Matroid-reachability-based decomposition into arborescences

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May 10, 2024

Abstract

The problem of matroid-reachability-based packing of arborescences was solved by Király. Here we solve the corresponding decomposition problem that turns out to be more complicated. The result is obtained from the solution of the more general problem of matroid-reachability-based (ℓ, ℓ')-limited packing of arborescences where we are given a lower bound ℓ and an upper bound ℓ' on the total number of arborescences in the packing. The problem is considered for branchings and in directed hypergraphs as well.

1 Introduction

Packing and Covering is an important and well-studied subject of Combinatorial Optimization. In graphs, packing problems consist of fitting as many non-overlapping subgraphs of a given type as possible in the input graph, while covering problems aim to cover the whole graph with such subgraphs possibly allowing overlaps. A packing which is also a covering is called a decomposition. Some of the classic results of the area are about packing trees and packing arborescences. Relevant applications include evacuation problems [16], rigidity problems [25], [17] and robustness problems in networks [4]. While Nash-Williams [20], and independently Tutte [26], characterized graphs having a packing of k spanning trees, Edmonds [4] characterized digraphs having a packing of k spanning arborescences. Frank noted in [8] that the result of Nash-Williams and Tutte can be obtained from the result of Edmonds via an orientation theorem. The covering problems, covering the edge set of a graph by forests and covering the arc set of a directed graph by branchings, were solved by Nash-Williams [21] and by Frank [9]. We mention that it is well-known that the previous corresponding packing and covering problems are equivalent (see Section 10 in [11]). When spanning arborescences do not exist one may instead be interested in packing reachability arborescences. Kamiyama, Katoh, and Takizawa [16] gave a surprising extension of Edmonds' theorem on packing reachability arborescences.

To solve a rigidity problem, Katoh and Tanigawa [17] introduced and solved the problem of matroid-based packing of rooted trees, in which given a graph and a matroid on a multiset of its vertices, we want a packing of rooted-trees such that for every vertex v of the graph, the root-set of the rooted-trees containing v forms a basis of the matroid. The corresponding problem in directed graphs, matroid-based packing of arborescences was solved by Durand de Gevigney, Nguyen, and Szigeti [5]. We pointed out in [5] how the result of Katoh and Tanigawa [17] can be obtained from its directed counterpart given in [5] via an orientation theorem of Frank. Katoh and Tanigawa [17] also solved the problem of matroid-based rooted tree decomposition. The problem of matroid-based decomposition into arborescences was not considered in [5], we will solve it in this paper. A common generalization of the results of Kamiyama, Katoh, and Takizawa [16] and Durand de Gevigney, Nguyen, and Szigeti [5] was given by Király [18], namely a characterization of the existence of a matroid-reachability-based packing of arborescences, where instead of the condition having the root-set of the arborescences containing any given vertex be a basis of the matroid, it must be a basis of the restriction of the matroid to the set of vertices from which that vertex is reachable in the original directed graph. Later the result of Király [18] was further refined by Gao and Yang [14]. We will use this refinement to get a TDI description of the polyhedron of the subgraphs that admit a matroid-reachability-based packing of arborescences. This and the strong duality theorem allow us to solve the problem of matroid-reachability-based (ℓ, ℓ')-limited packing of arborescences where we are given a lower bound ℓ and an upper bound ℓ' on the total number of arborescences in the packing. This in turn will easily imply the solution of the problem of matroid-reachability-based decomposition into arborescences.

We mention that all these results were extended to hypergraphs, namely packing spanning hypertrees by Frank, Király, and Kriesell [13], packing spanning hyperarborescences by Frank, Király, and Király [12], packing reachability hyperarborescences by Bérczi and Frank [1], matroid-based packing of rooted hypertrees, matroid-based packing of hyperarborescences, matroid-reachability-based packing of hyperarborescences by Fortier et al. [7]. The problems of matroid-based decomposition into hyperarborescences and matroid-reachability-based decomposition into hyperarborescences will be treated in this paper.

Along the presentation of our results, we will show how they imply the previous results of the field (see Figure 1). While describing those implications, we will only show the sufficiency as the necessity can easily be obtained directly.

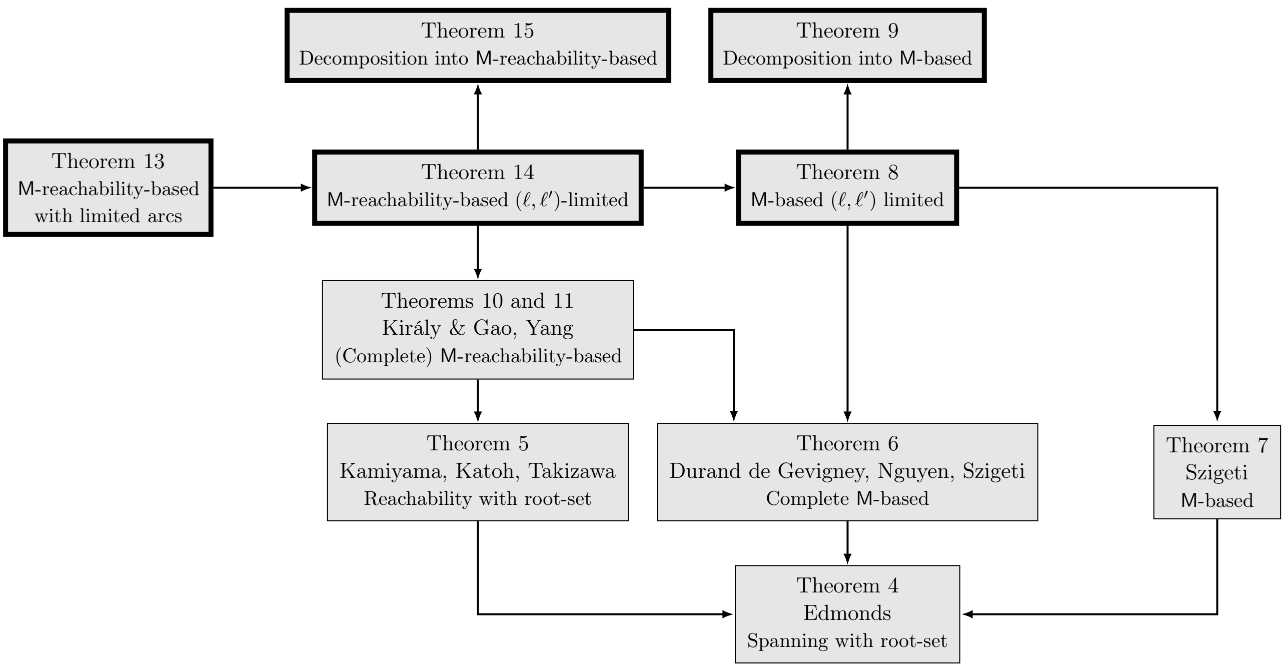


Figure 1: Our results (in bold) on packings of arborescences and their implications.

2 Definitions

Two subsets of a set V are called *intersecting* if their intersection is non-empty. A set of mutually disjoint subsets of V is called a *subpartition* of V. For a subpartition $\mathcal{P}$ of V, $\boldsymbol{\cup\mathcal{P}}$ denotes the set of elements of V that belong to some member of $\mathcal{P}$. For a multiset S of V and a subset X of V, $\boldsymbol{S_X}$ denotes the multiset consisting of the elements of X with the same multiplicities as in S. For a family $\mathcal{S}$ of subsets of V and a subset X of V, $\boldsymbol{\mathcal{S}_X}$ denotes the members of $\mathcal{S}$ that intersect X.

Let $D=(V,A)$ be a directed graph, shortly *digraph*. A subgraph of D that contains all the vertices of D is called a *spanning subgraph* of D. By a *packing* of subgraphs in D, we mean a set of subgraphs that are arc-disjoint. For a subset X of V, the *in-degree* of X, denote by $\boldsymbol{d_A^-(X)}$, is the number of arcs entering X. For a subpartition $\mathcal{P}$ of V, we denote by $\boldsymbol{e_A(\mathcal{P})}$ the set of arcs in A that enters at least one member of $\mathcal{P}$. By an *atom* of D we mean a strongly-connected component of D, a *subatom* is a non-empty subset of an atom of D.

A digraph (U,F) is called an S*-branching* if $S\subseteq U$ and there exists a unique (S,v)-path for every $v\in U$. The vertex set S is called the *root set* of the S-branching. If $S=\{s\}$, then the S-branching is an s*-arborescence* where the vertex s is called the *root* of the s-arborescence. A subgraph (U,F) of D is called *reachability* s*-arborescence* if it an s-arborescence and U is the set of vertices that can be attained from s by a path in D. For a subset X of V, we denote by $\boldsymbol{P_X}$ or $\boldsymbol{P_X^D}$ the set of vertices from which there exists a path to at least one vertex of X in D. For $\ell,\ell'\in\mathbb{Z}_+$, a packing of branchings is (ℓ,ℓ')*-limited* if the total number of the arborescences in the packing is at least ℓ and at most ℓ'.

A set function r on a set V is called *monotone* if for all $X\subseteq Y\subseteq V$, we have $r(X)\le r(Y)$. We say that r is *subcardinal* if $r(X)\le|X|$ for every $X\subseteq V$. Set functions b and p on V are called *submodular* and *supermodular* if for all $X,Y\subseteq V$, (1) and (2) hold, respectively. We say that b and p are *intersecting submodular* and *intersecting supermodular* if for all intersecting subsets X and Y of V, (1) and (2) hold, respectively.

$$b(X)+b(Y) \ \ge\ b(X\cap Y)+b(X\cup Y), \tag{1}$$
$$p(X)+p(Y) \ \le\ p(X\cap Y)+p(X\cup Y). \tag{2}$$

Let S be a finite ground set and $r:S\to\mathbb{Z}_+$ a non-negative integer valued function on S such that $r(\emptyset)=0$, r is subcardinal, monotone and submodular. Then $\mathsf{M}=(S,r)$ is called a *matroid*. The function r is the *rank function* of the matroid M. For a matroid M, its rank function will be denoted by $\boldsymbol{r_{\mathsf{M}}}$. An *independent set* of M is a subset X of S such that $r_{\mathsf{M}}(X)=|X|$. The set of independent sets of M is denoted by $\boldsymbol{\mathcal{I}_{\mathsf{M}}}$. A maximal independent set of M is called a *basis* of M. Every basis of M has size $r_{\mathsf{M}}(S)$. For a subset S' of S, a maximal independent set in S' is called a *basis* of S'. We say that two elements s and s' of S are *parallel* if $r_{\mathsf{M}}(s)=r_{\mathsf{M}}(s')=r_{\mathsf{M}}(\{s,s'\})$. The *free matroid* on S is the matroid where the only basis is the ground set S. For a given partition $\mathcal{P}$ of S and a positive integer a_i for every member X_i of $\mathcal{P}$, the *partition matroid* $\mathsf{M}^a_{\mathcal{P}}$ is the matroid whose rank function is $r_{\mathsf{M}^a_{\mathcal{P}}}(S')=\sum_{X_i\in\mathcal{P}}\min\{|S'\cap X_i|,a_i\}$ for every $S'\subseteq S$.

In a directed graph $D=(V,A)$, let S be a multiset of V and M a matroid on S. A packing $\mathcal{B}$ of arborescences in D is called M*-based* or *matroid-based* if every $s\in S$ is the root of at most one arborescence in the packing and for every vertex $v\in V$, the multiset $\boldsymbol{R_v^{\mathcal{B}}}$ of the roots of the arborescences containing v in the packing forms a basis of M. A packing $\mathcal{B}$ of arborescences in D is called M*-reachability-based* or *matroid-reachability-based* if every $s\in S$ is the root of at most one arborescence in the packing and for every vertex $v\in V$, the multiset $R_v^{\mathcal{B}}$ of the roots of the arborescences containing v in the packing forms a basis of $S_{P_v^D}$ in M. A packing of arborescences is *complete* if every $s\in S$ is the root of exactly one arborescence in the packing.

