -edge-connected.*

Theorem 3 (Nash-Williams). *Every graph has a best-balanced orientation.*

Theorem 4 (Nash-Williams). *Every subgraph H of G has a best-balanced orientation that can be extended to a best-balanced orientation of G .*

Theorem 5 (Nash-Williams). *Every graph has a feasible pairing.*

We present in the following claim the global case of the above “odd vertex pairing” theorem. A short proof of Claim 4 is given in the next section.

Claim 4. *Every $2k$ -edge-connected graph $G = (V, E)$ has a pairing M so that*

$$d_M(X) \leq d_G(X) - 2k \quad \text{for all } X \subset V, X \neq \emptyset. \quad (8)$$

By Claim 2, Theorem 5 is equivalent to Theorem 6.

Theorem 6. *Every graph has a strong pairing.*

By Theorem 3 and Claim 3, every graph has a well-orientable pairing. In the following theorem we generalize this result.

Theorem 7. *Every pairing is well-orientable.*

The main results of this paper are the following generalizations of Theorems 4 and 3.

Theorem 8. *Let $G = (V, E)$ be a graph, $\{E_1, \dots, E_k\}$ be an arbitrary partition of E and let $G_i := (V, E_i)$, $1 \leq i \leq k$. Then G has a best-balanced orientation $\vec{G}$ such that the inherited orientation $\vec{G}_i$ of each G_i is also best-balanced.*

Theorem 9. *For every partition $\{X_1, \dots, X_l\}$ of $V = V(G)$, G has an orientation $\vec{G}$ such that $\vec{G}$ and its contractions $((\vec{G}/X_1) \dots)/X_l$ and $\vec{G}/(V - X_i)$ for all $1 \leq i \leq l$, are best-balanced orientations of the corresponding graphs.*

For Eulerian graphs we have the following result.

Theorem 10. *Every Eulerian graph $G = (V, E)$ has a best-balanced orientation $\vec{G}$ such that $\vec{G} - v$ is a best-balanced orientation of $G - v$ for all $v \in V$.*

The statement of Theorem 10 is not necessarily true for non-Eulerian graphs, as the example of K_4 shows.

6. Proofs

In this section we apply Theorem 6 (or equivalently, Theorem 5) to prove all the other results in the previous section. For a relatively simple proof for Theorem 5 see [4]. A polynomial time algorithm to find a feasible pairing can be found in [6].

First, we mention that Theorems 2, 3 and 4 are easy consequences of Theorems 3, 6 and 8, respectively. We must emphasize that we do not have a new proof neither for Theorem 5 nor for Theorem 3. However, for Claim 4 we have the following simple proof.

Proof of Claim 4. If $k = 0$ then the claim is true by Proposition 3. From now on we assume that $k \geq 1$. We prove the statement by induction on $|E|$.

Case 1. There is $s \in V$ with $d(s)$ even. Then, by Lovász' splitting off theorem [8], there exists an edge-pairing $\{u_i s, s v_i\}_{i=1}^{d(s)/2}$ at s such that replacing each non-parallel pair $u_i s, s v_i$ by a new edge $u_i v_i$ and then deleting the vertex s , the new graph G' is $2k$ -edge-connected. Note that $T_{G'} = T_G$ and $|E(G')| < |E|$ so by induction there is a pairing M of G' that satisfies (8) for G' . Then M is a pairing of G and, since $d_{G'}(X) \leq d_G(X)$ for all $X \subset V$, clearly M satisfies (8) for G as well and we are done.

Case 2. Otherwise, $T_G = V$. By a result of Mader [9], since there is no vertex v with $d(v) = 2k$, there exists an edge $uv \in E$ such that $G' := G - uv$ is $2k$ -edge-connected. Note that $T_{G'} = T_G - \{u, v\}$ and $|E(G')| < |E|$ so by induction there is a pairing M' of G' so that (8) is satisfied for G' and M' . Let $M := M' \cup uv$. Then M is a pairing of G and for all $X \subseteq V$ either $d_M(X) = d_{M'}(X)$ and $d_G(X) = d_{G'}(X)$ or $d_M(X) = d_{M'}(X) + 1$ and $d_G(X) = d_{G'}(X) + 1$ so (8) is satisfied for G and M and this completes the proof. $\square$

Proof of Theorem 7. Let M_1 be an arbitrary pairing and M_2 be a strong pairing of G . M_2 exists by Theorem 6. The graph $M_1 \cup M_2$ is Eulerian so it has an Eulerian orientation $\vec{M}_1 \cup \vec{M}_2$. Then $f_{\vec{M}_1}(v) = f_{\vec{M}_2}(v)$ for all $v \in V$. Thus, by (1) and using that $\vec{M}_2$ is a good orientation of M_2 , $f_{\vec{M}_1}(X) = f_{\vec{M}_2}(X) \leq b_G(X)$ for all $X \subseteq V$, so $\vec{M}_1$ is a good orientation of M_1 . $\square$

By the above proof, if we know a feasible pairing, then for every pairing we can find a good orientation in polynomial time. Note that if we apply Theorem 4 with $H' = G$ and $G' = G + M$ we get another proof for Theorem 7.

