

Sandwich problems on orientations

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Received: 1 February 2012 / Accepted: 3 February 2012 / Published online: 28 February 2012
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Abstract The graph sandwich problem for property Π is defined as follows: Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, is there a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ which satisfies property Π ? We propose to study sandwich problems for properties Π concerning orientations, such as Eulerian orientation of a mixed graph and orientation with given in-degrees of a graph. We present a characterization and a polynomial-time algorithm for solving the m -orientation sandwich problem.

Keywords Sandwich problems · Orientation of graphs · Submodular flows

1 Introduction

Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with the same vertex set V and $E_1 \subseteq E_2$, a graph $G = (V, E)$ is called a *sandwich* graph for the pair G_1, G_2 if $E_1 \subseteq E \subseteq E_2$. The graph sandwich problem for property Π is defined as follows [12]:

GRAPH SANDWICH PROBLEM FOR PROPERTY Π

Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$.

Question: Is there a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ and G satisfies property Π ?

We call E_1 the *mandatory* edge set, $E_0 = E_2 \setminus E_1$ the *optional* edge set and E_3 the *forbidden* edge set, where E_3 denotes the set of edges of the complementary graph $\overline{G}_2$ of G_2 . Thus any sandwich graph $G = (V, E)$ for the pair G_1, G_2 must contain all mandatory edges, no forbidden edges and may contain a subset of the optional edges. Graph sandwich problems have attracted much attention lately arising from many applications and as a natural generalization of recognition problems [1–3, 7, 22, 24]. The recognition problem for a class of graphs $\mathcal{C}$ is equivalent to the graph sandwich problem in which $G_1 = G_2 = G$, where G is the graph we want to recognize and property Π is “to belong to class $\mathcal{C}$ ”.

In this paper we propose to study sandwich problems for properties Π concerning orientations, such as Eulerian orientation of a mixed graph and orientation with given in-degrees of a graph, or more generally of a mixed graph.

The paper is organized as follows: Sect. 2 contains some basic definitions, notations and results. Section 3 contains some known results on degree constrained sandwich problems. We consider the undirected version and the directed version, the complexity, the characterization and the related optimization problems. We also define a simultaneous version and discuss its complexity. Section 4 focuses on Eulerian sandwich problems. We consider first undirected graphs and then directed graphs. These problems were already solved in [12], here we point out that the undirected case reduces to T -joins, while the directed case to circulations. We discuss the complexity of the problems and their characterizations and we also propose some mixed versions.

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In Sect. 5 we consider sandwich problems regarding an m -orientation, i.e., given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$ and a non-negative integer vector m on V , we show that it is polynomial to decide whether there exists a sandwich graph $G = (V, E)$ ($E_1 \subseteq E \subseteq E_2$) that has an orientation $\vec{G}$ whose in-degree vector is m that is $d_{\vec{G}}^-(v) = m(v)$ for all $v \in V$. This result stands in contrast to the strongly connected m -orientation sandwich problem which we show is NP-complete. Section 6 is devoted to a new kind of sandwich problem where we may contract (and not delete) optional edges and property Π is being bipartite.

2 Definitions

Undirected graphs Let $G = (V, E)$ be an undirected graph. For vertex sets X and Y , the *cut* induced by X is defined to be the set of edges of G having exactly one end-vertex in X and is denoted by $\delta_G(X)$. The *degree* $d_G(X)$ (or $d_E(X)$) of X is the cardinality of the cut induced by X , that is, $d_G(X) = |\delta_G(X)|$. The number of edges between $X \setminus Y$ and $Y \setminus X$ is denoted by $d_G(X, Y)$. The number of edges of G having both (resp. at least one) end-vertices in X is denoted by $i_G(X)$ or $i_E(X)$ or simply $i(X)$ (resp. $e_G(X)$). It is well-known that (1) is satisfied for all $X, Y \subseteq V$,

$$d_G(X) + d_G(Y) = d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X, Y). \quad (1)$$

We say that a vector m on V is the *degree vector* of G if $d_G(v) = m(v)$ for all $v \in V$. For a vector m on V , we consider m as a *modular* function, that is, we use the notation $\mathbf{m}(X) = \sum_{v \in X} m(v)$. Let us recall that $d_G(X)$ is the *degree function* of G . We define $\hat{d}_G$ as the modular function defined by the degree vector $d_G(v)$ of G . Note that $\hat{d}_G(X) = d_G(X) + 2i_G(X) \forall X \subseteq V$.

We denote by T_G the set of vertices of G of odd degree. For an edge set F of G , the subgraph induced by F , that is, (V, F) , is denoted by $G(F)$. We say that G is *Eulerian* if the degree of each vertex is even, that is, if $T_G = \emptyset$. Note that we do not suppose the graph to be connected.

Let T be a vertex set in G . An edge set F of G is called *T-join* if the set of odd degree vertices in the subgraph induced by F coincide with T , that is if $T_{G(F)} = T$. Given a cost vector on the edge set of G , a minimum cost T -join can be found in polynomial time by Edmonds and Johnson's algorithm [5].

Let f be a non-negative integer vector on V . An edge set F of G is called an *f-factor* of G if f is the degree vector of $G(F)$, that is, $d_F(v) = f(v)$ for all $v \in V$. If $f(v) = 1$ for all $v \in V$, then we say that F is a 1-factor or a *perfect matching*. An *f-factor*—if it exists—can be found in polynomial time,

see [20]. The graph G is called *3-regular* if each vertex is of degree 3. Note that for a 3-regular graph, the existence of two edge-disjoint perfect matchings is equivalent to the existence of three edge-disjoint perfect matchings which is equivalent to the 3-edge-colorability of the graph.