A biset X on a set V is an ordered pair (X_O, X_I) of subsets of V such that $X_I \subseteq X_O$. We call X_O and X_I the *outer set* and the *inner set* of X, while $\boldsymbol{X_W} = X_O - X_I$ is called the *wall* of X. We say that an arc uv *enters* a biset X if $u \in V - X_O$ and $v \in X_I$. The set of arcs in A entering a biset X is denoted by $\boldsymbol{\delta_A^-(\mathsf{X})}$ and $\mathbf{d}_A^-(\mathsf{X}) = |\delta_A^-(\mathsf{X})|$. For two bisets $\mathsf{X} = (X_O, X_I)$ and $\mathsf{Y} = (Y_O, Y_I)$, the *intersection* $\mathsf{X} \cap \mathsf{Y}$ of X and Y is the biset $(X_O \cap Y_O, X_I \cap Y_I)$ and the *union* $\mathsf{X} \cup \mathsf{Y}$ of X and Y is the biset $(X_O \cup Y_O, X_I \cup Y_I)$. A function on bisets is called a *biset function*. Biset function p is called *positively intersecting supermodular* if (3) holds for all bisets X and Y with $\mathsf{p}(\mathsf{X}), \mathsf{p}(\mathsf{Y}) > 0$ and $X_I \cap Y_I \neq \emptyset$.

$$\mathsf{p}(\mathsf{X}) + \mathsf{p}(\mathsf{Y}) \quad \leq \quad \mathsf{p}(\mathsf{X} \cap \mathsf{Y}) + \mathsf{p}(\mathsf{X} \cup \mathsf{Y}). \tag{3}$$

In a directed graph $D = (V, A)$, we call a vertex set Z a *petal* if there exists an atom C of D such that $Z \cap C \neq \emptyset, Z \subseteq P_C$ and $d_A^-(Z - C) = 0$. Note that if Z is a petal, then the atom C_Z is uniquely defined and $P_{C_Z} = P_Z$. The *core* of a petal Z is $C_Z \cap Z$. Let $\boldsymbol{\hat{\mathcal{Z}}_D}$, shortly $\boldsymbol{\hat{\mathcal{Z}}}$, be the set of petals in D. We define the biset $\mathbf{X}_{\boldsymbol{Z}} = (Z, C_Z \cap Z)$ for every petal Z and we call it a *petal biset*. Let $\boldsymbol{\hat{\mathcal{Z}}_\mathsf{b}}$ be the set of petal bisets, that is $\hat{\mathcal{Z}}_\mathsf{b} = \{\mathsf{X}_Z : Z \in \hat{\mathcal{Z}}\}$. Note that for every $\mathsf{X} \in \hat{\mathcal{Z}}_\mathsf{b}$, X_I is the core of the petal X_O. More generally, let $\boldsymbol{\mathcal{X}}$ be the set of bisets X on V, called *generalized petal bisets*, such that X_O is a petal and X_I is a non-empty *subset* of the core of the petal X_O. Two petals Z and Z' are called *core-intersecting* if their cores intersect (see Figure 2a). Note that two petals may intersect without being core-intersecting (see Figure 2b). We say that a set $\mathcal{Z}$ of petals is *core-laminar* if for all core-intersecting $Z, Z' \in \mathcal{Z}$, we have $Z \subseteq Z'$ or $Z' \subseteq Z$. More generally, bisets $\mathsf{X}^1, \mathsf{X}^2 \in \mathcal{X}$ are called *core-intersecting* if their petals X_O^1 and X_O^2 are core-intersecting. We say that $\mathcal{P} \subseteq \mathcal{X}$ is *OW-laminar* if for all core-intersecting $\mathsf{X}^1, \mathsf{X}^2 \in \mathcal{P}$, we have $X_O^1 \subseteq X_W^2$ or $X_O^2 \subseteq X_W^1$. Note that a biset on an atom is an element X of $\mathcal{X}$ for which the petal X_O of X coincide with the core of X_O.

(a) Two core-intersecting petals (b) Two intersecting petals that are not core-intersecting

Figure 2: Examples of petals. The dashed areas correspond to the cores of the petals.

3 Total dual integrality

The solution of our problems will rely on the polyhedral description of the subgraphs of a given digraph D, that admit an M-reachability-based packing of arborescences. We will use a TDI description of the polyhedron in question. To be able to do that we need some properties of TDI systems.

A linear system $Ax \leq b$ where A is a rational matrix and b a rational vector is called *Totally Dual Integral* (**TDI**) if the dual linear program of $\max\{c^T x : Ax \leq b\}$ has an integer-valued optimal solution for every integral vector c for which the dual has a feasible solution.

The seminal result of Edmonds, Giles [6] on TDIness is the following.

Theorem 1 (Edmonds, Giles [6], Corollary 22.1b in [22])**.** *Let $Ax \leq b$ be a TDI-system where A is an integral matrix and b is an integral vector. If $\max\{c^T x : Ax \leq b\}$ is finite, then it has an integral optimal solution.*

The TDI description of the polyhedron we are interested in uses bisets. The following result result of Frank [10] will hence play an important role.

Theorem 2 (Frank, Theorem 5.3 in [10])**.** *Let $D = (V, A)$ be a digraph and p a positively intersecting supermodular biset function on V such that $\mathsf{d}^-_A(\mathsf{X}) \geq \mathsf{p}(\mathsf{X})$ for every biset X on V. Then the following linear system is TDI:*

$$\begin{aligned} x(\delta^-_A(\mathsf{X})) &\geq \mathsf{p}(\mathsf{X}) \qquad \text{for every biset } \mathsf{X} \text{ on } V, \\ \mathbb{1} \geq x &\geq \mathbb{0}. \end{aligned}$$

We also need the following simple observation on TDI systems given by Schrijver in [22].

Theorem 3 (Schrijver, (41) in [22])**.** *If $A_1x \leq b_1$ and $A_2x \leq b_2$ define the same polyhedron, and each inequality of $A_1x \leq b_1$ is a non-negative integral combination of inequalities in $A_2x \leq b_2$, then $A_1x \leq b_1$ is TDI implies that $A_2x \leq b_2$ is TDI.*

4 Packings in directed graphs

In this section we consider packings of arborescences in digraphs. We present known and new results on packing spanning arborescences, packing reachability arborescences, matroid-based packing of arborescences, and matroid-reachability-based packing of arborescences in different subsections.

4.1 Packing arborescences

Let us start with the seminal result of Edmonds on packing spanning arborescences.

Theorem 4 (Edmonds [4])**.** *Let $D = (V, A)$ be a digraph and S a multiset of vertices in V. There exists a packing of spanning s-arborescences ($s \in S$) in D if and only if*

$$|S_X| + d^-_A(X) \geq |S| \qquad \text{for every non-empty } X \subseteq V. \tag{4}$$

It is worth mentioning that to have a packing of spanning s-arborescences ($s \in S$) in D it is sufficient that (4) holds for every subatom.

The following nice extension of Theorem 4 about packing of reachability arborescences was given in [16].

Theorem 5 (Kamiyama, Katoh, Takizawa [16])**.** *Let $D = (V, A)$ be a digraph and S a multiset of vertices in V. There exists a packing of reachability s-arborescences ($s \in S$) in D if and only if*

$$|S_X| + d^-_A(X) \geq |S_{P^D_X}| \qquad \text{for every } X \subseteq V. \tag{5}$$

Theorem 5 implies Theorem 4 because (4) implies that each reachability s-arborescence is spanning and it also implies that (5) holds. We mention that Hörsch and Szigeti [15] pointed out that Theorem 5 can be obtained from Edmonds' result on packing spanning branchings (see Theorem 16) by an easy induction.

4.2 Matroid-based packing of arborescences

The directed counterpart of the problem of matroid-based packing of rooted trees of Katoh and Tanigawa [17], the problem of matroid-based packing of arborescences, was solved in [5].

Theorem 6 (Durand de Gevigney, Nguyen, Szigeti [5]). *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V, and $\mathsf{M} = (S, \mathcal{I}_{\mathsf{M}})$ a matroid with rank function r_{M}. There exists a complete M-based packing of arborescences in D if and only if*

$$
\begin{aligned}
S_v &\in \mathcal{I}_{\mathsf{M}} && \text{for every } v \in V, && (6)\\
r_{\mathsf{M}}(S_Z) + d_A^-(Z) &\geq r_{\mathsf{M}}(S) && \text{for every non-empty } Z \subseteq V. && (7)
\end{aligned}
$$

For the free matroid, Theorem 6 reduces to Theorem 4. We mention that (7) is equivalent to

$$r_{\mathsf{M}}(S_X) + d_A^-(X) \geq r_{\mathsf{M}}(S) \quad \text{for every subatom } X \text{ of } D. \tag{8}$$

Indeed, (7) trivially implies (8). To see that (8) implies (7), let Z be a non-empty subset of V. Let $V_1, \dots, V_t$ be a topological ordering of the atoms of D that is if an arc exists from V_i to V_j, then $i < j$. Let i be the smallest index such that $V_i \cap Z \neq \emptyset$. Since $Z \neq \emptyset$, i exists. Then $X = V_i \cap Z$ is a subset of the atom V_i, so X is a subatom. Since we have a topological ordering, every arc entering X enters Z, so we have $d^-(X) \leq d^-(Z)$. Further, by $X \subseteq Z$ and the monotonicity of r_{M}, we have $r_{\mathsf{M}}(S_X) \leq r_{\mathsf{M}}(S_Z)$. Then, by (8), we get $r_{\mathsf{M}}(S) \leq r_{\mathsf{M}}(S_X) + d_A^-(X) \leq r_{\mathsf{M}}(S_Z) + d_A^-(Z)$, so (7) holds.

In the previous theorem we were interested in a matroid-based packing of arborescences containing the largest possible number of arborescences, namely $|S|$. In the following result we consider a matroid-based packing of arborescences containing the smallest possible number of arborescences, namely $r_{\mathsf{M}}(S)$.

Theorem 7 (Szigeti [24]). *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. There exists an M-based packing of spanning arborescences in D if and only if*

$$r_{\mathsf{M}}(S_{\cup\mathcal{P}}) + e_A(\mathcal{P}) \geq r_{\mathsf{M}}(S)|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{9}$$

For the free matroid, Theorem 7 reduces to Theorem 4.

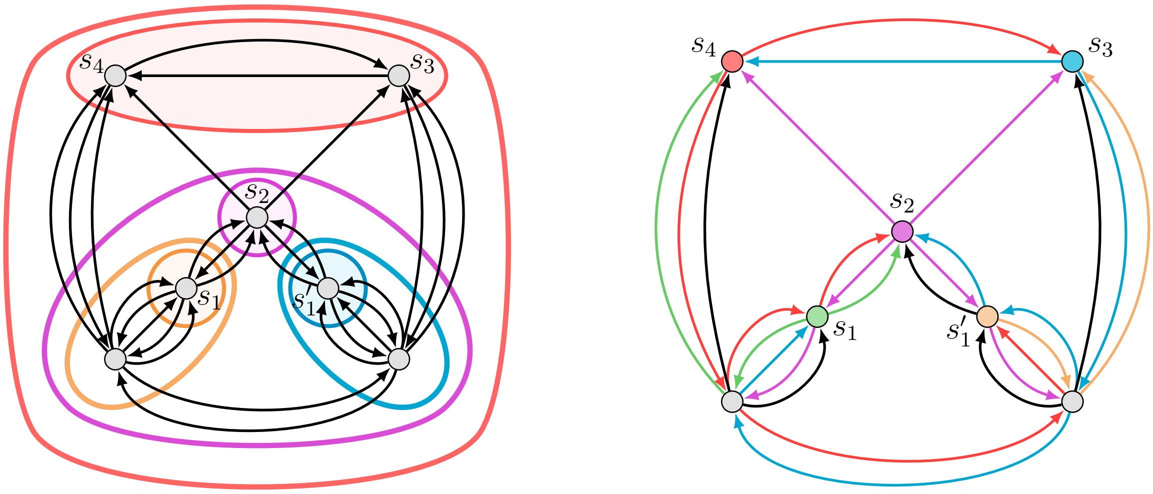
4.2.1 New results on matroid-based packing of arborescences

Here we propose to study a problem that is more general than the above two problems. We give the following common generalization of Theorems 6 and 7 where we have a lower bound and an upper bound on the number of arborescences in the packing.

Theorem 8. *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V, $\ell, \ell' \in \mathbb{Z}_+$, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. There exists an M-based (ℓ, ℓ')-limited packing of arborescences in D if and only if* (8) *holds and*

$$
\begin{aligned}
\ell &\leq \ell', && (10)\\
\sum_{v \in V} r_{\mathsf{M}}(S_v) &\geq \ell, && (11)
\end{aligned}
$$

$$\sum_{\mathsf{X} \in \mathcal{P}} (r_{\mathsf{M}}(S) - r_{\mathsf{M}}(S_{X_W}) - d_A^-(X_O)) \leq \ell' \quad \text{for every OW laminar biset family } \mathcal{P} \text{ of subatoms.} \tag{12}$$



(a) Family giving a lower bound of 5 arborescences. (b) A matroid-based packing of 5 arborescences.

Figure 3: Instance of Theorem 8.

We provide an instance of Theorem 8 in Figure 3 with the given digraph $D = (V, A)$ and the matroid $\mathsf{M} = (S, \mathcal{I}_\mathsf{M})$ where $S = \{s_1, s_1', s_2, s_3, s_4\}$ and a set is in $\mathcal{I}_\mathsf{M}$ if and only if it contains at most one of s_1 and s_1'. In Figure 3a, we give an OW laminar biset family of subatoms such that, for the function $\mathsf{f}(\mathsf{X}) = r_\mathsf{M}(S) - r_\mathsf{M}(S_{X_W}) - d_A^-(X_O)$, we have $\mathsf{f}(\mathsf{A}) = 2, \mathsf{f}(\mathsf{B}) = 1, \mathsf{f}(\mathsf{C}) = 1$, and $\mathsf{f}(\mathsf{D}) = 1$. Thus, by Theorem 8, the given digraph requires at least 5 arborescences and a matroid-based packing of 5 arborescences is shown in Figure 3b.

Theorem 8 will be later obtained from Theorem 14.

We now provide the answer to the decomposition problem for the matroid-based version. It will be obtained from Theorem 8.

Theorem 9. *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists a decomposition of A into an M-based packing of arborescences in D if and only if* (8) *holds and for every OW laminar biset family $\mathcal{P}$ of subatoms,*

$$\sum_{\mathsf{X}\in\mathcal{P}} (r_\mathsf{M}(S) - r_\mathsf{M}(S_{X_W}) - d_A^-(X_O)) \;\leq\; r_\mathsf{M}(S)|V| - |A|. \tag{13}$$

We conclude by showing that Theorem 8 implies all the results of Subsection 4.2.

