Packing forests

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Abstract

The seminal papers of Edmonds [6], Nash-Williams [29] and Tutte [34] have laid the foundations of the theories of packing arborescences and packing trees. The directed version has been extensively investigated, resulting in a great number of generalizations. In contrast, the undirected version has been marginally considered. The aim of this paper is to further develop the theory of packing trees and forests. We present a broad extension of an already generic theorem on packing spanning branchings of Bérczi, Frank [3], as well as its universal undirected counterpart which is our main result.

1 Introduction

The theory of packing arborescences started in the seventies with the results of Edmonds [6], Lovász [26], and Frank [10]. After a long silence, a new wave of results appeared in the aughts due to Frank, Király, Király [13], Kamiyama, Katoh, Takizawa [22], Frank [11], and Bérczi, Frank [1, 2]. The last ten years the development of the theory accelerated, great number of new results appeared: [3, 5, 8, 9, 16, 17, 20, 21, 24, 25, 28].

The theory of packing trees, the undirected counterparts of arborescences, started earlier in 1961 with the result of Nash-Williams [29] and Tutte [34]. Later, only a few results appeared, we can cite Peng, Chen, Koh [30], Frank, Király, Kriesell [14], and Katoh, Tanigawa [23]. The aim of this paper is to further develop the theory of packing trees.

Edmonds [6] characterized digraphs having a packing of k spanning arborescences with fixed roots. In this paper we consider packing arborescences with flexible roots. We will concentrate on a result of Bérczi, Frank [3] that characterizes digraphs having a packing of k spanning branchings $B_1, \dots, B_k$ such that the number of roots of each B_i is between a lower bound $\ell(i)$ and an upper bound $\ell'(i)$ and the total number of roots of all B_i's is also between a lower bound and an upper bound. We will consider the more general problem of h-regular packing of k branchings, that is, a packing of k branchings such that each vertex belongs to h of them. Note that a packing of k spanning branchings is equivalent to a k-regular packing of k branchings. We will show that the problem of h-regular packing of k branchings with the above mentioned conditions on the numbers of roots can also be solved, even in dypergraphs. We point out that the proof technic of Bérczi, Frank [3] also works for this problem. We will present not only this general result but also some interesting special cases. Four subsections are devoted to the results, depending on if the results are about packings of spanning branchings, regular packings of branchings, packings of spanning hyperbranchings or regular packings of hyperbranchings. In each subsection we will present four results. The first result is about arborescences, that is, branchings with one root, the second one is about branchings with the same given number of roots, the third one is about branchings with

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(not necessarily the same) given number of roots, and finally, the fourth one is about branchings satisfying bounds on the number of their roots and bounds on the total number of their roots.

In the second part of the paper we present the undirected counterparts of the previous sixteen results. We first prove, using matroid theory, a result on packings of k spanning forests in a graph with $\ell(1),\dots,\ell(k)$ connected components. It will be applied together with some new technics developed in [19] to obtain the result on h-regular packings of k forests in a graph with $\ell(1),\dots,\ell(k)$ connected components. This will imply the result on h-regular packings of k rooted forests in a graph satisfying bounds on the number of their roots and bounds on the total number of their roots. The main result of the present paper, on h-regular packings of k hyperforests rooted at $S_1,\dots,S_k$ in a hypergraph satisfying a lower bound $\ell(i)$ and an upper bound $\ell'(i)$ on $|S_i|$ and a lower bound and an upper bound on the total number of roots of the rooted hyperforests in the packing, is obtained from the graphic version by the technic of trimming.

Let us explain why we are obliged to work with rooted hyperforests. In [3], Bérczi, Frank considered bounds on the number of arcs (equivalently, number of connected components or number of roots) of the spanning branchings. The preliminary results presented in Theorem 2 say that the decision problem whether there exists an h-regular packing of k branchings (forests) in a digraph (graph) each containing ℓ arcs (edges, respectively) is NP-complete. This reveals that the result of Bérczi, Frank [3] can not be generalized for h-regular packings with bounds on the number of arcs. Hence the number of connected components are considered in our packing problems regarding graphs. However, we can not define a unique number of connected components for hyperbranchings. We will therefore consider the number of roots of the hyperbranchings. Similarly, in hypergraphs the correct notion to be used is rooted hyperforest.

The organization of the paper is as follows. In Section 2 we give all the definitions needed. In Section 3 we provide some preliminary results to be applied later in the proofs. Section 4 contains all the results on packing branchings and hyperbranchings. We present in Section 5 all the results on packing forests and hyperforests. In Section 6 we provide the proofs of the new results. In Section 7 we propose a common extension of some of the results to mixed hypergraphs and a conjecture for the simplest open problem in mixed graphs.

2 Definitions

In this section we provide all the definitions needed in the paper. For the basic definitions, see [12].

2.1 General definitions

The sets of integers, non-negative integers and positive integers are respectively denoted by $\mathbb{Z}, \mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$. For a set Q, a subset X of Q and a function $m : Q \to \mathbb{Z}$, we define $\boldsymbol{m(X)} = \sum_{x\in X} m(x)$. For $k \in \mathbb{Z}_{>0}$, $\boldsymbol{K}$ denotes the set $\{1,\dots,k\}$. For a function $\ell : K \to \mathbb{Z}$ and $p \in \mathbb{Z}$, we introduce the function $\boldsymbol{\ell_p}$ (that will be extensively employed throughout the paper) as

$$\ell_p(i) = \min\{\ell(i), p\} \text{ for } i \in K.$$

For a family $\mathcal{S}$ of subsets of V and a subset $X \subseteq V$, we denote by $\boldsymbol{\mathcal{S}_X} = \{S \in \mathcal{S} : X \cap S \neq \emptyset\}$ the family of members of $\mathcal{S}$ that intersect X. For a subset $X \subseteq V$, we denote $\overline{\boldsymbol{X}} = V - X$ (we use this notation only when it is clear what V is in the context).

A set function p on V is called *supermodular* if the following inequality holds for all $X, Y \subseteq V$,

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y). \tag{1}$$

We say that p is *intersecting supermodular* if (1) holds for all $X, Y \subseteq V$ such that $X \cap Y \neq \emptyset$. A set function b on V is called *submodular* if $-b$ is supermodular.