Directed graphs Let $D = (V, A)$ be a directed graph. For a vertex set X , the set of arcs of D entering (resp. leaving) X is denoted by $\varrho_D(X)$ (resp. $\delta_D(X)$). The *in-degree* $d_D^-(X)$ (resp. *out-degree* $d_D^+(X)$) of X is the number of arcs of D entering (resp. leaving) X , that is $d_D^-(X) = |\varrho_D(X)|$ (resp. $d_D^+(X) = |\delta_D(X)|$). The set of arcs of G having both end-vertices in X is denoted by $A(X)$. The following equality will be used frequently without reference:

$$d_D^-(X) - d_D^+(X) = \sum_{v \in X} (d_D^-(v) - d_D^+(v)). \quad (2)$$

We say that a vector m on V is the *in-degree vector* of D if $d_D^-(v) = m(v)$ for all $v \in V$. Let us recall that $d_D^-(X)$ is the *in-degree function* of D . Let f be a non-negative integer vector on V . An arc set F of D is called a *directed f-factor* of D if f is the in-degree vector of $D(F)$, that is, $d_F^-(v) = f(v)$ for all $v \in V$.

We say that D is *Eulerian* if the in-degree of v is equal to the out-degree of v for all $v \in V$, that is, $d_D^-(v) = d_D^+(v)$ for all $v \in V$. Note that we do not suppose the graph to be connected.

Let f and g be two vectors on the arcs of D such that $f(e) \leq g(e)$ for all $e \in A$. A vector x on the arcs of D is a *circulation* if (3) and (4) are satisfied.

$$x(\delta_D(v)) = x(\varrho_D(v)) \quad \forall v \in V, \quad (3)$$

$$f(e) \leq x(e) \leq g(e) \quad \forall e \in A. \quad (4)$$

Note that if $f(e) = g(e) = 1$ for all $e \in A$, then D is Eulerian if and only if f is a circulation. We will use the following characterization when a circulation exists.

Theorem 1 (Hoffmann [15]) *Let $D = (V, A)$ be a directed graph and f and g two vectors on A such that $f(e) \leq g(e) \forall e \in A$. There exists a circulation in D if and only if*

$$f(\varrho_D(X)) \leq g(\delta_D(X)) \quad \forall X \subseteq V. \quad (5)$$

We say that $H = (V, E \cup A)$ is a *mixed graph* if E is an edge set and A is an arc set on V . For an undirected graph $G = (V, E)$, if we replace each edge uv by the arc uv or vu , then we get the directed graph $\vec{G} = (V, \vec{E})$. We say that $\vec{G}$ is an *orientation* of G .

Mixed graphs having Eulerian orientations are characterized as follows.

Theorem 2 (Ford, Fulkerson [8]) *A mixed graph $H = (V, E \cup A)$ has an Eulerian orientation if and only if*

$$d_A^-(v) + d_A^+(v) + d_E(v) \text{ is even } \forall v \in V, \quad (6)$$

$$d_A^-(X) - d_A^+(X) \leq d_E(X) \quad \forall X \subseteq V. \quad (7)$$

The following theorem characterizes graphs having an orientation with a given in-degree vector.

Theorem 3 (Hakimi [13]) *Given an undirected graph $G = (V, E)$ and a non-negative integer vector m on V , there exists an orientation $\vec{G}$ of G whose in-degree vector is m if and only if*

$$m(X) \geq i(X) \quad \forall X \subseteq V, \quad (8)$$

$$m(V) = |E|. \quad (9)$$

Functions Let b be a set function on the subsets of V . We say that b is *submodular* if for all $X, Y \subseteq V$,

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y). \quad (10)$$

The function b is called *supermodular* if $-b$ is submodular. A function is *modular* if it is supermodular and submodular. We will use frequently in this paper the following facts.

Claim 1 *The degree function $d_G(Z)$ of an undirected graph G and the in-degree function $d_D^-(Z)$ of a directed graph D are submodular and the function $i(Z)$ is supermodular.*

Theorem 4 [17, 21] *The minimum value of a submodular function can be found in polynomial time.*

Theorem 5 (Frank [9]) *Let b and p be a submodular and a supermodular set function on V such that $p(X) \leq b(X)$ for all $X \subseteq V$. Then there exists a modular function m on V such that $p(X) \leq m(X) \leq b(X)$ for all $X \subseteq V$. If b and p are integer valued then m can also be chosen integer valued.*

A pair (p, b) of set functions on 2^V is a *strong pair* if p (resp. b) is supermodular (submodular) and they are *compliant*, that is, for every pairwise disjoint triple X, Y, Z ,

$$b(X \cup Z) - p(Y \cup Z) \geq b(X) - p(Y).$$

Note that a pair (α, β) of modular functions is a strong pair if and only if $\alpha \leq \beta$. If (p, b) is a strong pair then the polyhedron

$$Q(p, b) = \{x \in \mathbb{R}^V : p(X) \leq x(X) \leq b(X), \\ \text{for every } X \subseteq V\}$$

is called a *generalized polymatroid* (or a *g-polymatroid*). When $\alpha \leq \beta$ are modular, we also call the g-polymatroid $Q(\alpha, \beta)$ a *box*.