Claim 1. *Theorem 8 implies Theorem 6.*

Proof. Let (D, S, M) be an instance of Theorem 6 satisfying (6) and (7). Let $\ell = \ell' = |S|$. The condition (10) trivially holds. By (6), we have $\sum_{v\in V} r_\mathsf{M}(S_v) = \sum_{v\in V} |S_v| = |S|$, so (11) is satisfied. By (7), (8) holds. Further, let $\mathcal{P}$ be an OW laminar biset family of subatoms. Then the inner sets of the bisets in $\mathcal{P}$ are disjoint. By (7), the submodularity and the subcardinality of r_M, and since X_I's are disjoint, we have

$$\sum_{\mathsf{X}\in\mathcal{P}} (r_\mathsf{M}(S) - r_\mathsf{M}(S_{X_W}) - d_A^-(X_O)) \leq \sum_{\mathsf{X}\in\mathcal{P}} (r_\mathsf{M}(S_{X_O}) - r_\mathsf{M}(S_{X_W})) \leq \sum_{\mathsf{X}\in\mathcal{P}} r_\mathsf{M}(S_{X_I}) \leq \sum_{\mathsf{X}\in\mathcal{P}} |S_{X_I}| \leq |S|,$$

so (12) is satisfied. Hence, by Theorem 8, there exists an M-based packing of s-arborescences ($s \in S$) in D, that is, a complete M-based packing of arborescences in D. □

Claim 2. *Theorem 8 implies Theorem 7.*

Proof. Let (D, S, M) be an instance of Theorem 7 satisfying (9). Let $\ell = \ell' = r_{\mathsf{M}}(S)$. The condition (10) trivially holds. Then, by the submodularity of r_{M}, we have $\sum_{v\in V} r_{\mathsf{M}}(S_v) \geq r_{\mathsf{M}}(\bigcup_{v\in V} S_v) = r_{\mathsf{M}}(S)$, so (11) is satisfied. Applying (9) for every subatom as a subpartition we get that (8) holds. To show that (12) also holds, let $\mathcal{P}$ be an OW laminar biset family of subatoms. For all $\mathsf{Y} \in \mathcal{P}$, let

$$\begin{aligned} \boldsymbol{\mathcal{P}_{\mathsf{Y}}} &= \{\mathsf{Z} \in \mathcal{P} - \mathsf{Y} : Z_O \subseteq Y_W\}, \\ \boldsymbol{\mathcal{Q}_{\mathsf{Y}}} &= \{\mathsf{Z} \in \mathcal{P}_{\mathsf{Y}} : \text{ there exists no } \mathsf{Z}' \in \mathcal{P}_{\mathsf{Y}} - \mathsf{Z} \text{ such that } Z_O \subseteq Z'_W\}. \end{aligned}$$

Note that $\mathcal{P}_{\mathsf{Y}} = \bigcup_{Z\in\mathcal{Q}_{\mathsf{Y}}}(Z \cup \mathcal{P}_{\mathsf{Z}})$ and the outer sets of the bisets in $\mathcal{Q}_{\mathsf{Y}}$ are mutually disjoint. Indeed, if $\mathsf{Z}^1, \mathsf{Z}^2 \in \mathcal{Q}_{\mathsf{Y}}$ and $Z^1_O \cap Z^2_O \neq \emptyset$, then, since $\mathcal{P}$ is OW laminar, we have that $Z^1_O \subseteq Z^2_W$ or $Z^2_O \subseteq Z^1_W$, so either $Z^1 \notin \mathcal{Q}_{\mathsf{Y}}$ or $Z^2 \notin \mathcal{Q}_{\mathsf{Y}}$ which is a contradiction.

Let us introduce the following biset function: $\mathbf{f}(\mathsf{X}) = r_{\mathsf{M}}(S) - r_{\mathsf{M}}(S_{X_W}) - d^-_A(X_O)$ for all $\mathsf{X} \in \mathcal{P}$. We now claim that we have

$$\sum_{\mathsf{Z}\in\mathcal{P}_{\mathsf{Y}}} \mathsf{f}(\mathsf{Z}) \leq r_{\mathsf{M}}(S_{Y_W}) \text{ for every } \mathsf{Y} \in \mathcal{P}. \tag{14}$$

If $\mathcal{P}_{\mathsf{Y}} = \emptyset$, then, by the non-negativity of r_{M}, the (14) holds. Suppose that (14) holds for every $\mathsf{Z} \in \mathcal{P}_{\mathsf{Y}}$. We show that it also holds for Y. By the hypothesis, the definition of f, since the outer sets of the bisets in $\mathcal{Q}_{\mathsf{Y}}$ are mutually disjoint, by (9) applied for these sets, since $\mathcal{P}$ is OW laminar and by the monotonicity of r_{M}, we have (14) for Y :

$$\begin{aligned} \sum_{\mathsf{Z}\in\mathcal{P}_{\mathsf{Y}}} \mathsf{f}(\mathsf{Z}) &= \sum_{\mathsf{Z}\in\mathcal{Q}_{\mathsf{Y}}} (\mathsf{f}(\mathsf{Z}) + \sum_{\mathsf{X}\in\mathcal{P}_{\mathsf{Z}}} \mathsf{f}(\mathsf{X})) \leq \sum_{\mathsf{Z}\in\mathcal{Q}_{\mathsf{Y}}} (\mathsf{f}(\mathsf{Z}) + r_{\mathsf{M}}(S_{Z_W})) \\ &= \sum_{\mathsf{Z}\in\mathcal{Q}_{\mathsf{Y}}} (r_{\mathsf{M}}(S) - d^-_A(Z_O)) \leq r_{\mathsf{M}}(S_{\bigcup_{\mathsf{Z}\in\mathcal{Q}_{\mathsf{Y}}} Z_O}) \leq r_{\mathsf{M}}(S_{Y_W}). \end{aligned}$$

Let $\mathcal{P}' = \mathcal{P} \cup \{\mathsf{V}\}$ where $\mathsf{V} = (V, \emptyset)$. Then $\mathcal{P}'$ is also an OW laminar biset family of V. Since the above arguments also work for $\mathcal{P}'$, we have, by (14) for V, $\sum_{\mathsf{X}\in\mathcal{P}} \mathsf{f}(\mathsf{X}) = \sum_{\mathsf{X}\in\mathcal{P}'_{\mathsf{V}}} \mathsf{f}(\mathsf{X}) \leq r_{\mathsf{M}}(S_V) = r_{\mathsf{M}}(S)$, so (12) holds. Hence, by Theorem 8, there exists an M-based packing of s-arborescences ($s \in S^* \subseteq S$) in D with $|S^*| = r_{\mathsf{M}}(S)$, and hence each arborescence in the packing is spanning. ☐

Claim 3. *Theorem 8 implies Theorem 9.*

Proof. Let $(D = (V, A), S, \mathsf{M})$ be an instance of Theorem 9. To see the **necessity**, suppose that there exists a decomposition of A into an M-based packing of arborescences in D with root set S^*. Then, by Theorem 6, (7) and hence (8) holds. Since $|S^*| = \sum_{v\in V} |S^*_v| = \sum_{v\in V}(r_{\mathsf{M}}(S) - d^-_A(v)) = r_{\mathsf{M}}(S)|V| - |A|$, by Theorem 8, (12) holds for $\ell' = r_{\mathsf{M}}(S)|V| - |A|$ and hence (13) holds.

To see the **sufficiency**, suppose that (8) and (13) hold. Let $\ell = \ell' = r_{\mathsf{M}}(S)|V| - |A|$. Then conditions (10), (8) and (12) hold. By (8) applied for all $v \in V$, we get that $\sum_{v\in V} r_{\mathsf{M}}(S_v) \geq \sum_{v\in V}(r_{\mathsf{M}}(S) - d^-_A(v)) = r_{\mathsf{M}}(S)|V| - |A|$, so (11) also holds. Hence, by Theorem 8, there exists an M-based packing of arborescences in D with arc set B and root set S^* such that $|S^*| = r_{\mathsf{M}}(S)|V| - |A|$. Since $|B| = \sum_{v\in V} d^-_B(v) = \sum_{v\in V}(r_{\mathsf{M}}(S) - |S^*_v|) = r_{\mathsf{M}}(S)|V| - |S^*| = |A|$, we have in fact a decomposition of A, and the proof of Theorem 9 is complete. ☐

4.3 Matroid-reachability-based packing of arborescences

In this section we present results on matroid-reachability-based packing of arborescences.

We need the following two functions which are non-zero only on petals and petal bisets. Let $D = (V, A)$ be a digraph, S a multiset of vertices in V and M a matroid on S with rank function r_{M}. Recall that $\hat{\mathcal{Z}}$ is the set of petals and we have $P_{C_Z} = P_Z$ for all $Z \in \hat{\mathcal{Z}}$; $\hat{\mathcal{Z}}_{\mathsf{b}}$ is the set of petal bisets and we have $P_{C_{\mathsf{X}}} = P_{X_O}$ for all $\mathsf{X} \in \hat{\mathcal{Z}}_{\mathsf{b}}$. Let the set function $\hat{\boldsymbol{p}}$ and the biset function $\hat{\mathbf{p}}$ on V be defined as follows:

$$\hat{p}(Z) = \begin{cases} r_{\mathsf{M}}(S_{P_{C_Z}}) - r_{\mathsf{M}}(S_Z) & Z \in \hat{\mathcal{Z}}, \\ 0 & \text{otherwise,} \end{cases} \qquad \hat{\mathsf{p}}(\mathsf{X}) = \begin{cases} r_{\mathsf{M}}(S_{P_{C_{\mathsf{X}}}}) - r_{\mathsf{M}}(S_{X_O}) & \mathsf{X} \in \hat{\mathcal{Z}}_{\mathsf{b}}, \\ 0 & \text{otherwise.} \end{cases}$$

The following properties of these functions will be crucial.

Claim 4. *The following hold.*

(a) $\hat{p}$ is a supermodular function on core-intersecting sets of $\hat{\mathcal{Z}}$.

(b) $\hat{\mathsf{p}}$ is a positively intersecting supermodular biset function on V.

Proof. (a) Let Z and Z' be core-intersecting sets in $\hat{\mathcal{Z}}$. It follows that there exists an atom C of D such that

$$\hat{p}(Z) = r_{\mathsf{M}}(S_{P_C}) - r_{\mathsf{M}}(S_Z), \tag{15}$$
$$\hat{p}(Z') = r_{\mathsf{M}}(S_{P_C}) - r_{\mathsf{M}}(S_{Z'}), \tag{16}$$
$$\emptyset \neq Z \cap C, \qquad Z \subseteq P_C, \qquad d_A^-(Z - C) = 0, \tag{17}$$
$$\emptyset \neq Z' \cap C, \qquad Z' \subseteq P_C, \qquad d_A^-(Z' - C) = 0. \tag{18}$$

Then, by (17) and (18), we have

$$\emptyset \neq (Z \cap Z') \cap C, \quad Z \cap Z' \subseteq P_C, \quad d_A^-((Z \cap Z') - C) = 0, \tag{19}$$
$$\emptyset \neq (Z \cup Z') \cap C, \quad Z \cup Z' \subseteq P_C, \quad d_A^-((Z \cup Z') - C) = 0. \tag{20}$$

By (19) and (20), we have $Z \cap Z', Z \cup Z' \in \hat{\mathcal{Z}}$, so

$$\hat{p}(Z \cap Z') = r_{\mathsf{M}}(S_{P_C}) - r_{\mathsf{M}}(S_{Z \cap Z'}), \tag{21}$$
$$\hat{p}(Z \cup Z') = r_{\mathsf{M}}(S_{P_C}) - r_{\mathsf{M}}(S_{Z \cup Z'}). \tag{22}$$

Since $r_{\mathsf{M}}(S_Z) + r_{\mathsf{M}}(S_{Z'}) \geq r_{\mathsf{M}}(S_{Z \cap Z'}) + r_{\mathsf{M}}(S_{Z \cup Z'})$, we get, by (15),(16),(21),(22), that $\hat{p}(Z) + \hat{p}(Z') \leq \hat{p}(Z \cap Z') + \hat{p}(Z \cup Z')$, and (a) follows.

(b) The same proof as in (a) also works for (b). □

A common generalization of Theorems 5 and 6 was given by Király [18] where he characterized the existence of a complete matroid-reachability-based packing of arborescences.

Theorem 10 (Király [18])**.** *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V, and $\mathsf{M} = (S, \mathcal{I}_{\mathsf{M}})$ a matroid with rank function r_{M}. There exists a complete M-reachability-based packing of arborescences in D if and only if* (6) *holds and*

$$d_A^-(Z) \geq r_{\mathsf{M}}(S_{P_Z}) - r_{\mathsf{M}}(S_Z) \qquad \text{for every } Z \subseteq V. \tag{23}$$

For the free matroid, Theorem 10 reduces to Theorem 5. If $r_{\mathsf{M}}(S_{P_v}) = r_{\mathsf{M}}(S)$ for all $v \in V$, then Theorem 10 reduces to Theorem 6.

Another characterization of the existence of a matroid-reachability-based packing of arborescences was given by Gao and Yang [14].

Theorem 11 (Gao, Yang [14])**.** *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid.*

(a) There exists an M-reachability-based packing of arborescences in D if and only if

$$\mathsf{d}_A^-(\mathsf{X}) \;\geq\; \hat{\mathsf{p}}(\mathsf{X}) \quad \textit{for every biset } \mathsf{X} \textit{ on } V, \tag{24}$$

or equivalently

$$d_A^-(Z) \;\geq\; \hat{p}(Z) \quad \textit{for every } Z \in \hat{\mathcal{Z}}. \tag{25}$$

(b) There exists a complete M-reachability-based packing of arborescences in D if and only if (6) *and* (25) *hold.*

In [23] it was proved that Theorems 10 and 11 are equivalent.