Proof of Theorem 8. Let M_0 and M_i be strong pairings of G and of G_i for $1 \leq i \leq k$, provided by Theorem 6. Note that for $K := \bigcup_0^k M_i$, $d_K(v) = \sum_0^k d_{M_i}(v) \equiv d_G(v) + \sum_1^k d_{G_i}(v) = 2d_G(v)$ is even for all $v \in V$, so K has an Eulerian orientation $\vec{K} = \bigcup_0^k \vec{M}_i$ that is $\bigcup_1^k \vec{M}_i$ and $\vec{M}_0$ are compatible. For $1 \leq i \leq k$, $\vec{M}_i$ is a good orientation of M_i , so, by Claim 3, G_i has a best-balanced orientation $\vec{G}_i$ compatible with $\vec{M}_i$. Let $\vec{G} := \bigcup_1^k \vec{G}_i$. Then $\vec{G}$ and $\bigcup_1^k \vec{M}_i$ are compatible hence so are $\vec{G}$ and $\vec{M}_0$. Since the orientation $\vec{M}_0$ is good, $\vec{G}$ is a best-balanced orientation of G by Claim 3. $\square$

Proof of Theorem 9. Let $G_0 := (((G/X_1)/X_2)/\dots)/X_l$ and $G_i := G/(V - X_i)$ for $1 \leq i \leq l$. Let M_i be a strong pairing of G_i ($0 \leq i \leq l$) provided by Theorem 6. It is easy to see that G has a unique pairing M whose restriction in G_i is M_i for all $0 \leq i \leq l$. By Theorem 7, M has a good orientation $\vec{M}$. By Claim 3, G has a best-balanced orientation $\vec{G}$ compatible with $\vec{M}$. $\vec{G}$ and $\vec{M}$ define the orientations $\vec{G}_i$ of G_i and $\vec{M}_i$ of M_i for $0 \leq i \leq l$. Then, by Proposition 4, $\vec{G}_i$ and $\vec{M}_i$ are compatible. Since $\vec{M}_i$ is a good orientation of M_i , $\vec{G}_i$ is a best-balanced orientation of G_i by Claim 3. $\square$

Proof of Theorem 10. We define an edge-pairing for all $v \in V$ as follows. Take a maximum number of disjoint pairs of parallel edges incident to v . Since G is Eulerian, the other edges from v go to T_{G-v} . These edges can be naturally paired, defined by a strong pairing M_v of $G - v$, where M_v exists by Theorem 6. By Proposition 5 there is an admissible Eulerian orientation $\vec{G}$

of G . Let $\vec{M}_v$ be the natural orientation of M_v (for all $v \in V$) defined by $\vec{G}$; as M_v is strong, $\vec{M}_v$ is good. Now $\vec{G} - v + \vec{M}_v$ is an Eulerian orientation of $G - v + M_v$, so by Claim 3, $\vec{G} - v$ is a best-balanced orientation of $G - v$ for all $v \in V$. $\square$

7. Corollaries

Theorem 4 implies the following result for global edge-connectivity.

Corollary 1. *For a subgraph H of G , H has an l -arc-connected orientation that can be extended to a k -arc-connected orientation of G if and only if H is $2l$ -edge-connected and G is $2k$ -edge-connected.*

Note that the simple proof given for Claim 4, together with the short proof of Theorem 8 gives a direct proof for Corollary 1.

Corollary 2. *If H is an Eulerian subgraph of G , then any Eulerian orientation of H can be extended to a best-balanced orientation of G .*

Proof. By Theorem 4, H has a best-balanced orientation $\vec{H}$ that can be extended to a best-balanced orientation of G . Since $\vec{H}$ is smooth and H is Eulerian, $\vec{H}$ is an Eulerian orientation. By Proposition 2, any other Eulerian orientation of H can be reached by reversing directed cycles, and this operation cannot make the best-balanced orientation of G wrong by Claim 1. $\square$

More generally, we may consider the following problem: Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $E_1 \cap E_2 \neq \emptyset$, decide whether there exist simultaneous best-balanced orientations of G_1 and G_2 . This problem is NP-complete even if both G_1 and G_2 are restricted to be Eulerian [7]. By Corollary 2, if $E_1 \cap E_2$ defines an Eulerian graph then such orientations always exist.

Corollary 3. *Let $x, y \in V(G)$ with $\lambda_G(x, y) = 2k + 1$. Then G has a best-balanced orientation $\vec{G}$ such that $\lambda_{\vec{G}}(x, y) = k + 1$.*

Proof. Let $G' = G + xy$ and $H' = G$. Note that $\lambda_{G'}(x, y) = 2k + 2$. By applying Theorem 4 for G' and H' the corollary follows (either $\vec{G}$ or $\vec{G}$ is appropriate). $\square$

By Proposition 3 the edge-set of any undirected graph G can be decomposed into cycles and $|T_G|/2$ paths. Theorem 8 easily implies the following.