A pair X, Y of subsets of V is called *properly intersecting* if $X - Y, Y - X, X \cap Y \neq \emptyset$. A set $\mathcal{L}$ of subsets of V is called *laminar* if no pair $X, Y \in \mathcal{L}$ is properly intersecting. We say that a subset X of V *crosses* a partition $\mathcal{P}$ if X intersects at least 2 members of $\mathcal{P}$.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions of V. Let $\mathcal{P}$ be the family consisting of the sets in $\mathcal{P}_1$ and $\mathcal{P}_2$. While there exist properly intersecting sets X and Y in $\mathcal{P}$, we replace them by $X \cap Y$ and $X \cup Y$. This preserves the property that each vertex belongs to exactly two sets of $\mathcal{P}$. Furthermore, $\mathcal{P}$ is now a laminar family, therefore it can be partitioned into two partitions, one consisting of the maximal elements (we denote it by $\boldsymbol{\mathcal{P}_1 \sqcup \mathcal{P}_2}$) of $\mathcal{P}$ and the other consisting of the minimal elements (we denote it by $\boldsymbol{\mathcal{P}_1 \sqcap \mathcal{P}_2}$) of $\mathcal{P}$. Let us give a list of some properties of the partitions we have just defined.

$$\text{For every } X \in \mathcal{P}_1 \sqcap \mathcal{P}_2, \text{ there exist } U_1 \in \mathcal{P}_1 \text{ and } U_2 \in \mathcal{P}_2 \text{ such that } X = U_1 \cap U_2, \tag{2}$$

$$\text{For every } X \in \mathcal{P}_1 \cup \mathcal{P}_2, \text{ there exists } Y \in \mathcal{P}_1 \sqcup \mathcal{P}_2 \text{ such that } X \subseteq Y, \tag{3}$$

$$|\mathcal{P}_1 \sqcup \mathcal{P}_2| + |\mathcal{P}_1 \sqcap \mathcal{P}_2| = |\mathcal{P}_1| + |\mathcal{P}_2|, \tag{4}$$

$$|\mathcal{P}_1 \sqcap \mathcal{P}_2| \geq \max\{|\mathcal{P}_1|, |\mathcal{P}_2|\} \geq \min\{|\mathcal{P}_1|, |\mathcal{P}_2|\} \geq |\mathcal{P}_1 \sqcup \mathcal{P}_2|. \tag{5}$$

2.2 Digraphs and dypergraphs

Let $D = (V, A)$ be a directed graph, shortly *digraph*. We also denote its vertex set by $\boldsymbol{V(D)}$ and its arc set by $\boldsymbol{A(D)}$. For $X \subseteq V$ and $F \subseteq A$, the *in-degree* of X in F, denoted by $\boldsymbol{d_F^-(X)}$, is the number of arcs in F entering X. For $F \subseteq A$ and a subpartition $\mathcal{P}$ of V, we denote by $\boldsymbol{e_F(\mathcal{P})}$ the number of arcs in F that enter some member of $\mathcal{P}$. We say that a directed graph (U, F) is a *branching* with root set S, shortly S*-branching*, if $S \subseteq U$ and if there exists a unique path from S to every $u \in U$. When $S = \{s\}$, we call it an *arborescence* with root s. A subgraph D' of a digraph D is said to be *spanning* if $V(D') = V(D)$.

Let $\mathcal{D} = (V, \mathcal{A})$ be a directed hypergraph, shortly *dypergraph*. We also denote its vertex set by $\boldsymbol{V(\mathcal{D})}$ and its hyperarc set by $\boldsymbol{\mathcal{A}(\mathcal{D})}$, where a hyperarc has exactly one head and at least one tail. For $X \subseteq V$ and $\mathcal{F} \subseteq \mathcal{A}$, the *in-degree* of X in $\mathcal{F}$, denoted by $\boldsymbol{d_{\mathcal{F}}^-(X)}$, is the number of hyperarcs in $\mathcal{F}$ entering X. We denote by $\boldsymbol{V_{\geq 1}(\mathcal{D})}$ the set of vertices whose in-degree is at-least 1 in $\mathcal{D}$. For a subpartition $\mathcal{P}$ of V, we denote by $\boldsymbol{e_{\mathcal{F}}(\mathcal{P})}$ the number of hyperarcs in $\mathcal{F}$ that enter some member of $\mathcal{P}$. By *trimming* a hyperarc X in $\mathcal{A}$ we mean the operation that replaces X by an arc yx, where x is the head of X and y is one of the tails of X. We say that a dypergraph $\mathcal{B} = (U, \mathcal{F})$ is a *hyperbranching* with root set S, shortly S*-hyperbranching*, if $S \subseteq U$ and if the hyperarcs in $\mathcal{F}$ can be trimmed to obtain an arc set F such that the digraph $(V_{\geq 1}(\mathcal{B}) \cup S, F)$ is an S-branching. The *core* of an S-hyperbranching $\mathcal{B}$ is the vertex set $V_{\geq 1}(\mathcal{B}) \cup S$. When $S = \{s\}$, we call $\mathcal{B}$ a *hyperarborescence* with root s. A subdypergraph $\mathcal{B}$ of $\mathcal{D}$ is a *spanning* S-hyperbranching if it is an S-hyperbranching whose core is V.

2.3 Graphs and hypergraphs

Let $G = (V, E)$ be an undirected graph, shortly *graph*. We also denote its vertex set by $\boldsymbol{V(G)}$ and its edge set by $\boldsymbol{E(G)}$. For a set $\mathcal{K}$ of edge-disjoint graphs, we denote the union of their edge sets by $\boldsymbol{E(\mathcal{K})}$. For $X \subseteq V$ and $F \subseteq E$, we denote by $\boldsymbol{d_F(X)}$ the number of edges in F entering X. We denote by $\boldsymbol{\mathcal{P}(G)}$ the partition of V consisting of the connected components of G and by $\boldsymbol{c(G)} = \boldsymbol{c(V, E)} = |\mathcal{P}(G)|$ the number of connected components of G. For $F \subseteq E$, let $\boldsymbol{\mathcal{P}(F)} = \mathcal{P}((V, F))$. For $F \subseteq E$ and a partition $\mathcal{P}$ of V, we denote by $\boldsymbol{e_F(\mathcal{P})}$ the number of edges in F that enter some member of $\mathcal{P}$. A subgraph G' of a graph G is said to be *spanning* if $V(G') = V(G)$. A graph G is called *bipartite* if there exists a bipartition $\{A, B\}$ of its vertex set such that every edge of G connects a vertex of A to a vertex of B, it is then denoted by $\boldsymbol{(A, B; E)}$. For bipartite graph $G = (A, B; E)$ and $X \subseteq A$, we denote by $\boldsymbol{\Gamma(X)}$ the set of vertices that are connected to at least one vertex in X. The graph G is called a *forest* if it has no cycle. A connected forest is called a *tree*. We say that a spanning forest A *crosses* another spanning forest B if there is a connected component of A intersecting at least two connected components of B.

Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph. We also denote its vertex set by $\boldsymbol{V(\mathcal{G})}$ and its hyperedge set by $\boldsymbol{\mathcal{E}(\mathcal{G})}$, where a hyperedge consists of at least two vertices of V. For $X \subseteq V$ and $\mathcal{F} \subseteq \mathcal{E}$, we denote by $\boldsymbol{d_{\mathcal{F}}(X)}$ the number of hyperedges in $\mathcal{F}$ entering X. Further, for a subpartition $\mathcal{P}$ of V, we denote by $\boldsymbol{e_{\mathcal{F}}(\mathcal{P})}$ the number of hyperedges in $\mathcal{F}$ that enter some member of $\mathcal{P}$. By *trimming* a hyperedge X in $\mathcal{E}$ we mean the operation that replaces X by an edge between two different vertices in X. By *orienting* a hyperedge X of $\mathcal{E}$ we mean the operation that chooses a vertex of X to be its head and hence X becomes a hyperarc. A hypergraph $\mathcal{T}$ is called a *hyperforest* with root set S, shortly *S-hyperforest*, if the hyperedges of $\mathcal{T}$ can be oriented to obtain an S-hyperbranching. When $S = \{s\}$, we call $\mathcal{T}$ a *hypertree* with root s. An S-hyperforest $\mathcal{T}$ in $\mathcal{G}$ is said to be *spanning* if it can be oriented to a spanning S-hyperbranching.

2.4 Mixed graphs and mixed hypergraphs

Let $F = (V, E \cup A)$ be a mixed graph, where E is the set of edges and A is the set of arcs. F is called a *mixed branching* if there exists an orientation $\vec{E}$ of E such that $(V, \vec{E} \cup A)$ is a branching. A mixed subgraph F' of a mixed graph F is said to be *spanning* if $V(F') = V(F)$.

Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, where $\mathcal{E}$ is the set of hyperedges and $\mathcal{A}$ is the set of hyperarcs. $\mathcal{F}$ is called a *mixed hyperbranching* with root set S, shortly *mixed S-hyperbranching*, if there exists an orientation $\vec{\mathcal{E}}$ of $\mathcal{E}$ such that $(V, \vec{\mathcal{E}} \cup \mathcal{A})$ is an S-hyperbranching. We note that an S-hyperforest is a mixed S-hyperbranching.

By a *packing* of mixed hyperbranchings in a mixed hypergraph $\mathcal{F}$, we mean a set of hyperarc- and hyperedge-disjoint mixed hyperbranchings of $\mathcal{F}$. Such a packing is said to be *h-regular* ($h \in \mathbb{Z}_{\geq 0}$) if every mixed hyperbranching in the packing can be oriented to a hyperbranching such that every vertex of $\mathcal{F}$ belongs to exactly h of their cores.

2.5 Matroid theory

We use the usual notions from matroid theory. Let S be a finite ground set. A function $r : 2^S \to \mathbb{Z}_{\geq 0}$ is called the *rank function* of the *matroid* $\mathsf{M} = (S, r)$ if and only if $r(X) \leq |X|$ for every $X \subseteq S$, $X \subseteq Y \subseteq S$ implies $r(X) \leq r(Y)$, and r is submodular. A subset X of S is called an *independent set* of M if $r_{\mathsf{M}}(X) = |X|$.

(1) For an undirected graph $G = (V, E)$, the *graphic matroid* $\boldsymbol{\mathsf{M}_G} = (E, r_G)$ is defined such that $r_G(F) = |V| - c(V, F)$ for any subset F of E. It is well-known that the set of edge sets of spanning forests in G is exactly the set of independent sets of M_G.

(2) For a matroid $\mathsf{M} = (S, r)$ and $\ell \in \mathbb{Z}_{\geq 0}$, the *truncated matroid* of M at ℓ is $\mathsf{M}' = (S, r')$, where $r'(X) = \min\{r(X), \ell\}$ for every $X \subseteq S$.

(3) For k matroids $\mathsf{M}_1 = (S, r_1), \ldots, \mathsf{M}_k = (S, r_k)$ on the same ground set S, we define *the sum* of $\mathsf{M}_1, \ldots, \mathsf{M}_k$ to be the matroid M such that X is independent in M if and only if X can be partitioned into $X_1, \ldots, X_k$ such that X_i is independent in M_i for every $1 \leq i \leq k$.

Theorem 1 (Edmonds, Fulkerson [7]). *The rank function r of the sum matroid M of k matroids $\mathsf{M}_i = (S, r_i)$ is given by the following formula*

$$r(Z) = \min_{X \subseteq Z} \left\{ |Z - X| + \sum_{i=1}^{k} r_i(X) \right\} \qquad \text{for every } Z \subseteq S.$$

3 Preliminary results

In this section we provide preliminary results to be applied later.

We start with a simple technical claim.

Claim 1. *Let $a_1, a_2, a'_1, a'_2, \ell \in \mathbb{Z}_{>0}$ such that $a_1 + a_2 = a'_1 + a'_2$ and $\min\{a_1, a_2\} \geq a'_2$. Then*

$$\min\{\ell, a_1\} + \min\{\ell, a_2\} \geq \min\{\ell, a'_1\} + \min\{\ell, a'_2\}.$$

Proof. We may assume without loss of generality that $a'_1 \geq a_1 \geq a_2 \geq a'_2$.

If $\ell \geq a_1$, then $\min\{\ell, a_1\} + \min\{\ell, a_2\} = a_1 + a_2 = a'_1 + a'_2 \geq \min\{\ell, a'_1\} + \min\{\ell, a'_2\}$.

If $a_1 \geq \ell$, then $\min\{\ell, a_1\} + \min\{\ell, a_2\} \geq \ell + \min\{\ell, a'_2\} = \min\{\ell, a'_1\} + \min\{\ell, a'_2\}$. □

We need the submodularity of $e_{\mathcal{E}}$ on partitions.

Lemma 1. *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph and $\mathcal{P}_1$ and $\mathcal{P}_2$ partitions of V. The following holds*

$$e_{\mathcal{E}}(\mathcal{P}_1) + e_{\mathcal{E}}(\mathcal{P}_2) \geq e_{\mathcal{E}}(\mathcal{P}_1 \sqcap \mathcal{P}_2) + e_{\mathcal{E}}(\mathcal{P}_1 \sqcup \mathcal{P}_2). \tag{6}$$

Proof. The result will easily follow from the following claim.

Claim 2. *The following hold for all $X \in \mathcal{E}$.*

(a) If X crosses $\mathcal{P}_1 \sqcup \mathcal{P}_2$ then it crosses both $\mathcal{P}_1$ and $\mathcal{P}_2$.