4.3.1 New results on matroid-reachability-based packing of arborescences

In this subsection we provide our main results.

We start with the following polyhedral result.

Theorem 12. *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid such that* (25) *holds. Then the system defined by* (26) *and* (27) *is TDI.*

$$x(\delta_A^-(Z)) \;\geq\; \hat{p}(Z) \qquad \textit{for every } Z \in \hat{\mathcal{Z}}, \tag{26}$$

$$\mathbb{1} \;\geq\; x \;\geq\; \mathbb{0}. \tag{27}$$

Proof. It is clear that the polyhedron defined by (26) and (27) coincides with the polyhedron defined by (27) and

$$x(\delta_A^-(\mathsf{X})) \;\geq\; \hat{\mathsf{p}}(\mathsf{X}) \qquad \text{for every biset } \mathsf{X} \text{ on } V. \tag{28}$$

Claim 4(b), (24) and Theorem 2 immediately imply that the system defined by (27) and (28) is TDI. Let X be a biset on V. If $\mathsf{X} \notin \hat{\mathcal{Z}}_{\mathsf{b}}$, then $\hat{\mathsf{p}}(\mathsf{X}) = 0$, so the inequality $x(\delta_A^-(\mathsf{X})) \geq \hat{\mathsf{p}}(\mathsf{X}) = 0$ is the sum of the inequalities $x(a) \geq 0$ for all $a \in \delta_A^-(\mathsf{X})$. Otherwise, $\mathsf{X} \in \hat{\mathcal{Z}}_{\mathsf{b}}$ so $\hat{\mathsf{p}}(\mathsf{X}) = r_{\mathsf{M}}(S_{P_C}) - r_{\mathsf{M}}(S_{X_O})$ for an atom C of D. Then $Z = X_O \in \hat{\mathcal{Z}}$, hence (28) for X and (26) for Z coincide. Then, by Theorem 3, the system defined by (26) and (27) is also TDI. □

In order to characterize the existence of a matroid-reachability-based (ℓ, ℓ')-limited packing of arborescences, our strategy is to minimize the number of roots of the arborescences in the packing. To achieve this we consider the extended version of the problem where the elements of the matroid correspond to different vertices of the extended graph. For an instance $(D = (V, A), S, \ell, \ell', \mathsf{M} = (S, r_{\mathsf{M}}))$ of the problem, let $\boldsymbol{D'} = (V \cup S', A \cup A')$ be obtained from D by adding a new vertex set $\boldsymbol{S'}$ containing one vertex s' for every $s \in S$ and adding a new arc set $\boldsymbol{A'}$ containing one arc $s's$ for every $s \in S$. Let $\mathbf{M'}$ be a copy of M on S'. We say that a family $\mathcal{Z}$ of subsets of $V \cup S$ is *A'-disjoint* if every arc of A' enters at most one member $\mathcal{Z}$.

We are ready to present the following intermediary result, for the proof see Subsection 6.1.

Theorem 13. *Let $D = (V \cup S, A^*)$ be a digraph (A' being the set of arcs leaving S and $A = A^* - A'$) such that no arc enters s and exactly one arc leaves s for every vertex s of S, $\ell, \ell' \in \mathbb{Z}_+$, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. There exists an M-reachability-based packing of arborescences in D using at least ℓ and at most ℓ' arcs of A' if and only if* (10) *holds and*

$$\sum_{v \in V} r_{\mathsf{M}}(N_{A'}^-(v)) \;\geq\; \ell, \tag{29}$$

$$r_{\mathsf{M}}(S \cap P_Z) - r_{\mathsf{M}}(S \cap Z) \;\leq\; d_{A^*}^-(Z) \quad \textit{for every } Z \in \hat{\mathcal{Z}}, \tag{30}$$

$$\sum_{Z \in \mathcal{Z}} (r_{\mathsf{M}}(S \cap P_Z) - r_{\mathsf{M}}(S \cap Z) - d_A^-(Z)) \;\leq\; \ell' \quad \forall A'\textit{-disjoint core-laminar subset } \mathcal{Z} \textit{ of } \hat{\mathcal{Z}}. \tag{31}$$

Recall that $\mathcal{X}$ is the set of generalized petal bisets. We now present our main result. It will be obtained from Theorem 13, for the proof see Subsection 6.2.

Theorem 14. *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V, $\ell, \ell' \in \mathbb{Z}_+$, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists an M-reachability-based (ℓ, ℓ')-limited packing of arborescences in D if and only if* (10)*,* (11) *and* (24) *hold and*

$$\sum_{\mathsf{X}\in\mathcal{P}}(r_\mathsf{M}(S_{P_{X_I}}) - r_\mathsf{M}(S_{X_W}) - d_A^-(X_O)) \leq \ell' \quad \textit{for every OW laminar biset family } \mathcal{P} \textit{ of } \mathcal{X}. \quad (32)$$

We finally provide the answer for the decomposition problem for the matroid-reachability-based version. It will be easily obtained from the previous result.

Theorem 15. *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists a decomposition of A into an M-reachability-based packing of arborescences in D if and only if* (25) *holds and for every OW laminar biset family $\mathcal{P}$ of $\mathcal{X}$,*

$$\sum_{\mathsf{X}\in\mathcal{P}}(r_\mathsf{M}(S_{P_{X_I}}) - r_\mathsf{M}(S_{X_W}) - d_A^-(X_O)) \leq (\sum_{v\in V} r_\mathsf{M}(S_{P_v})) - |A|. \quad (33)$$

We conclude by showing that Theorem 14 implies all the results of Subsection 4.3.

Claim 5. *Theorem 14 implies Theorem 8.*

Proof. Let $(D = (V, A), \mathsf{M} = (S, r_\mathsf{M}), \ell, \ell')$ be an instance of Theorem 8 satisfying (8), (10), (11), and (12). We show that (12) implies (24) and (32). First we mention that

$$r_\mathsf{M}(S) \leq r_\mathsf{M}(S_Z) \qquad \text{for every non-empty } Z \subseteq V \text{ with } d_A^-(Z) = 0. \quad (34)$$

Indeed, by $Z \neq \emptyset$ and $d_A^-(Z) = 0$, there exists a smallest non-empty $Y \subseteq Z$ such that $d_A^-(Y) = 0$. Then Y is an atom of D. Thus, by (8) and since r_M is monotone, we get that (34) also holds.

To show (24), let $\mathsf{X} \in \hat{\mathcal{Z}}_\mathsf{b}$. If $X_W \neq \emptyset$, then, by the monotonicity of r_M, $d_A^-(X_W) = 0$ and (34), we get that $r_\mathsf{M}(S_{P_{X_I}}) - r_\mathsf{M}(S_{X_O}) \leq r_\mathsf{M}(S) - r_\mathsf{M}(S_{X_W}) \leq 0 \leq d_A^-(X_O)$, and (24) holds. If $X_W = \emptyset$, then, X_O is an subatom, so, by the monotonicity of r_M and (8), we get that $r_\mathsf{M}(S_{P_{X_I}}) - r_\mathsf{M}(S_{X_O}) \leq r_\mathsf{M}(S) - r_\mathsf{M}(S_{X_O}) \leq d_A^-(X_O)$, and (24) holds.

To show that (32) also holds let $\mathcal{P}$ be an OW laminar biset family of $\mathcal{X}$. For every $\mathsf{X} \in \mathcal{P}$, we may suppose without loss of generality, by the monotonicity of r_M, that we have $1 \leq r_\mathsf{M}(S_{P_{X_I}}) - r_\mathsf{M}(S_{X_W}) - d_A^-(X_O) \leq r_\mathsf{M}(S) - r_\mathsf{M}(S_{X_W - C_\mathsf{X}})$. Since $d_A^-(X_W - C_\mathsf{X}) = 0$, it follows, by (34), that $X_W - C_\mathsf{X} = \emptyset$, that is, $X_O \subseteq C_\mathsf{X}$, so X_O is a subatom. Then, since r_M is monotone and by (12), we have $\sum_{\mathsf{X}\in\mathcal{P}}(r_\mathsf{M}(S_{P_{X_I}}) - r_\mathsf{M}(S_{X_W}) - d_A^-(X_O)) \leq \sum_{\mathsf{X}\in\mathcal{P}}(r_\mathsf{M}(S) - r_\mathsf{M}(S_{X_W}) - d_A^-(X_O)) \leq \ell'$, so (32) holds.

By Theorem 14, there exists an M-reachability-based (ℓ, ℓ')-limited packing of arborescences in D. Since, by (34), we have $r_\mathsf{M}(S_{P_v}) \geq r_\mathsf{M}(S)$ for every $v \in V$, the packing is M-based and the proof of Theorem 8 is complete. □

Claim 6. *Theorem 14 implies Theorem 11.*

Proof. Let $(D = (V, A), \mathsf{M} = (S, r_\mathsf{M}))$ be an instance of Theorem 11 satisfying (24). Let $\ell = 0$ and $\ell' = |S|$. Then $(D = (V, A), \mathsf{M} = (S, r_\mathsf{M}), \ell, \ell')$ is an instance of Theorem 14. We now show that all the conditions of Theorem 14 hold. Conditions (10) and (11) trivially hold and (24) holds by assumption. To show that (32) also holds let $\mathcal{P}$ be an OW laminar biset family of $\mathcal{X}$. By (24), the submodularity and the subcardinality of r_{M}, and since X_I's are disjoint, we have

$$\begin{aligned}
0 &\geq \sum_{\mathsf{X}\in\mathcal{P}}(r_{\mathsf{M}}(S_{P_{X_O}}) - r_{\mathsf{M}}(S_{X_O}) - d_A^-(X_O)) \\
&\geq \sum_{\mathsf{X}\in\mathcal{P}}(r_{\mathsf{M}}(S_{P_{X_I}}) - r_{\mathsf{M}}(S_{X_W}) - |S_{X_I}| - d_A^-(X_O)) \\
&\geq \sum_{\mathsf{X}\in\mathcal{P}}(r_{\mathsf{M}}(S_{P_{X_I}}) - r_{\mathsf{M}}(S_{X_W}) - d_A^-(X_O)) - |S|,
\end{aligned}$$

so (32) holds. Then, by Theorem 14, there exists an M-reachability-based packing of arborescences in D (containing at most $|S|$ arborescences) and the proof of Theorem 11 is completed. □

Claim 7. *Theorem 14 implies Theorem 15.*

Proof. Let $(D=(V,A),S,\mathsf{M})$ be an instance of Theorem 15. To see the **necessity**, suppose that there exists a decomposition of A into an M-reachability-based packing of arborescences in D with root set S^*. Then, by Theorem 11, (25) holds. Since $|S^*| = \sum_{v\in V}|S^*_v| = \sum_{v\in V}(r_{\mathsf{M}}(S_{P_v}) - d_A^-(v)) = \sum_{v\in V} r_{\mathsf{M}}(S_{P_v}) - |A|$, by Theorem 14, (32) holds for $\ell' = \sum_{v\in V} r_{\mathsf{M}}(S_{P_v}) - |A|$ and hence (33) holds.

To see the **sufficiency**, suppose that (25) and (33) hold. Let $\ell = \ell' = \sum_{v\in V} r_{\mathsf{M}}(S_{P_v}) - |A|$. Then (10), (25) (and hence (24)) and (32) hold. By (25) applied for all $v \in V$, we get that $\sum_{v\in V} r_{\mathsf{M}}(S_v) \geq \sum_{v\in V}(r_{\mathsf{M}}(S_{P_v}) - d_A^-(v)) = \sum_{v\in V} r_{\mathsf{M}}(S_{P_v}) - |A|$, so (11) also holds. Hence, by Theorem 14, there exists an M-reachability-based packing of arborescences in D with arc set B and root set S^* such that $|S^*| = \sum_{v\in V}(r_{\mathsf{M}}(S_{P_v}) - |A|$. Since $|B| = \sum_{v\in V} d_B^-(v) = \sum_{v\in V}(r_{\mathsf{M}}(S_{P_v}) - |S^*_v|) = \sum_{v\in V} r_{\mathsf{M}}(S_{P_v}) - |S^*| = |A|$, we have in fact a decomposition of A, and the proof of Theorem 9 is complete. □

4.4 Packing of branchings

We complete the section on packings in digraphs by some results on packing branchings that can be derived from our results.

Let $D=(V,A)$ be a digraph, $\mathcal{S}$ a family of subsets of V, and $\mathsf{M}=(\mathcal{S}, r_{\mathsf{M}})$ a matroid. Let $\hat{\boldsymbol{S}} = \bigcup_{S\in\mathcal{S}} S$ where the union is taken by multiplicities, so $\hat{S}$ is a multiset of V and $\mathcal{S}$ is a partition of $\hat{S}$. Recall that $\mathbf{M}^{\mathbf{1}}_{\boldsymbol{\mathcal{S}}}$ is the partition matroid on $\hat{S}$ with value 1 on each $S\in\mathcal{S}$. Let $\hat{\mathbf{p}}_{\boldsymbol{\mathcal{S}}}(\mathbf{X}) = r_{\mathsf{M}}(\mathcal{S}_{P_{C_{\mathsf{X}}}}) - r_{\mathsf{M}}(\mathcal{S}_{X_O})$ if $\mathsf{X}\in\hat{\mathcal{Z}}_{\mathsf{b}}$ and 0 otherwise.