Corollary 4. *Let us fix a decomposition of the edge-set of an undirected graph G into cycles and paths. There exists a best-balanced orientation of G where all the prescribed cycles and paths become directed cycles and paths.*

As a counterpart to Theorem 9 we have the following result by Theorem 8.

Corollary 5. *For every partition $\{X_1, \dots, X_l\}$ of $V(G)$, G has an orientation $\vec{G}$ such that $\vec{G}$ and $\vec{G}[X_i]$ for all $1 \leq i \leq l$, are best-balanced orientations of the corresponding graphs.*

Finally we mention a conjecture on vertex-connectivity orientation (see in [5]), and prove a special case of it and some related statements.

Conjecture 1. *Let $G = (V, E)$ be an undirected graph with $|V| > k$. Then G has a k -vertex-connected orientation if and only if for all $X \subseteq V$ with $|X| < k$, $G - X$ is $(2k - 2|X|)$ -edge-connected.*

Corollary 1 implies at once the following.

Corollary 6. *Let $G = (V, E)$ be an undirected graph and $v \in V$. Then G has a k -arc-connected orientation $\vec{G}$ such that $\vec{G} - v$ is $(k - 1)$ -arc-connected if and only if G is $2k$ -edge-connected and $G - v$ is $(2k - 2)$ -edge-connected.*

Concerning global edge-connectivity we can replace Theorem 6 by Claim 4 in the proof of Theorem 10 and hence we have short simple proofs for the following corollaries of Theorem 10.

Corollary 7. *An Eulerian graph $G = (V, E)$ has a k -arc-connected orientation $\vec{G}$ such that $\vec{G} - v$ is $(k - 1)$ -arc-connected for all $v \in V$ if and only if $\vec{G}$ is $2k$ -edge-connected and $G - v$ is $(2k - 2)$ -edge-connected for all $v \in V$.*

The statement of Corollary 7 is not necessarily true for non-Eulerian graphs, as an example, consider the graph obtained from K_4 by replacing each edge by three parallel edges.

The following result was conjectured by Frank in [2].

Corollary 8. *An Eulerian graph $G = (V, E)$ has an Eulerian orientation $\vec{G}$ such that $\vec{G} - v$ is k -arc-connected for all $v \in V$ if and only if $G - v$ is $2k$ -edge-connected for all $v \in V$.*

For the special case of Conjecture 1 when the graph is Eulerian and $k = 2$, Berg and Jordán [1] provided a sophisticated proof. Their result below follows immediately from Corollary 8.

Corollary 9 (Berg–Jordán). *Let $G = (V, E)$ be a 4-edge-connected Eulerian graph such that $|V| \geq 3$ and $G - v$ is 2-edge-connected for all $v \in V$. Then G has a 2-vertex-connected Eulerian orientation.*

The interested readers may find many counter-examples for problems related to well-balanced orientations in [7].

Acknowledgment

We thank András Frank for many helpful discussions.

References

- [1] A.R. Berg, T. Jordán, Two-connected orientations of Eulerian graphs, J. Graph Theory, in press.
- [2] Egres Open Problems, Problem 4, http://www.cs.elte.hu/egres/problems/prob_04.
- [3] L.R. Ford, D.R. Fulkerson, Flows in Networks, Princeton Univ. Press, Princeton, 1962.
- [4] A. Frank, Applications of submodular functions, in: Surveys in Combinatorics, in: London Math. Soc. Lecture Note Ser., vol. 187, Cambridge Univ. Press, Cambridge, 1993, pp. 85–136.

- [5] A. Frank, Connectivity and network flows, in: *Handbook of Combinatorics*, vol. 1, Elsevier, Amsterdam, 1995, pp. 111–177.
- [6] H.N. Gabow, Efficient splitting off algorithms for graphs, in: *STOC '94: Proceedings of the Twenty-Sixth Annual ACM Symposium on Theory of Computing*, ACM Press, 1994, pp. 696–705.
- [7] S. Iwata, T. Király, Z. Király, Z. Szigeti, On well-balanced orientations, counter-examples for related problems, Egres Technical Reports, TR-2004-16, 2004, <http://www.cs.elte.hu/egres/www/tr-04-16.html>.
- [8] L. Lovász, *Combinatorial Problems and Exercises*, North-Holland, Amsterdam, 1979.
- [9] W. Mader, Minimale n -fach kantenzusammenhängende Graphen, *Math. Ann.* 191 (1971) 21–28.
- [10] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.* 10 (1927) 96–115.
- [11] C.St.J.A. Nash-Williams, On orientations, connectivity and odd-vertex-pairings in finite graphs, *Canad. J. Math.* 12 (1960) 555–567.
- [12] C.St.J.A. Nash-Williams, Well-balanced orientations of finite graphs and unobtrusive odd-vertex-pairings, in: *Recent Progress in Combinatorics*, Proc. Third Waterloo Conf. on Combinatorics, 1968, Academic Press, San Diego, 1969, pp. 133–149.
- [13] H.E. Robbins, A theorem on graphs with an application to a problem of traffic control, *Amer. Math. Monthly* 46 (1939) 281–283.