(b) If X crosses $\mathcal{P}_1 \sqcap \mathcal{P}_2$ then it crosses either $\mathcal{P}_1$ or $\mathcal{P}_2$.

Proof. (a) Suppose that X leaves $Y \in \mathcal{P}_1 \sqcup \mathcal{P}_2$, that is, $\emptyset \neq X \cap Y \neq X$. Let $Z \in \mathcal{P}_1$ be such that $X \cap Y \cap Z \neq \emptyset$ (Z exists because $\mathcal{P}_1$ is a partition of V). By (3) and $Y \cap Z \neq \emptyset$, we have $Z \subseteq Y$, thus X also leaves Z, that is, X crosses $\mathcal{P}_1$. The same holds for $\mathcal{P}_2$.

(b) Suppose that X leaves $Y \in \mathcal{P}_1 \sqcap \mathcal{P}_2$. By (2), there exist $U_1 \in \mathcal{P}_1$ and $U_2 \in \mathcal{P}_2$ such that $Y = U_1 \cap U_2$. Therefore X leaves either U_1 or U_2, that is, X crosses either $\mathcal{P}_1$ or $\mathcal{P}_2$. □

For any $X \in \mathcal{E}$, if $e_{\{X\}}(\mathcal{P}_1 \sqcup \mathcal{P}_2) = 1$ then, by Claim 2(a), we have $e_{\{X\}}(\mathcal{P}_1) + e_{\{X\}}(\mathcal{P}_2) = 2 \geq e_{\{X\}}(\mathcal{P}_1 \sqcap \mathcal{P}_2) + e_{\{X\}}(\mathcal{P}_1 \sqcup \mathcal{P}_2)$. If $e_{\{X\}}(\mathcal{P}_1 \sqcup \mathcal{P}_2) = 0$ then, by Claim 2(b), we have $e_{\{X\}}(\mathcal{P}_1) + e_{\{X\}}(\mathcal{P}_2) \geq e_{\{X\}}(\mathcal{P}_1 \sqcap \mathcal{P}_2) = e_{\{X\}}(\mathcal{P}_1 \sqcap \mathcal{P}_2) + e_{\{X\}}(\mathcal{P}_1 \sqcup \mathcal{P}_2)$. Hence (6) follows. □

We present the following negative results on packing branchings and forests that justify why we do not consider in our packing problems the number of arcs and edges.

Theorem 2. *Let D be a digraph, G a graph and $h, k, \ell \in \mathbb{Z}_{>0}$.*

(a) It is NP-complete to decide whether there exists an h-regular packing of k branchings in D each containing ℓ arcs, even for $h = 1$ and $k = 2$.

(b) It is NP-complete to decide whether there exists an h-regular packing of k forests in G each containing ℓ edges, even for $h = 1$ and $k = 2$.

Proof. We only prove (a), the same proof works for (b) as well. Let $\{a_i : i \in I\}$ be an instance of the problem Partition that is $a_i \in \mathbb{Z}_{>0}$ for all $i \in I$ and we have to decide whether there exists $J \subset I$ such that $\sum_{i \in J} a_i = \sum_{i \in I-J} a_i$. By Garey, Johnson [18], it is known that Partition is NP-complete. We define an instance (D, h, k, ℓ) of our problem Branch$hk\ell$. Let D be the vertex disjoint union of arborescences D_i containing a_i arcs ($i \in I$), $h = 1, k = 2$, and $\ell = \frac{1}{2}\sum_{i \in I} a_i$. It is evident that $\{a_i : i \in I\}$ is a positive instance of Partition if and only if (D, h, k, ℓ) is a positive instance of Branch$hk\ell$. □

Finally, we also need the following two results. The first one can be obtained by combining Theorems 16 and 17 in [3], while the second one is a special case of Corollary 1 of [8].

Theorem 3 (Bérczi, Frank [3])**.** *Let S and T be disjoint sets, p a positively intersecting supermodular set function on T, $f, g : T \cup S \to \mathbb{Z}_{\geq 0}$ and $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ such that $f \leq g$ and $\alpha \leq \beta$. There exists a simple bipartite graph $G = (S, T, E)$ such that*

$$\begin{array}{rcccll} & & |\Gamma_E(Y)| & \geq & p(Y) & \text{for every } Y \subseteq T, \\ f(v) & \leq & d_E(v) & \leq & g(v) & \text{for every } v \in T \cup S, \\ \alpha & \leq & |E| & \leq & \beta & \end{array}$$

if and only if for all $X \subseteq S, Y \subseteq T$ and subpartition $\mathcal{P}$ of $T - Y$, we have

$$f(Y) - |X||Y| + \sum_{P \in \mathcal{P}} p(P) - |X||\mathcal{P}| \leq g(S - X), \tag{7}$$

$$f(X) - |X||Y| + \sum_{P \in \mathcal{P}} p(P) - |X||\mathcal{P}| \leq g(T - Y), \tag{8}$$

$$\alpha - |X||Y| + \sum_{P \in \mathcal{P}} p(P) - |X||\mathcal{P}| \leq g((S - X) \cup (T - Y)), \tag{9}$$

$$f(X \cup Y) - |X||Y| + \sum_{P \in \mathcal{P}} p(P) - |X||\mathcal{P}| \leq \beta. \tag{10}$$

Theorem 4 (Fortier et al. [8])**.** *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, $\mathcal{S}$ a family of subsets of V and $h \in \mathbb{Z}_{>0}$. There exists an h-regular packing of S-hyperbranching ($S \in \mathcal{S}$) in $\mathcal{D}$ if and only if*

$$\begin{array}{rcll} |\mathcal{S}_X| + d^-_{\mathcal{A}}(X) & \geq & h & \text{for every } \emptyset \neq X \subseteq V, \\ |\mathcal{S}_v| & \leq & h & \text{for every } v \in V. \end{array}$$

4 Packings in digraphs and dypergraphs

The motivation of this paper comes from the results of this section on packing branchings. Subsection 4.1 contains known results on packings of spanning branchings. Subsection 4.2 contains their generalizations to regular packings of branchings. Subsections 4.3 and 4.4 contain the extensions of the results of Subsections 4.1 and 4.2 to dypergraphs. Every subsection contains four results, each of them generalizing the previous one:

1. a theorem on branchings with 1 root, that is, on arborescences,
2. a theorem on branchings with ℓ roots,
3. a theorem on branchings with $\ell(1), \ldots, \ell(k)$ roots respectively,
4. a theorem on branchings B_i with number of roots between $\ell(i)$ and $\ell'(i)$ for $i \in \{1, \ldots, k\}$ and a total number of roots between $\ell(0)$ and $\ell'(0)$.