Theorem 6 (Frank, Tardos [11]) *The intersection of an integral g-polymatroid $Q(p, b)$ and an integral box $Q(\alpha, \beta)$ is an integral g-polymatroid. It is nonempty if and only if $\alpha \leq b$ and $p \leq \beta$.*

Matroids A set system $M = (V, \mathcal{F})$ is called a *matroid* if $\mathcal{F}$ satisfies the following three conditions:

- (I1) $\emptyset \in \mathcal{F}$,
- (I2) if $F \in \mathcal{F}$ and $F' \subseteq F$, then $F' \in \mathcal{F}$,
- (I3) if $F, F' \in \mathcal{F}$ and $|F| > |F'|$, then there exists $f \in F \setminus F'$ such that $F' \cup f \in \mathcal{F}$.

A subset X of V is called *independent* in M if $X \in \mathcal{F}$, otherwise it is called *dependent*. The maximal independent sets of V are the *basis* of M . Let $\mathcal{B}$ be the set of basis of M . Then $\mathcal{B}$ satisfies the following two conditions:

- (B1) $\mathcal{B} \neq \emptyset$,
- (B2) if $B, B' \in \mathcal{B}$ and $b \in B \setminus B'$, then there exists $b' \in B' \setminus B$ such that $(B - b) \cup b' \in \mathcal{B}$.

Conversely, if a set system $(V, \mathcal{B})$ satisfies (B1) and (B2), then $M = (V, \mathcal{F})$ is a matroid, where $\mathcal{F} = \{F \subseteq V : \exists B \in \mathcal{B}, F \subseteq B\}$.

For $S \subset V$, the matroid $M \setminus S$ obtained from M by *deleting* S is defined as $M \setminus S = (V \setminus S, \mathcal{F}|_{V \setminus S})$, where $X \subseteq V \setminus S$ belongs to $\mathcal{F}|_{V \setminus S}$ if and only if $X \in \mathcal{F}$. For $S \in \mathcal{F}$, the matroid M/S obtained from M by *contracting* S is defined as $M/S = (V \setminus S, \mathcal{F}_S)$, where $X \subseteq V \setminus S$ belongs to $\mathcal{F}_S$ if and only if $X \cup S \in \mathcal{F}$. Let $\{V_1, \dots, V_l\}$ be a partition of V and $a_1, \dots, a_l$ a set of non-negative integers. Then $M = (V, \mathcal{F})$ is a matroid, where $\mathcal{F} = \{F \subseteq V : |F \cap V_i| \leq a_i\}$, we call it *partition matroid*. The *dual* matroid M^* of M is defined as follows: the basis of M^* are the complements of the basis of M .

Let $M = (V, \mathcal{F})$ be a matroid and c a cost vector on $V = \{v_1, \dots, v_n\}$. We can find a minimum cost basis F_n of M in polynomial time by the greedy algorithm: take a non-decreasing order of the elements of $V : c(v_1) \leq \dots \leq c(v_n)$. Let F_0 be empty and for $i = 1, \dots, n$, let $F_i = F_{i-1} + v_i$ if $F_{i-1} + v_i \in \mathcal{F}$, otherwise let $F_i = F_{i-1}$.

If M_1 and M_2 are two matroids on the same ground set V , then we can find a common basis of M_1 and M_2 in polynomial time (if there exists one) by the matroid intersection algorithm of Edmonds [4].

Theorem 7 (Edmonds, Rota [18]) *For an integer-valued, non-decreasing, submodular function b defined on a ground set S , the set $\{F \subseteq S : |F'| \leq b(F') \text{ for all } \emptyset \neq F' \subseteq F\}$ forms the set of independent sets of a matroid M_b whose rank function r_b is given by*

$$r_b(Z) = \min\{b(X) + |Z - X|, X \subseteq Z\}.$$

Given an undirected graph $G = (V, E)$ and a non-negative integer vector m on V , let $\bar{m}^G = \bar{m}$ be the set function defined on E by $\bar{m}(F) = m(V(F))$ where $V(F)$ is the set of vertices covered by F . One can easily check that $\bar{m}$ is integer valued, non-decreasing and submodular. Thus, by Theorem 7, $\bar{m}$ defines a matroid $M_{\bar{m}}$. The following claim is straightforward.

Claim 2 *The set $\{F \subseteq E : m(X) \geq i_F(X), \forall X \subseteq V\}$ is the set of independent sets of the matroid $M_{\bar{m}}$.*

3 Degree constrained sandwich problems

Before studying sandwich problems on orientations of given in-degrees, let us start as a warming up by considering sandwich problems for undirected and directed graphs of given degrees. These problems reduce to the undirected and directed f -factor problems. We mention that the directed case is much easier than the undirected case because the addition of an arc in a directed graph contributes only to the in-degree of the head and not of the tail, while the addition of an edge in an undirected graph contributes to the degree of both end-vertices. This section contains no new results, we added it for the sake of completeness.

3.1 Undirected graphs

UNDIRECTED DEGREE CONSTRAINED SANDWICH PROBLEM

Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$ and a non-negative integer vector f on V .

Question: Does there exist a sandwich graph $G = (V, E)$ ($E_1 \subseteq E \subseteq E_2$) such that $d_G(v) = f(v)$ for all $v \in V$?

Complexity: It is in P because the answer is YES if and only if there exists an $(f(v) - d_{G_1}(v))$ -factor in the optional graph $G_0 = (V, E_0)$.

Characterization: The general f -factor theorem due to Tutte [25] can be applied to get a characterization.

Optimization: The minimum cost f -factor problem in undirected graphs can be solved in polynomial time, see Schrijver [20].

SIMULTANEOUS UNDIRECTED DEGREE CONSTRAINED SANDWICH PROBLEM

Instance: Given two edge-disjoint graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ in $G_3 = (V, E_3)$ and two non-negative integer vectors f_1 and f_2 on V .