Theorem 16 (Edmonds [4])**.** *Let $D=(V,A)$ be a digraph and $\mathcal{S}$ a family of subsets of V. There exists a packing of spanning S-branchings ($S\in\mathcal{S}$) in D if and only if*

$$|\mathcal{S}_X| + d_A^-(X) \geq |\mathcal{S}| \quad \text{for every non-empty } X\subseteq V. \tag{35}$$

For $\mathcal{S} = \{\{s\} : s\in S\}$, Theorem 16 reduces to Theorem 4. Theorem 16 can be obtained from Theorem 6 for the matroid $\mathsf{M}^1_{\mathcal{S}}$. We note that to have a packing of spanning S-branchings ($S\in\mathcal{S}$) in D it is sufficient that (35) holds for every subatom.

We provide a generalization of Theorem 16 with bounds on the total number of roots.

Theorem 17. *Let $D=(V,A)$ be a digraph, $\mathcal{S}$ a family of subsets of V, and $\ell,\ell'\in\mathbb{Z}_+$. There exists an (ℓ,ℓ')-limited packing of spanning S'-branchings ($\emptyset\neq S'\subseteq S\in\mathcal{S}$) in D if and only (10) holds and*

$$\begin{aligned}\ell &\le \sum_{S\in\mathcal{S}}|S|, &(36)\\ |\mathcal{S}_X|+d_A^-(X) &\ge |\mathcal{S}| \quad \text{for every subatom } X \text{ of } D, &(37)\\ \sum_{\mathsf{X}\in\mathcal{P}}(|\mathcal{S}|-|\mathcal{S}_{X_W}|-d_A^-(X_O)) &\le \ell' \quad \text{for every OW laminar biset family } \mathcal{P} \text{ of subatoms.} &(38)\end{aligned}$$

Theorem 17 can be extended even to matroid-reachability-based packings as follows.

Theorem 18. *Let $D=(V,A)$ be a digraph, $\mathcal{S}$ a family of subsets of V, $\ell,\ell'\in\mathbb{Z}_+$, and $\mathsf{M}=(\mathcal{S},r_{\mathsf{M}})$ a matroid. There exists an M-reachability-based (ℓ,ℓ')-limited packing of S'-branchings ($S'\subseteq S\in\mathcal{S}$) in D if and only if (10) holds and*

$$\begin{aligned}\sum_{v\in V} r_{\mathsf{M}}(\mathcal{S}_v) &\ge \ell, &(39)\\ \mathsf{d}_A^-(\mathsf{X}) &\ge \hat{\mathsf{p}}_{\mathcal{S}}(\mathsf{X}) \quad \text{for every biset } \mathsf{X} \text{ on } V, &(40)\\ \sum_{\mathsf{X}\in\mathcal{P}}(r_{\mathsf{M}}(\mathcal{S}_{P_{X_I}})-r_{\mathsf{M}}(\mathcal{S}_{X_W})-d_A^-(X_O)) &\le \ell' \quad \text{for every OW laminar biset family } \mathcal{P} \text{ of } \mathcal{X}. &(41)\end{aligned}$$

Let us show some implications between these results.

Claim 8. *Theorem 8 implies Theorem 17.*

Proof. Let $(D=(V,A),\mathcal{S},\ell,\ell')$ be an instance of Theorem 17 satisfying (10), (36), (37) and (38). Then $(D,\hat{S},\ell,\ell',\mathsf{M}^1_{\mathcal{S}})$ is an instance of Theorem 8. We now show that it satisfies (8), (10), (11), and (12). By assumption, (10) holds. By (36), we have $\sum_{v\in V} r_{\mathsf{M}^1_{\mathcal{S}}}(\hat{S}_v)=\sum_{v\in V}|\hat{S}_v|=\sum_{S\in\mathcal{S}}|S|\ge\ell$, so (11) holds for $(D,\hat{S},\ell,\ell',\mathsf{M}^1_{\mathcal{S}})$. Since $r_{\mathsf{M}^1_{\mathcal{S}}}(\hat{S}_Z)=|\mathcal{S}_Z|$ for all $Z\subseteq V$, (37) implies that (8) holds and (38) implies that (12) holds for $(D,\hat{S},\ell,\ell',\mathsf{M}^1_{\mathcal{S}})$. Thus, by Theorem 8, there exists an $\mathsf{M}^1_{\mathcal{S}}$-based (ℓ,ℓ')-limited packing of s-arborescences ($s\in S^*\subseteq\hat{S}$) in D. For every $S\in\mathcal{S}$, by the definition of $\mathsf{M}^1_{\mathcal{S}}$, the s-arborescences in the packing with $s\in S$ are vertex disjoint and hence form an S'-branching with $S'\subseteq S$. Since the packing is $\mathsf{M}^1_{\mathcal{S}}$-based and $r_{\mathsf{M}^1_{\mathcal{S}}}(\hat{S})=|\mathcal{S}|$, each S'-branching is spanning. We hence have an (ℓ,ℓ')-limited packing of spanning S'-branchings ($\emptyset\neq S'\subseteq S\in\mathcal{S}$) in D. This completes the proof of Theorem 17. ☐

Claim 9. *Theorem 17 implies Theorem 16.*

Proof. Let $(D,\mathcal{S})$ be an instance of Theorem 16 such that (35) holds. Let $\ell=\ell'=\sum_{S\in\mathcal{S}}|S|$. Note that (10), (36) trivially hold and, by (35), (37) holds. Let $\mathcal{P}$ be any OW laminar biset family of subatoms. Then the inner sets of the bisets in $\mathcal{P}$ are disjoint. Thus, by (35), we have

$$\sum_{\mathsf{X}\in\mathcal{P}}(|\mathcal{S}|-|\mathcal{S}_{X_W}|-d_A^-(X_O))\le\sum_{\mathsf{X}\in\mathcal{P}}(|\mathcal{S}_{X_O}|-|\mathcal{S}_{X_W}|)\le\sum_{\mathsf{X}\in\mathcal{P}}|\mathcal{S}_{X_I}|\le\sum_{v\in\bigcup_{\mathsf{X}\in\mathcal{P}}X_I}|\mathcal{S}_v|\le\sum_{v\in V}|\mathcal{S}_v|=\sum_{S\in\mathcal{S}}|S|,$$

so (38) also holds. Hence, by Theorem 17, there exists a packing of spanning S'-branchings ($\emptyset\neq S'\subseteq S\in\mathcal{S}$) in D such that $\sum_{S\in\mathcal{S}}|S'|=\sum_{S\in\mathcal{S}}|S|$, and Theorem 16 follows. ☐

Claim 10. *Theorem 14 implies Theorem 18.*

Proof. Let $(D, \mathsf{M} = (\mathcal{S}, r_{\mathsf{M}}), \ell, \ell')$ be an instance of Theorem 18 satisfying (10), (39), (40), and 41. Let $\mathsf{M}'_{\mathcal{S}}$ be the matroid on the multiset $\hat{S} = \bigcup_{S \in \mathcal{S}} S$ of vertices obtained from M by replacing each $S \in \mathcal{S}$ by parallel elements on all $s \in S$, that is $r_{\mathsf{M}'_{\mathcal{S}}}(S') = r_{\mathsf{M}}(\mathcal{S}_{S'})$ for every $S' \subseteq \hat{S}$. Then $(D, \mathsf{M}'_{\mathcal{S}} = (\hat{S}, r_{\mathsf{M}'_{\mathcal{S}}}), \ell, \ell')$ is an instance of Theorem 14 satisfying the conditions (10), (11), (24), and (32) for $\hat{S}$. Thus, by Theorem 14, there exists an $\mathsf{M}'_{\mathcal{S}}$-reachability-based (ℓ, ℓ')-limited packing of arborescences in D. By the construction of $\mathsf{M}'_{\mathcal{S}}$, this provides an M-reachability-based (ℓ, ℓ')-limited packing of S'-branchings ($S' \subseteq S \in \mathcal{S}$) in D. ☐

We conclude this section by mentioning an NP-complete result on packing branchings.

Theorem 19. *Let $D = (V, A)$ be a digraph, $\mathcal{S}$ a family of subsets of V, M a matroid on $\mathcal{S}$. It is NP-complete to decide whether there exists an M-based packing of S-branchings ($S \in \mathcal{S}'$) in D with $\mathcal{S}' \subseteq \mathcal{S}$ and $|\mathcal{S}'| = k$, even for the uniform matroid of rank k.*

Proof. In the special case when M is the uniform matroid of rank k, the problem is equivalent to whether there exists a packing of spanning S-branching ($S \in \mathcal{S}'$) in D with $\mathcal{S}' \subseteq \mathcal{S}$ and $|\mathcal{S}'| = k$, which is known to be NP-complete, see Theorem 3.6 in Bérczi and Frank [2]. ☐

5 Packings in directed hypergraphs

Let $\mathcal{D} = (V, \mathcal{A})$ be a directed hypergraph, shortly *dypergraph*, where $\mathcal{A}$ is the set of dyperedges of $\mathcal{D}$. A *dyperedge* e is an ordered pair (Z, z), where $z \in V$ is the *head* of e and $\emptyset \neq Z \subseteq V - z$ is the set of *tails* of e. All of our previous results can be extended to directed hypergraphs from the corresponding graphic versions, by applying the gadget of [7].

6 Proofs

In this section we provide the main proofs of the paper.

6.1 Proof of Theorem 13

Proof. To prove the **necessity**, let $B_1, \dots, B_k$ be an M-reachability-based packing of arborescences in D using a subset A'' of A' of size at least ℓ at most ℓ'. Then (10) holds. By the monotonicity of r_{M} and since $N^-_{A''}(v)$ is independent in M, we have

$$\sum_{v \in V} r_{\mathsf{M}}(N^-_{A'}(v)) \geq \sum_{v \in V} r_{\mathsf{M}}(N^-_{A''}(v)) = \sum_{v \in V} d^-_{A''}(v) = |A''| \geq \ell,$$

so (29) also holds. Let $\mathcal{Z}$ be an A'-disjoint subset of $\hat{\mathcal{Z}}$. By the necessity of Theorem 10 for $(V, A \cup A'')$ and $A'' \subseteq A'$, we have for every $Z \in \mathcal{Z}$,

$$r_{\mathsf{M}}(S \cap P_Z) - r_{\mathsf{M}}(S \cap Z) \leq d^-_{A \cup A''}(Z) \leq d^-_{A \cup A'}(Z), \tag{42}$$

so (30) holds. By (42) and since every arc of $A'' \subseteq A'$ enters at most one set in $\mathcal{Z}$, we get that

$$\sum_{Z \in \mathcal{Z}} (r_{\mathsf{M}}(S \cap P_Z) - r_{\mathsf{M}}(S \cap Z) - d^-_A(Z)) \leq \sum_{Z \in \mathcal{Z}} d^-_{A''}(Z) \leq |A''| \leq \ell',$$

so (31) holds.

To prove the **sufficiency**, let us suppose that (10), (29), (30) and (31) hold. The proof relies on the polyhedral description, obtained from Theorem 11(a), of the subgraphs that admit an

M-reachability-based packing of arborescences in D. First we focus on the upper bound ℓ'. To this purpose we find a subgraph admitting an M-reachability-based packing of arborescences in D that contains the minimum number of arcs in A'. The lower bound ℓ is achieved through Theorem 11(b). Let us hence consider the following dual linear programs where $\boldsymbol{c}(a) = 1$ if $a \in A'$ and 0 if $a \in A$:

$$(P)\quad \begin{array}{rcll} x(\delta^-_{A^*}(Z)) & \geq & \hat{p}(Z) & Z \in \hat{\mathcal{Z}} \\ -x(a) & \geq & -1 & a \in A^* \\ x & \geq & \mathbb{0} & \\ c^T x & = & w(\min) & \end{array} \qquad (D)\quad \begin{array}{rcll} -q(a) + \sum\limits_{a\in\delta^-(Z)} y(Z) & \leq & c(a) & a \in A^* \\ y, q & \geq & \mathbb{0} & \\ \hat{p}^T y - \mathbb{1}^T q & = & z(\max) & \end{array}$$

Note that since S is a set of vertices, that is every element of S has multiplicity one, $S \cap P_Z = S_{P_Z}$ and $S \cap Z = S_Z$ for every $Z \subseteq V \cup S$. Thus, by (30), the vector $\mathbb{1}$ is a feasible solution of (P) and, by $c \geq 0$, the vector $\mathbb{0}$ is a feasible solution of (D). The complementary slackness theorem says that feasible solutions $\overline{x}$ and $\binom{y}{q}$ of (P) and (D) are optimal if and only if

$$\overline{x}(a) > 0 \implies -\overline{q}(a) + \sum_{a\in\delta^-(Z)} \overline{y}(Z) = c(a), \tag{43}$$

$$\overline{y}(Z) > 0 \implies \overline{x}(\delta^-_{A^*}(Z)) = \hat{p}(Z), \tag{44}$$

$$\overline{q}(a) > 0 \implies \overline{x}(a) = 1. \tag{45}$$

By Theorems 12 and 1, there exist integral optimal solutions $\overline{\boldsymbol{x}}$ and $\binom{\overline{\boldsymbol{y}}}{\overline{\boldsymbol{q}}}$ of (P) and (D) that minimizes $\mathbb{1}^T\binom{\overline{y}}{\overline{q}}$ and then that maximizes $\sum_{Z\in\hat{\mathcal{Z}}} |Z|^2\overline{y}(Z)$. Let $\boldsymbol{\mathcal{Z}} = \{Z \in \hat{\mathcal{Z}} : \overline{y}(Z) > 0\}$.