In this section the most general result, Theorem 20, implies all the others. We prove it by imitating the proof of Theorem 8 by Bérczi, Frank [3].

4.1 Packing of spanning branchings

The first result is about packing spanning arborescences.

Theorem 5 (Frank [10]). *Let $D = (V, A)$ be a digraph and $k \in \mathbb{Z}_{>0}$. There exists a packing of k spanning arborescences in D if and only if*

$$e_A(\mathcal{P}) \geq k(|\mathcal{P}| - 1) \quad \text{for every subpartition } \mathcal{P} \text{ of } V.$$

Theorem 5 can be generalized for spanning branchings each having the same number of roots as follows.

Theorem 6 ([3]). *Let $D = (V, A)$ be a digraph, $k, \ell \in \mathbb{Z}_{>0}$. There exists a packing of k spanning branchings in D each having ℓ roots if and only if*

$$\begin{aligned} |V| &\geq \ell, & (11)\\ e_A(\mathcal{P}) &\geq k(|\mathcal{P}| - \ell) \quad \text{for every subpartition } \mathcal{P} \text{ of } V. & (12)\end{aligned}$$

For $\ell = 1$, Theorem 6 reduces to Theorem 5.

The following result extends Theorem 6 in the sense that the number of roots of the spanning branchings in the packing are not necessarily the same.

Theorem 7 ([3]). *Let $D = (V, A)$ be a digraph, $k \in \mathbb{Z}_{>0}$, and $\ell : K \to \mathbb{Z}_{>0}$. There exists a packing of k spanning branchings in D with $\ell(1), \dots, \ell(k)$ roots if and only if*

$$\begin{aligned} |V| &\geq \ell(i) \quad \text{for every } 1 \leq i \leq k, & (13)\\ \ell_{|\mathcal{P}|}(K) + e_A(\mathcal{P}) &\geq k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. & (14)\end{aligned}$$

For $\ell(i) = \ell$ for $i = 1, \dots, k$, Theorem 7 reduces to Theorem 6.

Now an even more general form follows.

Theorem 8 (Bérczi, Frank [3]). *Let $D = (V, A)$ be a digraph, $k \in \mathbb{Z}_{>0}$, and $\ell, \ell' : K \cup \{0\} \to \mathbb{Z}_{>0}$ such that*

$$\begin{aligned} \ell'(K) \geq \ell'(0) \geq \ell(0) &\geq \ell(K), & (15)\\ |V| \geq \ell'(i) &\geq \ell(i) \quad \text{for every } 1 \leq i \leq k. & (16)\end{aligned}$$

There exists a packing of k spanning S_i-branchings B_i in D such that

$$\begin{aligned} \ell(i) \leq |S_i| &\leq \ell'(i) \quad \text{for every } 1 \leq i \leq k, & (17)\\ \ell(0) \leq \sum_{i=1}^{k} |S_i| &\leq \ell'(0) & (18)\end{aligned}$$

if and only if

$$\begin{aligned} \ell'(0) - \ell(K) + \ell_{|\mathcal{P}|}(K) + e_A(\mathcal{P}) &\geq k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V,\\ \ell'_{|\mathcal{P}|}(K) + e_A(\mathcal{P}) &\geq k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V.\end{aligned}$$

For $\ell(i) = \ell'(i)$ for $i = 1, \dots, k$ and $\ell(0) = \ell'(0) = \ell(K)$, Theorem 8 reduces to Theorem 7.

4.2 Regular packing of branchings

Theorem 5 can be generalized for regular packings.

Theorem 9. *Let $D = (V, A)$ be a digraph and $h, k \in \mathbb{Z}_{>0}$. There exists an h-regular packing of k arborescences in D if and only if*

$$\begin{aligned} h|V| &\geq k, & (19)\\ e_A(\mathcal{P}) &\geq h|\mathcal{P}| - k \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V. & (20) \end{aligned}$$

For $h = k$, Theorem 9 reduces to Theorem 5. Theorem 8 easily implies Theorem 9.

Theorems 6 and 9 can be both generalized as follows.

Theorem 10. *Let $D = (V, A)$ be a digraph and $h, k, \ell \in \mathbb{Z}_{>0}$. There exists an h-regular packing of k branchings in D each having ℓ roots if and only if*

$$\begin{aligned} k &\geq h, & (21)\\ h|V| &\geq k\ell, & (22)\\ e_A(\mathcal{P}) + k\ell &\geq h|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V. & (23) \end{aligned}$$

For $h = k$, Theorem 10 reduces to Theorem 6. For $\ell = 1$, Theorem 10 reduces to Theorem 9.

We now provide a common generalization of Theorems 7 and 10.

Theorem 11. *Let $D = (V, A)$ be a digraph, $h, k \in \mathbb{Z}_{>0}$ and $\ell : K \to \mathbb{Z}_{>0}$. There exists an h-regular packing of k branchings in D with $\ell(1), \dots, \ell(k)$ roots if and only if* (13) *holds and*

$$\begin{aligned} h|V| &\geq \ell(K), & (24)\\ \ell_{|\mathcal{P}|}(K) + e_A(\mathcal{P}) &\geq h|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V. & (25) \end{aligned}$$

For $h = k$, Theorem 11 reduces to Theorem 7. For $\ell(i) = \ell$ for $i = 1, \dots, k$, Theorem 11 reduces to Theorem 10.

A common extension of Theorems 8 and 11 follows.

Theorem 12. *Let $D = (V, A)$ be a digraph, $h, k \in \mathbb{Z}_{>0}$, and $\ell, \ell' : K \cup \{0\} \to \mathbb{Z}_{>0}$ such that* (15) *and* (16) *hold. There exists an h-regular packing of k S_i-branchings B_i in D such that* (17) *and* (18) *hold if and only if*

$$\begin{aligned} h|V| &\geq \ell(0), & (26)\\ \ell'(0) - \ell(K) + \ell_{|\mathcal{P}|}(K) + e_A(\mathcal{P}) &\geq h|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V, & (27)\\ \ell'_{|\mathcal{P}|}(K) + e_A(\mathcal{P}) &\geq h|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V. & (28) \end{aligned}$$

For $h = k$, Theorem 12 reduces to Theorem 8. For $\ell(i) = \ell'(i)$ for $i = 1, \dots, k$, $\ell(0) = \ell'(0) = \ell(K)$, Theorem 12 reduces to Theorem 11. Theorem 12 follows from its dypergraphic version, Theorem 20.