Question: Do there exist simultaneously sandwich graphs $\hat{G}_1 = (V, \hat{E}_1)$ ($E_1 \subseteq \hat{E}_1 \subseteq E_3$) and $\hat{G}_2 = (V, \hat{E}_2)$ ($E_2 \subseteq$

$\hat{E}_2 \subseteq E_3$) such that $\hat{E}_1 \cap \hat{E}_2 = \emptyset$ and $d_{\hat{G}_1}(v) = f_1(v)$ and $d_{\hat{G}_2}(v) = f_2(v)$ for all $v \in V$?

Complexity: It is NP-complete because it contains as a special case whether there exist two edge-disjoint perfect matchings so 3-edge-colorability of 3-regular graphs. Indeed, let $G = (V, E)$ be an arbitrary 3-regular graph. Let G_1 and G_2 be the edgeless graph on V , $G_3 = G$ and $f_1(v) = f_2(v) = 1$ for all $v \in V$. Then the sandwich graphs $\hat{G}_1$ and $\hat{G}_2$ exist if and only if $\hat{E}_1$ and $\hat{E}_2$ are edge-disjoint perfect matchings of G or equivalently, if there exists a 3-edge-coloring of G . Since the problem of 3-edge-colorability of 3-regular graphs is NP-complete [16], so is our problem.

3.2 Directed graphs

DIRECTED DEGREE CONSTRAINED SANDWICH PROBLEM

Instance: Given directed graphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ with $A_1 \subseteq A_2$ and a non-negative integer vector f on V .

Question: Does there exist a sandwich graph $D = (V, A)$ ($A_1 \subseteq A \subseteq A_2$) such that $d_D^-(v) = f(v)$ for all $v \in V$?

Complexity + Characterization: It is in P because the answer is YES if and only if there exists a directed $(f(v) - d_{D_1}^-(v))$ -factor in the optional directed graph $D_0 = (V, A_0)$, hence we have the following.

Theorem 8 *The DIRECTED DEGREE CONSTRAINED SANDWICH PROBLEM has a YES answer if and only if $d_{D_2}^-(v) \geq f(v) \geq d_{D_1}^-(v)$ for all $v \in V$.*

Optimization: The feasible arc sets form the basis of a partition matroid, so the greedy algorithm provides a minimum cost solution.

SIMULTANEOUS DIRECTED DEGREE CONSTRAINED SANDWICH PROBLEM 1

Instance: Given two arc-disjoint directed graphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ in $D_3 = (V, A_3)$ and two non-negative integer vectors f_1 and f_2 on V .

Question: Do there exist simultaneously sandwich graphs $\hat{D}_1 = (V, \hat{A}_1)$ ($A_1 \subseteq \hat{A}_1 \subseteq A_3$) and $\hat{D}_2 = (V, \hat{A}_2)$ ($A_2 \subseteq \hat{A}_2 \subseteq A_3$) such that $\hat{A}_1 \cap \hat{A}_2 = \emptyset$ and $d_{\hat{D}_1}^-(v) = f_1(v)$ and $d_{\hat{D}_2}^-(v) = f_2(v)$ for all $v \in V$?

Complexity: It is in P because the answer is YES if and only if $d_{D_3}^-(v) \geq f_1(v) + f_2(v)$, $f_1(v) \geq d_{D_1}^-(v)$ and $f_2(v) \geq d_{D_2}^-(v)$ for all $v \in V$.

SIMULTANEOUS DIRECTED DEGREE CONSTRAINED SANDWICH PROBLEM 2

Instance: Given directed graphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ with $A_1 \subseteq A_2$ and two non-negative integer vectors f and g on V .

Question: Does there exist a sandwich graph $D = (V, A)$ ($A_1 \subseteq A \subseteq A_2$) such that $d_D^-(v) = f(v)$ and $d_D^+(v) = g(v)$ for all $v \in V$.

Complexity: The feasible arc sets for the in-degree constraint form the basis of a partition matroid and the feasible arc sets for the out-degree constraint form the basis of a partition matroid. The answer is YES if and only if there exists a common basis in these two matroids. Thus it is in P by the matroid intersection algorithm of Edmonds [4].

4 Eulerian sandwich problems

In this section we consider first two problems that were already solved in [12]: Eulerian sandwich problems for undirected and directed graphs. We point out that the undirected case reduces to T -joins, while the directed case to circulations. We show that in both cases the simultaneous versions are NP-complete.

Then we propose to study the problem in mixed graphs. We show two cases that can be solved. The first case will be solved by the Discrete Separation Theorem 5 of Frank [9], while the second case reduces to the DIRECTED EULERIAN SANDWICH PROBLEM. The general case, however, remains open.

4.1 Undirected graphs

UNDIRECTED EULERIAN SANDWICH PROBLEM

Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$.

Question: Does there exist a sandwich graph $G = (V, E)$ ($E_1 \subseteq E \subseteq E_2$) that is Eulerian?

Complexity: It is in P because the answer is YES if and only if there exists a T_{G_1} -join in the optional graph G_0 .

Characterization: The answer is YES if and only if each connected component of G_0 contains an even number of vertices of T_{G_1} .

Optimization: The minimum cost T -join problem can be solved in polynomial time [5].

SIMULTANEOUS UNDIRECTED EULERIAN SANDWICH PROBLEM

Instance: Given two edge-disjoint graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ in $G_3 = (V, E_3)$.

Question: Do there exist simultaneously Eulerian sandwich graphs $\hat{G}_1 = (V, \hat{E}_1)$ ($E_1 \subseteq \hat{E}_1 \subseteq E_3$) and $\hat{G}_2 = (V, \hat{E}_2)$ ($E_2 \subseteq \hat{E}_2 \subseteq E_3$) such that $\hat{E}_1 \cap \hat{E}_2 = \emptyset$?