We want to prove that $c^T\overline{x} \leq \ell'$. To obtain it we need some properties of $\mathcal{Z}$.

Lemma 1. *The following hold:*

(a) if $Z \in \mathcal{Z}$, then there exists an arc a entering Z with $\overline{q}(a) = 0$,

(b) if $a \in A^*$ enters $Z \in \mathcal{Z}$ and $\overline{q}(a) = 0$, then $a \in A'$, a enters no other member of $\mathcal{Z}$, and $\overline{y}(Z) = 1$,

(c) if $Z \in \mathcal{Z}$, then $\overline{y}(Z) = 1$,

(d) if $a \in A^*$ enters $Z \in \mathcal{Z}$ and $\overline{q}(a) \neq 0$, then $a \in A$ and $\overline{q}(a) = \sum_{a\in\delta^-(Z')} \overline{y}(Z')$,

(e) $\mathcal{Z}$ is an A'-disjoint core-laminar subset of $\hat{\mathcal{Z}}$,

(f) if $a \in A^*$ enters no $Z \in \mathcal{Z}$, then $\overline{q}(a) = 0$,

(g) $\sum_{a\in A^*} \overline{q}(a) = \sum_{Z\in\mathcal{Z}} d^-_A(Z)$,

(h) $c^T\overline{x} \leq \ell'$.

Proof. (a) Suppose that there exists $\boldsymbol{Z} \in \mathcal{Z}$ such that $\overline{q}(a) \neq 0$ for all $a \in \delta^-_{A^*}(Z)$. Let

$$\overline{\boldsymbol{y}}' = \overline{y} - \chi^{\hat{\mathcal{Z}}}_{\{Z\}} \text{ and } \overline{\boldsymbol{q}}' = \overline{q} - \chi^{A^*}_{\delta^-_{A^*}(Z)}.$$

Since $\overline{y} \in \mathbb{Z}^{\hat{\mathcal{Z}}}_+$ and $Z \in \mathcal{Z}$ (and hence $\overline{y}(Z) \geq 1$), we have $\overline{y}' \in \mathbb{Z}^{\hat{\mathcal{Z}}}_+$. Since $\overline{q} \in \mathbb{Z}^{A^*}_+$ and $\overline{q}(a) \neq 0$ (and hence $\overline{q}(a) \geq 1$) for all $a \in \delta^-_{A^*}(Z)$, we have $\overline{q}' \in \mathbb{Z}^{A^*}_+$. Further, for all $a \in \delta^-_{A^*}(Z)$, we have

$$-\overline{q}'(a) + \sum_{a\in\delta^-(Z'')} \overline{y}'(Z'') = -(\overline{q}(a) - 1) + (\sum_{a\in\delta^-(Z'')} \overline{y}(Z'')) - 1 = -\overline{q}(a) + \sum_{a\in\delta^-(Z'')} \overline{y}(Z'').$$

Since $\overline{x}$ and $\binom{\overline{y}}{\overline{q}}$ are optimal solutions of (P) and (D), it follows that $\binom{\overline{y}'}{\overline{q}'}$ is a feasible solution of (D) and that $\overline{x}$ and $\binom{\overline{y}'}{\overline{q}'}$ satisfy the complementary slackness conditions, so $\binom{\overline{y}'}{\overline{q}'}$ is an optimal solution of (D). However, $\mathbb{1}^T\binom{\overline{y}'}{\overline{q}'} < \mathbb{1}^T\binom{\overline{y}}{\overline{q}}$, that contradicts the choice of $\binom{\overline{y}}{\overline{q}}$.

(b) Let $\boldsymbol{a}$ be an arc with $\overline{q}(a) = 0$ that enters $\boldsymbol{Z} \in \mathcal{Z}$. Then, since $\binom{\overline{y}}{\overline{q}}$ is an integral feasible solution of (D) and $c(a) \leq 1$, we have

$$0 = \overline{q}(a) \geq -c(a) + \sum_{a\in\delta^-(Z')} \overline{y}(Z') \geq -c(a) + \overline{y}(Z) \geq -1 + 1 = 0,$$

hence equality holds everywhere, thus $\overline{y}(Z) = 1, c(a) = 1$ and $\overline{y}(Z') = 0$ for all $Z' \neq Z$ entered by a, so (b) holds.

(c) It immediately follows from (a) and (b).

(d) Let $\boldsymbol{a}$ be an arc with $\overline{q}(a) \neq 0$ that enters $\boldsymbol{Z} \in \mathcal{Z}$. Then, since $\overline{x}$ and $\binom{\overline{y}}{\overline{q}}$ are optimal solutions of (P) and (D), we get, by (45) and (43), that $-\overline{q}(a) + \sum_{a\in\delta^-(Z')} \overline{y}(Z') = c(a)$. To finish the proof we have to show that $a \in A$ (and hence $c(a) = 0$). Suppose for a contradiction that $a = \boldsymbol{sv} \in A'$, with $s \in S$ and $v \in V$. Let $\boldsymbol{Z'} = Z \cup \{s\}$. Since $Z \in \hat{\mathcal{Z}}$, we have $Z' \in \hat{\mathcal{Z}}$. Let

$$\boldsymbol{\overline{y}'} = \overline{y} - \chi^{\hat{\mathcal{Z}}}_{\{Z\}} + \chi^{\hat{\mathcal{Z}}}_{\{Z'\}} \text{ and } \boldsymbol{\overline{q}'} = \overline{q} - \chi^{A^*}_a.$$

Since $\overline{y} \in \mathbb{Z}^{\hat{\mathcal{Z}}}_+$ and $Z \in \mathcal{Z}$, we have $\overline{y}' \in \mathbb{Z}^{\hat{\mathcal{Z}}}_+$. As $\overline{q} \in \mathbb{Z}^{A^*}_+$ and $\overline{q}(a) \neq 0$, we have $\overline{q}' \in \mathbb{Z}^{A^*}_+$. Since no arc enters s, only a leaves s, $\overline{q}(a) > 0$ and (45), we get

$$\overline{x}(\delta^-_{A^*}(Z')) = \overline{x}(\delta^-_{A^*}(Z)) - \overline{x}(a) = \overline{x}(\delta^-_{A^*}(Z)) - 1. \tag{46}$$

Since $Z' \in \hat{\mathcal{Z}}$, $\overline{x}$ is a feasible solution of (P), by (46), $Z \in \mathcal{Z}$, (44), and $r_{\mathsf{M}}(S\cap Z') \leq r_{\mathsf{M}}(S\cap Z)+1$, we have

$$\hat{p}(Z') \leq \overline{x}(\delta^-_{A^*}(Z')) = \overline{x}(\delta^-_{A^*}(Z)) - 1 = \hat{p}(Z) - 1 \leq \hat{p}(Z').$$

It follows that equality holds everywhere, that is $\hat{p}(Z') = \overline{x}(\delta^-_{A^*}(Z'))$.

For all $b \in \delta^-_{A^*}(Z')$,

$$-\overline{q}'(b) + \sum_{b\in\delta^-(Z'')} \overline{y}'(Z'') = -\overline{q}(b) + (\sum_{b\in\delta^-(Z'')} \overline{y}(Z'')) - 1 + 1 = -\overline{q}(b) + \sum_{b\in\delta^-(Z'')} \overline{y}(Z'').$$

For a,

$$-\overline{q}'(a) + \sum_{a\in\delta^-(Z'')} \overline{y}'(Z'') = -(\overline{q}(a) - 1) + (\sum_{a\in\delta^-(Z'')} \overline{y}(Z'')) - 1 = -\overline{q}(a) + \sum_{a\in\delta^-(Z'')} \overline{y}(Z'').$$

Since $\overline{x}$ and $\binom{\overline{y}}{\overline{q}}$ are optimal solutions of (P) and (D), the above arguments show that $\binom{\overline{y}'}{\overline{q}'}$ is a feasible solution of (D) and $\overline{x}$ and $\binom{\overline{y}'}{\overline{q}'}$ satisfy the complementary slackness conditions, so $\binom{\overline{y}'}{\overline{q}'}$ is an optimal solution of (D). However, $\mathbb{1}^T\binom{\overline{y}'}{\overline{q}'} < \mathbb{1}^T\binom{\overline{y}}{\overline{q}}$ contradicts the choice of $\binom{\overline{y}}{\overline{q}}$.

(e) It immediately follows from (b) and (d) that $\mathcal{Z}$ is an A'-disjoint subset of $\hat{\mathcal{Z}}$. We prove by the usual uncrossing technique that $\mathcal{Z}$ is core-laminar. Suppose for a contradiction that there exist core-intersecting Z_1 and Z_2 in $\mathcal{Z}$ such that $Z_1 - Z_2 \neq \emptyset$ and $Z_2 - Z_1 \neq \emptyset$. Let $\boldsymbol{\overline{y}'} = \overline{y} - \chi^{\hat{\mathcal{Z}}}_{\{Z_1\}} - \chi^{\hat{\mathcal{Z}}}_{\{Z_2\}} + \chi^{\hat{\mathcal{Z}}}_{\{Z_1\cap Z_2\}} + \chi^{\hat{\mathcal{Z}}}_{\{Z_1\cup Z_2\}}$. Since $\overline{y} \in \mathbb{Z}^{\hat{\mathcal{Z}}}_+$ and $Z_1, Z_2 \in \mathcal{Z}$, we have $\overline{y}' \in \mathbb{Z}^{\hat{\mathcal{Z}}}_+$. Since $\binom{\overline{y}}{\overline{q}}$ is a feasible solution of (D) and $\sum_{a\in\delta^-(Z)} \overline{y}'(Z) \leq \sum_{a\in\delta^-(Z)} \overline{y}(Z)$, $\binom{\overline{y}'}{\overline{q}}$ is also a feasible solution of (D). Then, since $\binom{\overline{y}}{\overline{q}}$ is an optimal solution of (D), Z_1 and Z_2 are core-intersecting, and by Claim 4(a), we have

$$0 \leq (\hat{p}^T\overline{y} - \mathbb{1}^T\overline{q}) - (\hat{p}^T\overline{y}' - \mathbb{1}^T\overline{q}) = \hat{p}(Z_1) + \hat{p}(Z_2) - \hat{p}(Z_1 \cap Z_2) - \hat{p}(Z_1 \cup Z_2) \leq 0,$$

so $\binom{\overline{y}'}{\overline{q}}$ is an optimal solution of (D). Note that $\mathbb{1}^T\binom{\overline{y}'}{\overline{q}} = \mathbb{1}^T\binom{\overline{y}}{\overline{q}}$. However, by $Z_1 - Z_2 \neq \emptyset \neq Z_2 - Z_1$, we have

$$\sum_{Z\in\hat{\mathcal{Z}}} |Z|^2\overline{y}(Z) - \sum_{Z\in\hat{\mathcal{Z}}} |Z|^2\overline{y}'(Z) = |Z_1|^2 + |Z_2|^2 - |Z_1 \cap Z_2|^2 - |Z_1 \cup Z_2|^2 < 0$$

that contradicts the choice of $\binom{\overline{y}}{\overline{q}}$.

(f) Suppose that there exists an arc $\boldsymbol{a}$ with $\overline{q}(a) \neq 0$ that enters no $Z \in \mathcal{Z}$. Since $\overline{q}(a) \geq 0$, we have $\overline{q}(a) > 0$. Then, by (45), (43), and $c(a) \geq 0$, we have a contradiction:

$$0 = \sum_{a\in\delta^-(Z)} \overline{y}(Z) = c(a) + \overline{q}(a) > 0 + 0.$$

(g) By (f), (d), and (c), we have

$$\sum_{a\in A^*} \overline{q}(a) = \sum_{\substack{a\in A^*\\ \overline{q}(a)>0}} \overline{q}(a) = \sum_{\substack{a\in A^*\\ \overline{q}(a)>0}} \sum_{a\in\delta^-_A(Z)} \overline{y}(Z) = \sum_{\substack{a\in A^*\\ \overline{q}(a)>0}} \sum_{\substack{a\in\delta^-_A(Z)\\ Z\in\mathcal{Z}}} 1 = \sum_{Z\in\mathcal{Z}} \sum_{a\in\delta^-_A(Z)} 1 = \sum_{Z\in\mathcal{Z}} d^-_A(Z).$$

(h) Since $\overline{x}$ and $\binom{\overline{y}}{\overline{q}}$ are optimal solutions of (P) and (D), we have, by strong duality, that $c^T\overline{x} = \hat{p}^T\overline{y} - \mathbb{1}^T\overline{q}$, which, by (c) and (g), is equal to $\sum_{Z\in\mathcal{Z}}(r_{\mathsf{M}}(S \cap P_Z) - r_{\mathsf{M}}(S \cap Z) - d^-_A(Z))$. By (e) and (31), this sum is at most ℓ'. □

Let $\boldsymbol{A_1} = \{a \in A^* : \overline{x}(a) = 1\}$. Note that, by Lemma 1(h), we have

$$|A' \cap A_1| = c^T\overline{x} \leq \ell'. \tag{47}$$

Note that $N^-_{A_1\cap A'}(v)$ is independent in M for all $v \in V$. Indeed, let $D(A_1) = (V \cup S, A_1)$. Since $\overline{x}$ is a feasible solution of (P), (25) holds for $(D(A_1), S, \mathsf{M})$. Then, by Theorem 11(b), there exists an M-reachability-based packing $\mathcal{B}_1$ of arborescences in $D(A_1)$. Hence $\chi^{A^*}_{A(\mathcal{B}_1)}$ is a feasible solution of (P). Since $\overline{x}$ is an optimal solution of (P), $|A_1 \cap A'| = c^T\overline{x} \leq c^T\chi^{A^*}_{A(\mathcal{B}_1)} = |A(\mathcal{B}_1) \cap A_1 \cap A'|$ so $A_1 \cap A' \subseteq A(\mathcal{B}_1)$. Then, for each $v \in V$, the set R_v of roots of the arborescences in $\mathcal{B}_1$ containing v contains $N^-_{A_1\cap A'}(v)$. Since $\mathcal{B}_1$ is an M-reachability-based packing $\mathcal{B}_1$ of arborescences, R_v is independent in M and thus $N^-_{A_1\cap A'}(v)$ is independent in M.