4.3 Packing of spanning hyperbranchings

An extension of Theorem 5 for dypergraphs can be obtained from either Fortier et al. [8, Corollary 1] or Hörsch, Szigeti [21, Theorem 8].

Theorem 13 ([8, 21]). *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph and $k \in \mathbb{Z}_{>0}$. There exists a packing of k spanning hyperarborescences in $\mathcal{D}$ if and only if*

$$e_{\mathcal{A}}(\mathcal{P}) \geq k(|\mathcal{P}| - 1) \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V.$$

If $\mathcal{D}$ is a digraph then Theorem 13 reduces to Theorem 5.

We first propose a common generalization of Theorems 6 and 13.

Theorem 14. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph and $k, \ell \in \mathbb{Z}_{>0}$. There exists a packing of k spanning hyperbranchings in $\mathcal{D}$ each having ℓ roots if and only if* (11) *holds and*

$$e_{\mathcal{A}}(\mathcal{P}) \quad \geq \quad k(|\mathcal{P}| - \ell) \qquad \text{for every subpartition } \mathcal{P} \text{ of } V.$$

If $\mathcal{D}$ is a digraph then Theorem 14 reduces to Theorem 6. For $\ell = 1$, Theorem 14 reduces to Theorem 13.

A common generalization of Theorems 7 and 14 follows.

Theorem 15. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, $k \in \mathbb{Z}_{>0}$ and $\ell : K \to \mathbb{Z}_{>0}$. There exists a packing of k spanning hyperbranchings in $\mathcal{D}$ with $\ell(1), \ldots, \ell(k)$ roots if and only if* (13) *holds and*

$$\ell_{|\mathcal{P}|}(K) + e_{\mathcal{A}}(\mathcal{P}) \quad \geq \quad k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V.$$

If $\mathcal{D}$ is a digraph then Theorem 15 reduces to Theorem 7. For $\ell(i) = \ell$ for $i = 1, \ldots, k$, Theorem 15 reduces to Theorem 14.

Finally, we present a common extension of Theorems 8 and 15.

Theorem 16. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, $k \in \mathbb{Z}_{>0}$, $\ell, \ell' : K \cup \{0\} \to \mathbb{Z}_{>0}$ such that* (15) *and* (16) *hold. There exists a packing of k spanning S_i-hyperbranchings B_i in $\mathcal{D}$ such that* (17) *and* (18) *hold if and only if*

$$\begin{aligned} \ell'(0) - \ell(K) + \ell_{|\mathcal{P}|}(K) + e_{\mathcal{A}}(\mathcal{P}) \quad &\geq \quad k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V, \\ \ell'_{|\mathcal{P}|}(K) + e_{\mathcal{A}}(\mathcal{P}) \quad &\geq \quad k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \end{aligned}$$

If $\mathcal{D}$ is a digraph then Theorem 16 reduces to Theorem 8. If $\ell(i) = \ell'(i)$ for $i = 1, \ldots, k$, $\ell(0) = \ell'(0) = \ell(K)$ then Theorem 16 reduces to Theorem 15. Theorem 16 follows from its extension to regular packings, Theorem 20.

4.4 Regular packing of hyperbranchings

Theorem 9 was generalized to dypergraphs in [32], also giving a generalization of Theorem 13 to regular packings.

Theorem 17 ([32])**.** *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph and $h, k \in \mathbb{Z}_{>0}$. There exists an h-regular packing of k hyperarborescences in $\mathcal{D}$ if and only if* (19) *holds and*

$$e_{\mathcal{A}}(\mathcal{P}) \quad \geq \quad h|\mathcal{P}| - k \qquad \text{for every subpartition } \mathcal{P} \text{ of } V.$$

If $\mathcal{D}$ is a digraph then Theorem 17 reduces to Theorem 9. For $h = k$, Theorem 17 reduces to Theorem 13.

We first propose a common generalization of Theorems 10, 14 and 17.

Theorem 18. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph and $h, k, \ell \in \mathbb{Z}_{>0}$. There exists an h-regular packing of k hyperbranchings in $\mathcal{D}$ each having ℓ roots if and only if* (21) *and* (22) *hold and*

$$e_{\mathcal{A}}(\mathcal{P}) \quad \geq \quad h|\mathcal{P}| - k\ell \qquad \text{for every subpartition } \mathcal{P} \text{ of } V.$$

If $\mathcal{D}$ is a digraph then Theorem 18 reduces to Theorem 10. For $h = k$, Theorem 18 reduces to Theorem 14. For $\ell = 1$, Theorem 18 reduces to Theorem 17.

A common generalization of Theorems 11, 15 and 18 follows.

Theorem 19. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, $h, k \in \mathbb{Z}_{>0}$ and $\ell : K \to \mathbb{Z}_{>0}$. There exists an h-regular packing of k hyperbranchings in $\mathcal{D}$ with $\ell(1), \dots, \ell(k)$ roots if and only if* (13) *and* (24) *hold and*

$$\ell_{|\mathcal{P}|}(K) + e_{\mathcal{A}}(\mathcal{P}) \quad \geq \quad h|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V.$$

If $\mathcal{D}$ is a digraph then Theorem 19 reduces to Theorem 11. For $h = k$, Theorem 19 reduces to Theorem 15. For $\ell(i) = \ell$ for $i = 1, \dots, k$, Theorem 19 reduces to Theorem 18.

The main contribution of this section is the following common extension of Theorems 12, 16 and 19.

Theorem 20. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, $h, k \in \mathbb{Z}_{>0}$, $\ell, \ell' : K \cup \{0\} \to \mathbb{Z}_{>0}$ such that* (15) *and* (16) *hold. There exists an h-regular packing of k S_i-hyperbranchings B_i in $\mathcal{D}$ such that* (17) *and* (18) *hold if and only if* (26) *holds and*

$$\ell'(0) - \ell(K) + \ell_{|\mathcal{P}|}(K) + e_{\mathcal{A}}(\mathcal{P}) \quad \geq \quad h|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V, \tag{29}$$

$$\ell'_{|\mathcal{P}|}(K) + e_{\mathcal{A}}(\mathcal{P}) \quad \geq \quad h|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \tag{30}$$

If $\mathcal{D}$ is a digraph then Theorem 20 reduces to Theorem 12. For $h = k$, Theorem 20 reduces to Theorem 16. If $\ell(i) = \ell'(i)$ for $i = 1, \dots, k$ and $\ell(0) = \ell'(0) = \ell(K)$ then Theorem 20 reduces to Theorem 19. Note that Theorem 20 implies all the results of Section 4. Theorem 20 is proved in Subsection 6.1. We mention that even the $\mathsf{M}_{\oplus}$-restricted (f, g)-bounded version can be proved the same way, see [33].