Complexity: It is NP-complete because it contains as a special case whether there exist two edge-disjoint perfect matchings so 3-colorability of 3-regular graphs. Indeed, let $G = (V, E)$ be an arbitrary 3-regular graph. Let G_3 be obtained from G by adding 2 edge-disjoint perfect matchings M_1 and M_2 to G , let $G_1 = (V, M_1)$ and $G_2 = (V, M_2)$. Then the Eulerian sandwich graphs $\hat{G}_1$ and $\hat{G}_2$ exist if and only if $\hat{E}_1 \setminus M_1$ and $\hat{E}_2 \setminus M_2$ are edge-disjoint perfect matchings of G or equivalently, if there exists a 3-edge-coloring of G . Since the problem of 3-edge-colorability of 3-regular graphs is NP-complete [16], so is our problem.

4.2 Directed graphs

DIRECTED EULERIAN SANDWICH PROBLEM

Instance: Given directed graphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ with $A_1 \subseteq A_2$.

Question: Does there exist a sandwich graph $D = (V, A)$ ($A_1 \subseteq A \subseteq A_2$) that is Eulerian?

Complexity: It is in P because it can be reformulated as a circulation problem: let $f(e) = 1, g(e) = 1$ if $e \in A_1$ and $f(e) = 0, g(e) = 1$ if $e \in A_0$. This way the arcs of A_1 are forced and the arcs of A_0 can be chosen if necessary.

Characterization: The answer is YES if and only if $d_{D_1}^-(X) \leq d_{D_2}^+(X)$ for all $X \subseteq V$ by Theorem 1.

Optimization: The minimum cost circulation problem can be solved in polynomial time, see Tardos [23].

SIMULTANEOUS DIRECTED EULERIAN SANDWICH PROBLEM

Instance: Given two arc-disjoint directed graphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ in $D_3 = (V, A_3)$.

Question: Do there exist simultaneously Eulerian sandwich graphs $\hat{D}_1 = (V, \hat{A}_1)$ ($A_1 \subseteq \hat{A}_1 \subseteq A_3$) and $\hat{D}_2 = (V, \hat{A}_2)$ ($A_2 \subseteq \hat{A}_2 \subseteq A_3$) such that $\hat{A}_1 \cap \hat{A}_2 = \emptyset$?

Complexity: It is NP-complete, it contains as a special case ($D_1 = (V, t_1 s_1)$, $D_2 = (V, t_2 s_2)$ and $D_3 = D$) the following directed 2-commodity integral flow problem that is NP-complete [6]: Given a directed graph D and two pairs of vertices, s_1, t_1 and s_2, t_2 , decide whether there exist a path from s_1 to t_1 and a path from s_2 to t_2 that are arc-disjoint.

4.3 Mixed graphs

MIXED EULERIAN SANDWICH PROBLEM

Instance: Given mixed graphs $H_1 = (V, E_1 \cup A_1)$ and $H_2 = (V, E_2 \cup A_2)$ with $E_1 \subseteq E_2, A_1 \subseteq A_2$.

Question: Does there exist a sandwich mixed graph $H = (V, E \cup A)$ ($E_1 \subseteq E \subseteq E_2, A_1 \subseteq A \subseteq A_2$) that has an Eulerian orientation?

Complexity: We provide two special cases that can be treated, while the general problem remains open.

SPECIAL CASE 1: $E_1 = E_2 = E$ and $d_{A_2}^+(X) - d_{A_1}^-(X) + \hat{d}_E(X)$ is even for all $X \subseteq V$.

Characterization + Complexity: We show that the problem is in P and we provide a characterization.

Theorem 9 *The MIXED EULERIAN SANDWICH PROBLEM with $E_1 = E_2 = E$ and $d_{A_2}^+(X) - d_{A_1}^-(X) + \hat{d}_E(X)$ is even for all $X \subseteq V$ has a YES answer if and only if*

$$d_{A_1}^-(X) - d_{A_2}^+(X) \leq d_E(X) \forall X \subseteq V. \quad (11)$$

In particular, this problem is in P.

Proof By the result of Sect. 4.2, the answer is YES if and only if there exists an orientation $\vec{E}$ of E such that, $\forall X \subseteq V$, $d_{A_1 \cup \vec{E}}^-(X) \leq d_{A_2 \cup \vec{E}}^+(X)$ or equivalently

$$d_{\vec{E}}^-(X) - d_{\vec{E}}^+(X) \leq d_{A_2}^+(X) - d_{A_1}^-(X). \quad (12)$$

Let m be the in-degree vector of $\vec{E}$. Then $d_{\vec{E}}^-(X) - d_{\vec{E}}^+(X) = \sum_{v \in X} (d_{\vec{E}}^-(v) - d_{\vec{E}}^+(v)) = \sum_{v \in X} (2d_{\vec{E}}^-(v) - d_E(v)) = 2m(X) - \hat{d}_E(X)$, and (12) becomes

$$2m(X) \leq d_{A_2}^+(X) - d_{A_1}^-(X) + \hat{d}_E(X). \quad (13)$$

Let $b(X) = \frac{1}{2}(d_{A_2}^+(X) - d_{A_1}^-(X) + \hat{d}_E(X))$. Then b , being the sum of a modular function and a submodular function ($b(X) = \frac{1}{2} \sum_{v \in X} (d_{A_1}^+(v) - d_{A_1}^-(v) + d_E(v)) + d_{A_0}^+(X)$), is a submodular function and, by the assumption, it is integer valued. By Theorem 3, an orientation $\vec{E}$ satisfying (12) exists if and only if there exists a vector m such that $i_E(X) \leq m(X) \leq b(X)$, that is, by Claim 1 and Theorem 5, if and only if $i_E(X) \leq b(X)$. This is equivalent to (11) and can be decided in polynomial time by Theorem 4, namely the submodular function $b'(X) = b(X) - i_E(X)$ must have minimum value 0. $\square$

SPECIAL CASE 2: $E_1 = \emptyset$.