Let $\boldsymbol{A_2}$ be obtained by adding to A_1 a smallest arc set in A' such that $N^-_{A_2}(v)$ is independent in M and $\sum_{v\in V} |N^-_{A_2}(v)| \geq \ell$. By (29), this arc set exists. Let $\boldsymbol{D'} = (V, A_1 \cap A)$, $\boldsymbol{S'}$ the multiset of V such that $S'_v = N^-_{A_2}(v)$ for every $v \in V$ and $\boldsymbol{\mathsf{M}'}$ the restriction of M on S'. Let us check that the conditions of Theorem 11(b) are satisfied for (D', S', M'). First observe that

$$S'_v \in \mathcal{I}_{\mathsf{M}'} \quad \text{for every } v \in V. \tag{48}$$

For $Z' \in \hat{\mathcal{Z}}_{D'}$, let $Z = Z' \cup (S \cap N^-_{A_2}(Z'))$. By the definition of Z and the assumption on S, we have $d^-_{A_1\cap A}(Z') = d^-_{A_1}(Z) = \overline{x}(\delta^-_{A^*}(Z))$. Note that $Z' \in \hat{\mathcal{Z}}_{D'}$ implies that $Z \in \hat{\mathcal{Z}}_D$. Since $\overline{x}$ is a feasible solution of (P), $\overline{x}(\delta^-_{A^*}(Z)) \geq r_{\mathsf{M}}(S \cap P^D_Z) - r_{\mathsf{M}}(S \cap Z)$. By the constructions of D' and Z, we have $r_{\mathsf{M}}(S \cap P^D_Z) \geq r_{\mathsf{M}'}(S'_{P^{D'}_{Z'}})$ and $r_{\mathsf{M}}(S \cap Z) = r_{\mathsf{M}'}(S'_{Z'})$. Hence we have

$$d^-_{A_1\cap A}(Z') = \overline{x}(\delta^-_{A^*}(Z)) \geq r_{\mathsf{M}}(S \cap P^D_Z) - r_{\mathsf{M}}(S \cap Z) \geq r_{\mathsf{M}'}(S'_{P^{D'}_{Z'}}) - r_{\mathsf{M}'}(S'_{Z'}) \quad \forall Z' \in \hat{\mathcal{Z}}_{D'}. \tag{49}$$

By (48) and (49), the conditions of Theorem 11(b) are satisfied for (D', S', M'), and hence there exists a complete M'-reachability-based packing $\mathcal{B}'$ of arborescences in D'. By adding to each s'-arborescence in $\mathcal{B}'$ the arc from the corresponding vertex of S to s' and adding to the packing each other vertex s of S with $r_{\mathsf{M}}(s) = 1$ as an arborescence, we obtain a packing $\mathcal{B}$ of arborescences in D. Since $\mathcal{B}'$ is an M'-reachability-based packing of arborescences in D',

$$|R_v^{\mathcal{B}'}| = r_{\mathsf{M}'}(S'_{P_v^{D'}}) \text{ for every } v \in V. \tag{50}$$

Since $\mathcal{B}'$ is complete, $\mathcal{B}$ uses all the arcs in $A_2 \cap A'$. Further, $|A_2 \cap A'|$ is equal to either ℓ and so, by (10), is at most ℓ' or $|A_1 \cap A'|$ which is then at least ℓ and, by (47), at most ℓ'.

For every $v \in V$, let $Z'_v = P_v^{D'}$ and $Z_v = Z'_v \cup (S \cap N^-_{A_2}(Z'_v))$. We show that $Z'_v \in \hat{\mathcal{Z}}_{D'}$. Indeed, let C'_v be the strongly-connected component of D' containing v. Then $Z'_v \cap C'_v \neq \emptyset, Z'_v = P_{C'}^{D'}$ and $d^-_{A_1 \cap A}(Z'_v - C'_v) = 0$, so $Z'_v \in \hat{\mathcal{Z}}_{D'}$. As above, this implies that $Z_v \in \hat{\mathcal{Z}}_D$. Note also that $v \in Z_v$ and $S \cap Z_v = S \cap N^-_{A_2}(Z'_v)$. Then, by (49) applied for Z'_v and the monotonicity of r_{M}, we have

$$0 = d^-_{A_1 \cap A}(Z'_v) \geq r_{\mathsf{M}}(S \cap P^D_{Z_v}) - r_{\mathsf{M}}(S \cap Z_v) \geq r_{\mathsf{M}}(S \cap P_v^D) - r_{\mathsf{M}}(S \cap N^-_{A_2}(P_v^{D'})) \geq 0.$$

Hence equality holds everywhere, so $r_{\mathsf{M}}(S \cap P_v^D) = r_{\mathsf{M}}(S \cap N^-_{A_2}(P_v^{D'}))$. Further, by the construction of D', we have $r_{\mathsf{M}}(S \cap N^-_{A_2}(P_v^{D'})) = r_{\mathsf{M}'}(S'_{P_v^{D'}})$. Hence, by the construction of $\mathcal{B}$ and (50), we get that $|R_v^{\mathcal{B}}| = |R_v^{\mathcal{B}'}| = r_{\mathsf{M}'}(S'_{P_v^{D'}}) = r_{\mathsf{M}}(S \cap P_v^D)$ for every $v \in V$. Further, $|R_s^{\mathcal{B}}| = r_{\mathsf{M}}(s) = r_{\mathsf{M}}(S \cap P_s^D)$ for every $s \in S$, so the packing $\mathcal{B}$ is M-reachability-based. □

6.2 Proof of Theorem 14

Proof. To prove the **necessity**, take an M-reachability-based packing $\mathcal{B}$ of s-arborescences ($s \in S^*$) in D for some $\boldsymbol{S^*} \subseteq S$ with $\ell \leq |S^*| \leq \ell'$. Then, clearly, (10) holds. Also holds (11) because, since r_{M} is non-decreasing and S^*_v is independent in M, we have

$$\sum_{v \in V} r_{\mathsf{M}}(S_v) \geq \sum_{v \in V} r_{\mathsf{M}}(S^*_v) = \sum_{v \in V} |S^*_v| = |S^*| \geq \ell.$$

By the necessity of Theorem 11, (24) also holds. Let $\mathsf{X} \in \mathcal{X}$ and $v \in X_I$. Let $R^v = R_v^{\mathcal{B}}$.

$$|R^v| = |R^v_{X_I}| + |R^v_{X_W}| + |R^v_{\overline{X_O}}|. \tag{51}$$

Since the packing is M-reachability-based, R^v is a base of S_{P_v}. Since $v \in X_I \subseteq C_{\mathsf{X}}$ and C_{X} is strongly-connected, we have $P_v \subseteq P_{X_I} \subseteq P_{C_{\mathsf{X}}} \subseteq P_v$. Then

$$|R^v| = r_{\mathsf{M}}(S_{P_v}) = r_{\mathsf{M}}(S_{P_{X_I}}). \tag{52}$$

Since $R^v_{X_W} \subseteq R^v$, R^v is independent in M and r_{M} is monotone, we have

$$|R^v_{X_W}| = r_{\mathsf{M}}(R^v_{X_W}) \leq r_{\mathsf{M}}(S_{X_W}). \tag{53}$$

Since the arborescences are arc-disjoint and $v \in X_O$, we have

$$|R^v_{\overline{X_O}}| \leq d^-_A(X_O). \tag{54}$$

It follows from (51)–(54) that

$$|S^*_{X_I}| \geq |R^v_{X_I}| \geq r_{\mathsf{M}}(S_{P_{X_I}}) - r_{\mathsf{M}}(S_{X_W}) - d^-_A(X_O). \tag{55}$$

Let $\mathcal{P}$ be an OW laminar biset family of $\mathcal{X}$. Then, by $\ell' \geq |S^*|$, since X_I's are disjoint, and by (55), we have

$$\ell' \geq |S^*| \geq \sum_{\mathsf{X}\in\mathcal{P}} |S^*_{X_I}| \geq \sum_{\mathsf{X}\in\mathcal{P}} (r_{\mathsf{M}}(S_{P_{X_I}}) - r_{\mathsf{M}}(S_{X_W}) - d^-_A(X_O)),$$

so (32) holds.

To prove the **sufficiency**, suppose that for an instance $(D = (V, A), S, \ell, \ell', \mathsf{M} = (S, r_{\mathsf{M}}))$ of Theorem 14, the conditions (10), (11), (24) and (32) hold. To be able to apply Theorem 13 we have to consider the extended version. Let $\boldsymbol{D'} = (V \cup S', A \cup A')$ hence be obtained from D by adding a new vertex set $\boldsymbol{S'}$ containing one vertex s' for every $s \in S$ and adding a new arc set $\boldsymbol{A'}$ containing one arc $s's$ for every $s \in S$. Let $\mathbf{M'}$ be a copy of M on S'. Note that A' is the set of arcs leaving S', no arc enters s' and exactly one arc leaves s' for every vertex s' of S'. Then $(D', \ell, \ell', \mathsf{M}')$ is an instance of Theorem 13. We now show that all the conditions of Theorem 13 hold. First, (10) holds by assumption. Note that (11) implies (29) and (24) implies (30). Finally, the following lemma implies that (31) also holds.

Lemma 2. $\sum_{Z\in\mathcal{Z}}(r_{\mathsf{M}'}(S' \cap P^{D'}_Z) - r_{\mathsf{M}'}(S' \cap Z) - d^-_A(Z)) \leq \ell'$ *for all A'-disjoint core-laminar subset $\mathcal{Z}$ of $\hat{\mathcal{Z}}_{D'}$.*

Proof. Let $\boldsymbol{\mathcal{Z}}$ be an A'-disjoint core-laminar subset $\mathcal{Z}$ of $\hat{\mathcal{Z}}_{D'}$. For every $Z^i \in \mathcal{Z}$, we may suppose without loss of generality that $r_{\mathsf{M}'}(S' \cap P^{D'}_{Z^i}) - r_{\mathsf{M}'}(S' \cap Z^i) - d^-_A(Z^i) \geq 1$ and hence, by (30), that $d^-_{A'}(Z^i) \geq 1$. Let $\mathbf{X^i}$ be the biset on V with $\boldsymbol{X^i_O} = V \cap Z^i$ and $\boldsymbol{X^i_I} = N^+_{A'}(S' - Z^i) \cap Z^i$ and $\boldsymbol{\mathcal{P}} = \{\mathsf{X}^i : Z^i \in \mathcal{Z}\}$.

Proposition 1. *$\mathcal{P}$ is an OW laminar biset family of $\mathcal{X}$.*

Proof. To show that $\mathcal{P} \subseteq \mathcal{X}$, let $\mathsf{X}^i \in \mathcal{P}$. Since $Z^i \in \mathcal{Z} \subseteq \hat{\mathcal{Z}}_{D'}$, there exists an atom C of D' such that $Z^i \cap C \neq \emptyset, Z^i \subseteq P^{D'}_C$ and $d^-_{A\cup A'}(Z^i - C) = 0$. Then, since $d^-_{A'}(Z^i) \geq 1$, C is an atom of D, $\emptyset \neq X^i_I \subseteq C$, $X^i_W \subseteq P^D_C$ and $d^-_A(X_W - C) = 0$, so we have $\mathsf{X}^i \in \mathcal{X}$.