5 Packings in graphs and hypergraphs

The aim of this section is to present the undirected counterparts of the theorems in Section 4. The structure of this section is the same as that of the previous section: it contains four subsections each containing four results that are increasingly more general. The most general one, Theorem 36, implies all the results of this section.

5.1 Packing of spanning forests

The first result, the undirected counterpart of Theorem 5, is a well-known result about packing spanning trees.

Theorem 21 (Nash-Williams [29], Tutte [34])**.** *Let $G = (V, E)$ be a graph and $k \in \mathbb{Z}_{>0}$. There exists a packing of k spanning trees in G if and only if*

$$e_E(\mathcal{P}) \quad \geq \quad k(|\mathcal{P}| - 1) \qquad \textit{for every partition } \mathcal{P} \textit{ of } V.$$

The undirected counterpart of Theorem 6 that extends Theorem 21 is as follows.

Theorem 22 (Peng, Chen, Koh [30])**.** *Let $G = (V, E)$ be a graph, $k, \ell \in \mathbb{Z}_{>0}$. There exists a packing of k spanning forests in G each having ℓ connected components if and only if* (11) *holds and*

$$e_E(\mathcal{P}) \quad \geq \quad k(|\mathcal{P}| - \ell) \qquad \textit{for every partition } \mathcal{P} \textit{ of } V.$$

For $\ell = 1$, Theorem 22 reduces to Theorem 21.

The following result is the undirected counterpart of Theorem 7 and a generalization of Theorem 22. It can easily be proved using matroid theory.

Theorem 23. *Let $G = (V, E)$ be a graph, $k \in \mathbb{Z}_{>0}$ and $\ell : K \to \mathbb{Z}_{>0}$. There exists a packing of k spanning forests in G with $\ell(1), \dots, \ell(k)$ connected components if and only if* (13) *holds and*

$$\ell_{|\mathcal{P}|}(K) + e_E(\mathcal{P}) \quad \geq \quad k|\mathcal{P}| \qquad \text{for every partition } \mathcal{P} \text{ of } V.$$

For $\ell(i) = \ell$ for $i = 1, \dots, k$, Theorem 23 reduces to Theorem 22. For the sake of completeness we provide a proof of Theorem 23 in Subsection 6.2.

The following new result, the undirected counterpart of Theorem 8, extends Theorem 23. The result comes from [27].

Theorem 24. *Let $G = (V, E)$ be a graph and $k \in \mathbb{Z}_{>0}$, $\ell, \ell' : K \cup \{0\} \to \mathbb{Z}_{>0}$ such that* (15) *and* (16) *hold. There exists a packing of k spanning S_i-forests B_i in G such that* (17) *and* (18) *hold if and only if*

$$\begin{aligned} \ell'(0) - \ell(K) + \ell_{|\mathcal{P}|}(K) + e_E(\mathcal{P}) \quad &\geq \quad k|\mathcal{P}| \quad \text{for every partition } \mathcal{P} \text{ of } V, \\ \ell'_{|\mathcal{P}|}(K) + e_E(\mathcal{P}) \quad &\geq \quad k|\mathcal{P}| \quad \text{for every partition } \mathcal{P} \text{ of } V. \end{aligned}$$

For $\ell(i) = \ell'(i)$ for $i = 1, \dots, k$ and $\ell(0) = \ell'(0) = \ell(K)$, Theorem 24 reduces to Theorem 23. Theorem 24 follows from its extension to regular packings, Theorem 28.

5.2 Regular packing of forests

The undirected counterpart of Theorem 9, which generalizes Theorem 21, is as follows.

Theorem 25 ([31])**.** *Let $G = (V, E)$ be a graph and $h, k \in \mathbb{Z}_{>0}$. There exists an h-regular packing of k trees in G if and only if* (19) *holds and*

$$e_E(\mathcal{P}) \quad \geq \quad h|\mathcal{P}| - k \qquad \text{for every partition } \mathcal{P} \text{ of } V.$$

For $h = k$, Theorem 25 reduces to Theorem 21.

The undirected counterpart of Theorem 10, which generalizes Theorems 22 and 25, is as follows.

Theorem 26 ([33])**.** *Let $G = (V, E)$ be a graph and $h, k, \ell \in \mathbb{Z}_{>0}$. There exists an h-regular packing of k forests in G each having ℓ connected components if and only if* (21) *and* (22) *hold and*

$$e_E(\mathcal{P}) \quad \geq \quad h|\mathcal{P}| - k\ell \qquad \text{for every partition } \mathcal{P} \text{ of } V.$$

For $h = k$, Theorem 26 reduces to Theorem 22. For $\ell = 1$, Theorem 26 reduces to Theorem 25.

We now provide a common generalization of Theorems 23 and 26 which is the undirected counterpart of Theorem 11. The result and its proof come from [19].

Theorem 27. *Let $G = (V, E)$ be a graph, $h, k \in \mathbb{Z}_{>0}$ and $\ell : K \to \mathbb{Z}_{>0}$. There exists an h-regular packing of k forests in G with $\ell(1), \dots, \ell(k)$ connected components if and only if* (13) *and* (24) *hold and*

$$\ell_{|\mathcal{P}|}(K) + e_E(\mathcal{P}) \quad \geq \quad h|\mathcal{P}| \qquad \text{for every partition } \mathcal{P} \text{ of } V. \tag{31}$$

For $h = k$, Theorem 27 reduces to Theorem 23. For $\ell(i) = \ell$ for $i = 1, \dots, k$, Theorem 27 reduces to Theorem 26. The necessity of Theorem 27 follows from the necessity of its extension, Theorem 28. The sufficiency of Theorem 27 is proved in Subsection 6.3. This is the main proof of the present paper, it allows us to obtain Theorem 28.

The second main contribution of this paper is the following common extension of Theorems 24 and 27 which is the undirected counterpart of Theorem 12.

Theorem 28. *Let $G = (V, E)$ be a graph, $h, k \in \mathbb{Z}_{>0}$, and $\ell, \ell' : K \cup \{0\} \to \mathbb{Z}_{>0}$ such that* (15) *and* (16) *hold. There exists an h-regular packing of k S_i-forests B_i in G such that* (17) *and* (18) *hold if and only if* (26) *holds and*

$$\begin{aligned} \ell'(0) - \ell(K) + \ell_{|\mathcal{P}|}(K) + e_E(\mathcal{P}) &\geq h|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V, && (32) \\ \ell'_{|\mathcal{P}|}(K) + e_E(\mathcal{P}) &\geq h|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V. && (33) \end{aligned}$$

For $h = k$, Theorem 28 reduces to Theorem 24. For $\ell(i) = \ell'(i)$ for $i = 1, \dots, k$, $\ell(0) = \ell'(0) = \ell(K)$, Theorem 28 reduces to Theorem 27. The necessity of Theorem 28 follows from the necessity of its hypergraphic version, Theorem 36. The sufficiency of Theorem 28 is proved in Subsection 6.4. It will follow from Theorem 27.