Characterization + Complexity: It is in P because it can be reformulated as the following problem: We create two copies of each edge in E_2 and orient them in opposite directions. Denote this arc set by $\vec{E}_2$. It is not difficult to see that the graph $(V, E_2 \cup A_2)$ has a subgraph containing (V, A_1) with an Eulerian orientation if and only if the graph $(V, \vec{E}_2 \cup A_2)$ has a directed Eulerian subgraph containing (V, A_1) . Indeed, in such a graph, if every edge of E_2 is used at most once, we are done. If some edge of E_2 is used twice, as two arcs in opposite directions, we can just remove

these two arcs, the obtained graph remaining Eulerian and containing (V, A_1) . Now applying the result for DIRECTED EULERIAN SANDWICH PROBLEM we have

Theorem 10 *The MIXED EULERIAN SANDWICH PROBLEM with $E_1 = \emptyset$ has a YES answer if and only if*

$$d_{A_1}^-(X) - d_{A_2}^+(X) \leq d_{E_2}(X) \forall X \subseteq V. \quad (14)$$

In particular, this problem is in P.

Proof Let $D_1 = (V, A_1)$ and $D_2 = (V, A_2 \cup \vec{E}_2)$. By the arguments above, the MIXED EULERIAN SANDWICH PROBLEM with $E_1 = \emptyset$ has a solution if and only if there is an Eulerian sandwich graph for D_1 and D_2 or equivalently, $d_{D_1}^-(X) \leq d_{D_2}^+(X)$ for all $X \subseteq V$. By $d_{D_2}^+(X) = d_{A_2}^+(X) + d_{E_2}(X)$, we have $d_{A_1}^-(X) - d_{A_2}^+(X) \leq d_{E_2}(X)$ for all $X \subseteq V$. Note that $d_{E_2}(X) + d_{A_2}^+(X) - d_{A_1}^-(X)$ is a submodular function, and hence by Theorem 4, (14) can be verified in polynomial time. $\square$

5 m -orientation sandwich problems

In this section we consider the sandwich problem where the property Π is to have an orientation of given in-degrees.

5.1 m -Orientation

m -ORIENTATION SANDWICH PROBLEM

Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$ and a non-negative integer vector m on V .

Question: Does there exist a sandwich graph $G = (V, E)$ ($E_1 \subseteq E \subseteq E_2$) that has an orientation $\vec{G}$ whose in-degree vector is m that is $d_{\vec{G}}^-(v) = m(v)$ for all $v \in V$?

Characterization: We prove the following theorem.

Theorem 11 *The following assertions are equivalent.*

- The m -ORIENTATION SANDWICH PROBLEM has a YES answer.*
- E_1 is independent in $M_{\vec{m}}$ and $M_{\vec{m}}$ has an independent set of size $m(V)$.*
- $r_{\vec{m}}(E_1) = |E_1|$ and $r_{\vec{m}}(E_2) \geq m(V)$.*
- $i_{E_1}(X) \leq m(X) \leq e_{E_2}(X)$ for all $X \subseteq V$.*

Proof

(a) Implies (d). Let $X \subseteq V$. Since each edge of G_1 in X contributes 1 to $m(X)$, we have $i_{E_1}(X) \leq m(X)$. On the other hand, the edges of G_2 that have no end-vertex in X cannot contribute 1 to $m(X)$, so we have $m(X) \leq e_{E_2}(X)$.

(d) Implies (c). Let F be a subset of E_1 and $X = V(F)$. The condition $i_{E_1}(X) \leq m(X)$ implies $|F| \leq m(V(F)) = \bar{m}(F)$, that is, $|E_1| \leq \bar{m}(F) + |E_1 \setminus F|$. By Theorem 7, $r_{\bar{m}}(E_1) \geq |E_1|$, or equivalently $r_{\bar{m}}(E_1) = |E_1|$. Let now F be a subset of E_2 and $X = V \setminus V(F)$. The condition $m(X) \leq e_{E_2}(X)$ implies that $m(V) \leq m(V(F)) + e_{E_2}(V - V(F)) \leq \bar{m}(F) + |E_2 \setminus F|$. By Theorem 7, $r_{\bar{m}}(E_2) \geq m(V)$.

(c) Implies (b). By definition.

(b) Implies (a). By (b), E_1 is independent in $M_{\bar{m}}$ and there exists an independent in $M_{\bar{m}}$ of size $m(V)$. Therefore, by (I3), there exists an independent set E of size $m(V)$ that contains E_1 . By Theorem 3 and Claim 2, E is a solution of the m -ORIENTATION SANDWICH PROBLEM. $\square$

We say that a subset F of E_0 is *feasible* if $(V, F \cup E_1)$ has an m -orientation. The next corollary of Theorem 11 characterizes the feasible sets.

Corollary 1 *If the m -ORIENTATION SANDWICH PROBLEM has a YES answer, then a subset F of E_0 is feasible if and only if F is a base of the matroid $M_{\bar{m}}/E_1$.*

Complexity: The condition (d) of Theorem 11 can be verified in polynomial time by Theorem 4, so the m -ORIENTATION SANDWICH PROBLEM is in P.