We now show that $\mathcal{P}$ is OW laminar. Suppose there exist two core-intersecting bisets $\mathsf{X}^i, \mathsf{X}^j$ in $\mathcal{P}$ such that $X^i_O - X^j_W \neq \emptyset \neq X^j_O - X^i_W$. Then Z^i and Z^j are $\hat{\mathcal{Z}}_{D'}$-intersecting. Since $\mathcal{Z}$ is D'-core-laminar, we get that $Z^i \subseteq Z^j$ or $Z^j \subseteq Z^i$, say $Z^i \subseteq Z^j$, so $X^i_O \subseteq X^j_O$. Since $X^i_O - X^j_W \neq \emptyset$, it follows that there exists a vertex $v \in X^i_O \cap X^j_I$. Then, by definition, there exist $s'_j \in S' \setminus Z^j$ such that $s'_j v \in A'$. Since $v \in X^i_O \cap X^j_I \subseteq Z^i \subseteq Z^j$ and $s'_j \in S' \setminus Z^j \subseteq S' \setminus Z^i$, we get that the arc $s'_j v \in A'$ enters Z^i and Z^j, that contradicts the fact that $\mathcal{Z}$ is A'-disjoint. □

Note that, by construction of D', we have $S' \cap P^{D'}_{Z^i} = S_{P^D_{X_O}}$. Since there exists $v \in X_I \subseteq C_{\mathsf{X}}$ and C_{X} is strongly-connected, we have $P^D_v \subseteq P^D_{X_I} \subseteq P^D_{X_O} \subseteq P^D_{C_{\mathsf{X}}} \subseteq P^D_v$. Then

$$r_{\mathsf{M}'}(S' \cap P^{D'}_{Z^i}) = r_{\mathsf{M}}(S_{P^D_{X^i_I}}). \tag{56}$$

For every $s \in S_{X^i_W}$, by the definition of X^i_W, we have $s' \in S' \cap Z^i$. Hence the elements of S' corresponding to $S_{X^i_W}$ are contained in $S' \cap Z^i$. Then, by the monotonicity of r_{M}, we have

$$r_{\mathsf{M}}(S_{X^i_W}) \leq r_{\mathsf{M}'}(S' \cap Z^i). \tag{57}$$

Since no arc enters $Z^i \cap S'$, we have

$$d^-_A(Z^i) = d^-_A(X^i_O). \tag{58}$$

Thus, by (56)–(58), Proposition 1 and (32), we get that

$$\sum_{Z^i\in\mathcal{Z}}(r_{\mathsf{M}'}(S'\cap P^{D'}_{Z^i})-r_{\mathsf{M}'}(S'\cap Z^i)-d^-_A(Z^i))\leq\sum_{\mathsf{X}^i\in\mathcal{P}}(r_{\mathsf{M}}(S_{P^D_{X^i_I}})-r_{\mathsf{M}}(S_{X^i_W})-d^-_A(X^i_O))\leq\ell',$$

and the proof of the lemma is completed. □

By Theorem 13, there exists an M'-reachability-based packing of arborescences in D' using at least ℓ and at most ℓ' arcs of A'. By deleting the roots s' of the arborescences in the packing, we obtain s-arborescences in D. Hence we get an M-reachability-based packing of at least ℓ and at most ℓ' arborescences in D that completes the proof of Theorem 14. □

7 Algorithmic aspects

In this section we assume that a matroid is given by an oracle for the rank function and, under this assumption, we point out that all the problems which derive from the problem of Theorem 13 can be solved in polynomial time. Since all the reductions we presented are done in polynomial time, it is enough to show that the problem of Theorem 13 can be solved in polynomial time. The main algorithmic difficulty in the proof of Theorem 13 is to find an arc set containing a minimum number of arcs leaving S, that admits a matroid-reachability-based packing of arborescences. This can be done by finding the arc set of a matroid-reachability-based packing of arborescences of minimum weight where the weight is 1 for every arc leaving S and 0 for the other arcs. This latter problem can be solved in polynomial time due to Bérczi, Király, Kobayashi [3] or Király, Szigeti, and Tanigawa [19]. Then, we mention that the last part in the proof of Theorem 13, that is finding a complete matroid-reachability-based packing of arborescences, is polynomial (see Király [18], Hörsch and Szigeti [15]) hence we can get the required packing in polynomial time.

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A Detailed results on packings in directed hypergraphs

The aim of this section is to present the extensions of the previous results to directed hypergraphs. We start with the necessary definitions on directed hypergraphs.

Let $\boldsymbol{\mathcal{D}} = (V, \mathcal{A})$ be a directed hypergraph, shortly *dypergraph*, where $\boldsymbol{\mathcal{A}}$ is the set of dyperedges of $\mathcal{D}$. A *dyperedge* e is an ordered pair (Z, z), where $z \in V$ is the *head* of e and $\emptyset \neq Z \subseteq V - z$ is the set of *tails* of e. For a subset X of V, a dyperedge (Z, z) *enters* X if $z \in X$ and $Z - X \neq \emptyset$. The *in-degree* $\boldsymbol{d^-_{\mathcal{A}}(X)}$ of X is the number of dyperedges in $\mathcal{A}$ *entering* X. For a subpartition $\mathcal{P}$ of V, we denote by $\boldsymbol{e_{\mathcal{A}}(\mathcal{P})}$ the set of dyperedges in $\mathcal{A}$ that enters at least one member of $\mathcal{P}$. By *trimming* a dyperedge $e = (Z, z)$, we mean the operation that replaces e by an arc yz where $y \in Z$. A dypergraph $\mathcal{D}$ is called a *(spanning) s-hyperarborescence* if $\mathcal{D}$ can be trimmed to a (spanning) s-arborescence. A dypergraph $\mathcal{D}$ is called a *dyperpath from s to t* if $\mathcal{D}$ can be trimmed to a path from s to t. A *subatom* of $\mathcal{D}$ is a non-empty subset C of vertices such that for every ordered pair $(u, v) \in C \times C$, there exists a dyperpath from u to v in $\mathcal{D}$. An *atom* of $\mathcal{D}$ is a maximal subatom of $\mathcal{D}$. For a subset X of V, we denote by $\boldsymbol{P^{\mathcal{D}}_X}$ the set of vertices from which there exists a dyperpath to at least one vertex of X.

Let S be a multiset of V and M a matroid on S. A packing $\mathcal{B}$ of hyperarborescences in $\mathcal{D}$ is called M-*based* or *matroid-based* if every $s \in S$ is the root of at most one hyperarborescence in the packing and for every vertex $v \in V$, the multiset $\boldsymbol{R^{\mathcal{B}}_v}$ of roots of hyperarborescences in the packing in which v can be reached from the root forms a basis of M. A packing $\mathcal{B}$ of hyperarborescences in $\mathcal{D}$ is called M-*reachability-based* or *matroid-reachability-based* if every $s \in S$ is the root of at most one hyperarborescence in the packing and for every vertex $v \in V$, the multiset $R^{\mathcal{B}}_v$ forms a basis of $S_{P^{\mathcal{D}}_v}$ in M. A packing of hyperarborescences is *complete* if every $s \in S$ is the root of exactly one hyperarborescence in the packing.

A.1 Packing of hyperarborescences

Theorem 4 was generalized to dypergraphs in [12].

Theorem 20 (Frank, Király, Király [12]). *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph and S a multiset of vertices in V. There exists a packing of spanning s-hyperarborescences $(s \in S)$ in $\mathcal{D}$ if and only if*

$$|S_X| + d^-_{\mathcal{A}}(X) \quad \geq \quad |S| \qquad \text{for every } \emptyset \neq X \subseteq V. \tag{59}$$

If $\mathcal{D}$ is a digraph, then Theorem 20 reduces to Theorem 4.

The following common extension of Theorems 5 and 20 was given in [1].

Theorem 21 (Bérczi, Frank [1]). *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph and S a multiset of vertices in V. There exists a packing of reachability s-hyperarborescences $(s \in S)$ in $\mathcal{D}$ if and only if*

$$|S_X| + d^-_{\mathcal{A}}(X) \quad \geq \quad |S_{P^{\mathcal{D}}_X}| \qquad \text{for every } X \subseteq V. \tag{60}$$

If $\mathcal{D}$ is a digraph, then Theorem 21 reduces to Theorem 5. The same way as Theorem 5 implies Theorem 4, Theorem 21 implies Theorem 20.

A.2 Matroid-based packing of hyperarborescences

Theorem 6 was generalized to dypergraphs in [7] as follows. It was obtained from the graphic version, Theorem 6, by a simple gadget.

Theorem 22 (Fortier, Király, Léonard, Szigeti, Talon [7]). *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, S a multiset of vertices in V, and $\mathsf{M} = (S, \mathcal{I}_\mathsf{M})$ a matroid with rank function r_M. There exists a complete M-based packing of hyperarborescences in $\mathcal{D}$ if and only if* (6) *holds and*

$$d^-_\mathcal{A}(Z) \geq r_\mathsf{M}(S) - r_\mathsf{M}(S_Z) \qquad \text{for every } Z \subseteq V. \tag{61}$$

If $\mathcal{D}$ is a digraph, then Theorem 22 reduces to Theorem 6. For the free matroid, Theorem 22 reduces to Theorem 20.

Theorem 7 was extended to dypergraphs in [24].

Theorem 23 (Szigeti [24]). *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, S a multiset of vertices in V, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists an M-based packing of spanning hyperarborescences in $\mathcal{D}$ if and only if*

$$r_\mathsf{M}(S_{\cup\mathcal{P}}) + e_\mathcal{A}(\mathcal{P}) \quad \geq \quad r_\mathsf{M}(S)|\mathcal{P}| \qquad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{62}$$

If $\mathcal{D}$ is a digraph, then Theorem 23 reduces to Theorem 7. For the free matroid, Theorem 23 reduces to Theorem 22.

The following generalizations to dypergraphs follow from the corresponding graphic versions, by applying the gadget of [7].

Theorem 24. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, S a multiset of vertices in V, $\ell, \ell' \in \mathbb{Z}_+$, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists an M-based (ℓ, ℓ')-limited packing of hyperarborescences in $\mathcal{D}$ if and only if* (10) *and* (11) *hold and*

$$r_\mathsf{M}(S_X) + d^-_\mathcal{A}(X) \quad \geq \quad r_\mathsf{M}(S) \quad \text{for every subatom } X \text{ of } \mathcal{D}, \tag{63}$$

$$\sum_{\mathsf{X}\in\mathcal{P}} (r_\mathsf{M}(S) - r_\mathsf{M}(S_{X_W}) - d^-_\mathcal{A}(X_O)) \quad \leq \quad \ell' \qquad \forall \text{ OW laminar biset family } \mathcal{P} \text{ of subatoms.} \tag{64}$$

If $\mathcal{D}$ is a digraph, then Theorem 24 reduces to Theorem 8. If $\ell = \ell' = |S|$, then Theorem 24 reduces to Theorem 22. If $\ell = \ell' = r_\mathsf{M}(S)$, then Theorem 24 reduces to Theorem 23.

Theorem 25. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, S a multiset of vertices in V, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists a decomposition of $\mathcal{A}$ into an M-based packing of hyperarborescences in $\mathcal{D}$ if and only if* (63) *holds and for every OW laminar biset family $\mathcal{P}$ of subatoms,*

$$\sum_{\mathsf{X}\in\mathcal{P}} (r_\mathsf{M}(S) - r_\mathsf{M}(S_{X_W}) - d^-_\mathcal{A}(X_O)) \quad \leq \quad r_\mathsf{M}(S)|V| - |\mathcal{A}|. \tag{65}$$

If $\mathcal{D}$ is a digraph, then Theorem 25 reduces to Theorem 9.

A.3 Matroid-reachability-based packing of hyperarborescences

Theorem 10 was generalized to dypergraphs in [7] as follows.

Theorem 26 (Fortier, Király, Léonard, Szigeti, Talon [7]). *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, S a multiset of vertices in V, and $\mathsf{M} = (S, \mathcal{I}_\mathsf{M})$ a matroid with rank function r_M. There exists a complete M-reachability-based packing of hyperarborescences in $\mathcal{D}$ if and only if* (6) *holds and*

$$d^-_\mathcal{A}(Z) \geq r_\mathsf{M}(S_{P_Z}) - r_\mathsf{M}(S_Z) \qquad \text{for every } Z \subseteq V. \tag{66}$$

If $\mathcal{D}$ is a digraph, then Theorem 26 reduces to Theorem 10. If $r_\mathsf{M}(S_{P_v}) = r_\mathsf{M}(S)$ for all $v \in V$, then Theorem 26 reduces to Theorem 22.

The following generalizations to dypergraphs follow from the corresponding graphic versions, by applying the gadget of [7].

Theorem 27. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, S a multiset of vertices in V, $\ell, \ell' \in \mathbb{Z}_+$, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists an M-reachability-based (ℓ, ℓ')-limited packing of hyperarborescences in $\mathcal{D}$ if and only if* (10) *and* (11) *hold and*

$$r_\mathsf{M}(S_{P_{\mathsf{X}_I}}) - r_\mathsf{M}(S_{X_O}) \leq d^-_\mathcal{A}(X_O) \quad \text{for every biset } \mathsf{X} \text{ on } V, \tag{67}$$

$$\sum_{\mathsf{X}\in\mathcal{P}} (r_\mathsf{M}(S_{P_{\mathsf{X}_I}}) - r_\mathsf{M}(S_{X_W}) - d^-_\mathcal{A}(X_O)) \leq \ell' \quad \text{for every OW laminar biset family } \mathcal{P} \text{ of } \mathcal{X}. \tag{68}$$

If $\mathcal{D}$ is a digraph, then Theorem 27 reduces to Theorem 14. If $r_\mathsf{M}(S_{P_v}) = r_\mathsf{M}(S)$ for all $v \in V$, then Theorem 27 reduces to Theorem 24. If $\ell = \ell' = |S|$, then Theorem 27 reduces to Theorem 26.

Theorem 28. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, S a multiset of vertices in V, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists a decomposition of $\mathcal{A}$ into an M-reachability-based packing of hyperarborescences in $\mathcal{D}$ if and only if* (67) *holds and for every OW laminar biset family $\mathcal{P}$ of $\mathcal{X}$,*

$$\sum_{\mathsf{X}\in\mathcal{P}} (r_\mathsf{M}(S_{P_{\mathsf{X}_I}}) - r_\mathsf{M}(S_{X_W}) - d^-_\mathcal{A}(X_O)) \leq r_\mathsf{M}(S)|V| - |\mathcal{A}|. \tag{69}$$

If $\mathcal{D}$ is a digraph, then Theorem 28 reduces to Theorem 15. If $r_\mathsf{M}(S_{P_v}) = r_\mathsf{M}(S)$ for all $v \in V$, then Theorem 28 reduces to Theorem 25.