Optimization: The minimum cost version of the problem can be solved in polynomial time. First, we find an optimal feasible subset F by greedy algorithm. Then we can orient the edges of $F \cup E_1$ using a known algorithm. (See [10] for example.)

Corollary 1 and the matroid intersection algorithm of Edmonds [4] imply that the two following simultaneous versions of the m -ORIENTATION SANDWICH PROBLEM are also in P.

SIMULTANEOUS m -ORIENTATION SANDWICH PROBLEM 1

Instance: Given two edge-disjoint undirected subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ of an undirected graph $G_3 = (V, E_3)$ and two non-negative integer vectors m_1 and m_2 on V .

Question: Do there exist simultaneously edge-disjoint sandwich graphs $\hat{G}_1 = (V, \hat{E}_1)$ ($E_1 \subseteq \hat{E}_1 \subseteq E_3$) and $\hat{G}_2 = (V, \hat{E}_2)$ ($E_2 \subseteq \hat{E}_2 \subseteq E_3$) such that $\hat{G}_i$ has an orientation whose in-degree vector is m_i for $i \in \{1, 2\}$?

Note that the two input matroids for the matroid intersection algorithm must be taken as $(M_{\bar{m}_1}^{G_1}/E_1) \setminus E_2$ and the dual matroid of $(M_{\bar{m}_2}^{G_2}/E_2) \setminus E_1$.

SIMULTANEOUS m -ORIENTATION SANDWICH PROBLEM 2

Instance: Given two undirected subgraphs $G_1 = (V, E_1)$

and $G_2 = (V, E_2)$ of an undirected graph $G_3 = (V, E_3)$ and two non-negative integer vectors m_1 and m_2 on V .

Question: Does there exist an edge set F in $E_3 \setminus (E_1 \cup E_2)$ such that the graph $G_i = (V, E_i \cup F)$ admits an orientation whose in-degree vector is m_i for $i \in \{1, 2\}$?

5.2 Strongly connected m -orientation

STRONGLY CONNECTED m -ORIENTATION SANDWICH PROBLEM

Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$ and a non-negative integer vector m on V .

Question: Does there exist a sandwich graph $G = (V, E)$ ($E_1 \subseteq E \subseteq E_2$) that has a strongly connected orientation $\vec{G}$ whose in-degree function is m ?

Complexity: It is NP-complete because the special case $E_1 = \emptyset, m(v) = 1 \forall v \in V$ is equivalent to decide if G_2 has a Hamiltonian cycle.

5.3 (m_1, m_2) -orientation

(m_1, m_2) -ORIENTATION SANDWICH PROBLEM

Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$ and non-negative integer vectors m_1 and m_2 on V .

Question: Does there exist a sandwich graph $G = (V, E)$ ($E_1 \subseteq E \subseteq E_2$) that has an orientation $\vec{G}$ whose in-degree vector is m_1 and whose out-degree vector is m_2 ?

Complexity: The problem is NP-complete since it contains as a special case ($E_1 = \emptyset$) the NP-complete problem of [19].

5.4 Mixed m -orientation

MIXED m -ORIENTATION SANDWICH PROBLEM

Instance: Given mixed graphs $G_1 = (V, E_1 \cup A_1)$ and $G_2 = (V, E_2 \cup A_2)$ with $E_1 \subseteq E_2, A_1 \subseteq A_2$ and an non-negative integer vector m on V .

Question: Does there exist a sandwich mixed graph $G = (V, E \cup A)$ with $E_1 \subseteq E \subseteq E_2$ and $A_1 \subseteq A \subseteq A_2$ that has an orientation $\vec{G} = (V, \vec{E} \cup A)$ whose in-degree vector is m ?

Characterization: Suppose that $E_1 \subseteq E \subseteq E_2$ has been chosen and oriented, then the problem is reduced to the DIRECTED DEGREE CONSTRAINED SANDWICH PROBLEM with $m_1(v) = m(v) - d_{\vec{E}}^-(v)$ which, by Theorem 8, has a solution if and only if $d_{A_2}^-(v) \geq m(v) - d_{\vec{E}}^-(v) \geq d_{A_1}^-(v)$ for all $v \in V$. Hence the MIXED m -ORIENTATION SANDWICH PROBLEM has a solution if and only if there exists $E_1 \subseteq E \subseteq E_2$ which admits an orientation $\vec{E}$ with

$m(v) - d_{A_1}^-(v) \geq d_{\vec{E}}^-(v) \geq m(v) - d_{A_2}^-(v)$ for all $v \in V$. Let $m_2 : V \rightarrow \mathbb{Z}$ satisfy $m(v) - d_{A_2}^-(v) \leq m_2(v) \leq m(v) - d_{A_1}^-(v)$. By Theorem 11, there exists $E_1 \subseteq E \subseteq E_2$ which admits an orientation $\vec{E}$ with $d_{\vec{E}}^-(v) = m_2(v)$ if and only if $i_{E_1}(X) \leq m_2(X) \leq e_{E_2}(X)$ for all $X \subseteq V$. Therefore we have

Claim 3 *The MIXED m -ORIENTATION SANDWICH PROBLEM has a YES answer if and only if there exists an integer-valued function $m_2 : V \rightarrow \mathbb{Z}$ such that, $\forall v \in V$ and $\forall X \subseteq V$,*

$$m(v) - d_{A_2}^-(v) \leq m_2(v) \leq m(v) - d_{A_1}^-(v),$$

$$i_{E_1}(X) \leq m_2(X) \leq e_{E_2}(X).$$

Claim 4 *The pair (i_{E_1}, e_{E_2}) is a strong pair.*

Proof Let X, Y, Z be three pairwise disjoint subset of V . We show that $e_{E_2}(X \cup Z) - i_{E_1}(Y \cup Z) \geq e_{E_2}(X) - i_{E_1}(Y)$. In fact, we have $i_{E_1}(Y \cup Z) - i_{E_1}(Y) = i_{E_1}(Z) + d_{E_1}(Y, Z) \leq i_{E_2}(Z) + d_{E_2}(Y, Z)$, and $e_{E_2}(X \cup Z) - e_{E_2}(X) = i_{E_2}(Z) + d_{E_2}(Z) - d_{E_2}(X, Z)$. As X, Y, Z are pairwise disjoint, $d_{E_2}(Y, Z) + d_{E_2}(X, Z) \leq d_{E_2}(Z)$. The claim follows by Claim 1. $\square$

By Claim 3, 4 and Theorem 6 applied for $\alpha(v) = m(v) - d_{A_2}^-(v)$, $\beta(v) = m(v) - d_{A_1}^-(v)$, $p = i_{E_1}$, $b = e_{E_2}$, we have

Theorem 12 *The MIXED m -ORIENTATION SANDWICH PROBLEM has a YES answer if and only if*

$$i_{E_1}(X) + \hat{d}_{A_1}^-(X) \leq m(X) \leq e_{E_2}(X) + \hat{d}_{A_2}^-(X) \quad (15)$$

for every subset X of V .

Note that Theorem 12 implies Theorems 8 and 11.

Complexity: The condition (15) can be verified in polynomial time by Theorem 4. If it is satisfied, then a vector m_2 satisfying the conditions in Claim 3 can be found using a greedy algorithm for g-polymatroids. Then we find and orient an edge set E ($E_1 \subseteq E \subseteq E_2$) with in-degree m_2 (m -ORIENTATION SANDWICH PROBLEM). Last, we choose an arc set A ($A_1 \subseteq A \subseteq A_2$) such that $d_A^-(v) = m_1(v) = m(v) - m_2(v)$, for all $v \in V$ (DIRECTED DEGREE CONSTRAINED SANDWICH PROBLEM).

6 Contracting sandwich problems

In this section, we propose to consider a new type of sandwich problem. Instead of deleting edges from the optional graph, we are interested in contracting edges. We solve the problem for the property Π being a bipartite graph.

CONTRACTING SANDWICH PROBLEM

Instance: Given an undirected graph $G = (V, E)$ and $E_0 \subseteq E$.

Question: Does there exist $F \subseteq E_0$ such that contracting F results in a bipartite graph?

Complexity: Since a graph is bipartite if and only if all its cycles have an even length, the problem is equivalent to finding $F \subseteq E_0$ such that, for all cycles C , $|C \cap F| \equiv |C| \pmod{2}$.

Fix a spanning forest T of G . For $e \in E \setminus T$, denote $C(T, e)$ the unique cycle contained in $T \cup e$. By [18, Theorem 9.1.2], if C is a cycle of G then $C = \Delta_{e \in C} C(T, e)$, where Δ denotes the symmetric difference of sets. Therefore, $|C \cap F| \equiv \sum_{e \in C} |C(T, e) \cap F| \pmod{2}$. Let $\mathcal{C}_T$ denote the collection of cycles $C(T, e)$ of G . The problem is reduced to finding $F \subseteq E_0$ such that, for all $C \in \mathcal{C}_T$, $|C \cap F| \equiv |C| \pmod{2}$, or equivalently, finding an $F' (= E \setminus F) \supseteq E_1 = E \setminus E_0$ such that $|F' \cap C| \equiv 0 \pmod{2}$, for all $C \in \mathcal{C}_T$.

Consider now the matrix M defined as the following. The rows of M correspond to $C \in \mathcal{C}_T$ and the columns correspond to the edges of G ; the entry M_{Ce} is 1 if $e \in C$ and is 0 otherwise. For $X \subseteq E$, let χ_X denote the characteristic vector of X . For a vector $x \in \{0, 1\}^E$, let $x|_X$ denote the projection of x on X . Let $\mathbf{1}$ be the all-one vector in $\{0, 1\}^E$. A subset $F' \subseteq E$ satisfies $|F' \cap C| \equiv 0 \pmod{2}$, for all $C \in \mathcal{C}_T$, if and only if $\chi_{F'} \in \text{Ker} M$ in $\mathbb{F}_2$. Such an F' is the solution of the CONTRACTING SANDWICH PROBLEM if and only if $\chi_{F'|E_1} = \mathbf{1}|_{E_1}$.

Let B be a basis of the kernel of M in $\mathbb{F}_2$. (This can be computed in polynomial time using the Gauss elimination.) Consider the projections B' of B on E_1 . Then the CONTRACTING SANDWICH PROBLEM has a solution if and only if $\mathbf{1}|_{E_1}$ is in the subspace of $\{0, 1\}^{E_1}$ spanned by B' , that is, $\text{rank} B' = \text{rank} B' \cup \mathbf{1}|_{E_1}$. This can be decided in polynomial time using the Gauss elimination. We conclude that the CONTRACTING SANDWICH PROBLEM is in P.

We finish with a related problem. For a fixed integer k , solving the CONTRACTING SANDWICH PROBLEM when $E_0 = E$ with extra requirement $|F| \leq k$ is known to be tractable in polynomial time [14]. However, the authors mention that finding a solution of minimum cardinality is NP-complete.

Acknowledgements This research was partially supported by CNPq, CAPES (Brazil)/COFECUB (France), FAPERJ.

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