5.3 Packing of spanning hyperforests

Theorem 21 was generalized to hypergraphs in [14], it is the undirected counterpart of Theorem 13.

Theorem 29 (Frank, Király, Kriesell [14])**.** *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph and $k \in \mathbb{Z}_{>0}$. There exists a packing of k spanning hypertrees in $\mathcal{G}$ if and only if*

$$e_{\mathcal{E}}(\mathcal{P}) \quad \geq \quad k(|\mathcal{P}| - 1) \qquad \textit{for every partition } \mathcal{P} \textit{ of } V.$$

If $\mathcal{G}$ is a graph then Theorem 29 reduces to Theorem 21.

The other results of this subsection are from [27].

The following result is a common generalization of Theorems 22 and 29 and the undirected counterpart of Theorem 14.

Theorem 30. *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph and $k, \ell \in \mathbb{Z}_{>0}$. There exists a packing of k spanning rooted hyperforests in $\mathcal{G}$ each having ℓ roots if and only if* (11) *holds and*

$$e_{\mathcal{E}}(\mathcal{P}) \quad \geq \quad k(|\mathcal{P}| - \ell) \qquad \textit{for every partition } \mathcal{P} \textit{ of } V.$$

If $\mathcal{G}$ is a graph then Theorem 30 reduces to Theorem 22. For $\ell = 1$, Theorem 30 reduces to Theorem 29.

The undirected counterpart of Theorem 15, a common generalization of Theorems 23 and 30 follows.

Theorem 31. *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph, $k \in \mathbb{Z}_{>0}$ and $\ell : K \to \mathbb{Z}_{>0}$. There exists a packing of k spanning rooted hyperforests in $\mathcal{G}$ with $\ell(1), \dots, \ell(k)$ roots if and only if* (13) *holds and*

$$\ell_{|\mathcal{P}|}(K) + e_{\mathcal{E}}(\mathcal{P}) \quad \geq \quad k|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V.$$

If $\mathcal{G}$ is a graph then Theorem 31 reduces to Theorem 23. For $\ell(i) = \ell$ for $i = 1, \dots, k$, Theorem 31 reduces to Theorem 30.

The third main contribution of this paper is the following extension of Theorems 24 and 31 which is the undirected counterpart of Theorem 16.

Theorem 32. *Let $\mathcal{G}=(V,\mathcal{E})$ be a hypergraph, $k\in\mathbb{Z}_{>0}$, $\ell,\ell':K\cup\{0\}\to\mathbb{Z}_{>0}$ such that* (15) *and* (16) *hold. There exists a packing of k spanning S_i-hyperforests B_i in $\mathcal{G}$ such that* (17) *and* (18) *hold if and only if*

$$\begin{aligned}\ell'(0)-\ell(K)+\ell_{|\mathcal{P}|}(K)+e_{\mathcal{E}}(\mathcal{P}) &\geq k|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V,\\ \ell'_{|\mathcal{P}|}(K)+e_{\mathcal{E}}(\mathcal{P}) &\geq k|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V.\end{aligned}$$

If $\mathcal{G}$ is a graph then Theorem 32 reduces to Theorem 24. If $\ell(i)=\ell'(i)$ for $i=1,\dots,k$, $\ell(0)=\ell'(0)=\ell(K)$ then Theorem 32 reduces to Theorem 31. Theorem 32 follows from its extension to regular packings, Theorem 36. We mention that Theorem 32 was proved in [27] by applying the theory of generalized polymatroids.

5.4 Regular packing of hyperforests

Theorems 25 and 29 were generalized in [31], giving the following undirected counterpart of Theorem 17.

Theorem 33 ([31])**.** *Let $\mathcal{G}=(V,\mathcal{E})$ be a hypergraph and $h,k\in\mathbb{Z}_{>0}$. There exists an h-regular packing of k hypertrees in $\mathcal{G}$ if and only if* (19) *holds and*

$$e_{\mathcal{E}}(\mathcal{P}) \geq h|\mathcal{P}|-k \qquad \textit{for every partition } \mathcal{P} \textit{ of } V.$$

If $\mathcal{G}$ is a graph then Theorem 33 reduces to Theorem 25. For $h=k$, Theorem 33 reduces to Theorem 29.

The following result from [33] is a common generalization of Theorems 26, 30 and 33 and the undirected counterpart of Theorem 18.

Theorem 34 ([33])**.** *Let $\mathcal{G}=(V,\mathcal{E})$ be a hypergraph and $h,k,\ell\in\mathbb{Z}_{>0}$. There exists an h-regular packing of k rooted hyperforests in $\mathcal{G}$ each having ℓ roots if and only if* (21) *and* (22) *hold and*

$$e_{\mathcal{E}}(\mathcal{P}) \geq h|\mathcal{P}|-k\ell \qquad \textit{for every partition } \mathcal{P} \textit{ of } V.$$

If $\mathcal{G}$ is a graph then Theorem 34 reduces to Theorem 26. For $h=k$, Theorem 34 reduces to Theorem 30. For $\ell=1$, Theorem 34 reduces to Theorem 33.

The undirected counterpart of Theorem 19, a common generalization of Theorems 27, 31 and 34, follows.

Theorem 35. *Let $\mathcal{G}=(V,\mathcal{E})$ be a hypergraph, $h,k\in\mathbb{Z}_{>0}$ and $\ell:K\to\mathbb{Z}_{>0}$. There exists an h-regular packing of k rooted hyperforests in $\mathcal{G}$ with $\ell(1),\dots,\ell(k)$ roots if and only if* (13) *and* (24) *hold and*

$$\ell_{|\mathcal{P}|}(K)+e_{\mathcal{E}}(\mathcal{P}) \geq h|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V.$$

If $\mathcal{G}$ is a graph then Theorem 35 reduces to Theorem 27. For $h=k$, Theorem 35 reduces to Theorem 31. For $\ell(i)=\ell$ for $i=1,\dots,k$, Theorem 35 reduces to Theorem 34.

The main result of this paper is the following undirected counterpart of Theorem 20, which is a common extension of Theorems 28, 32, 35.

Theorem 36. *Let $\mathcal{G}=(V,\mathcal{E})$ be a hypergraph, $h,k\in\mathbb{Z}_{>0}$, $\ell,\ell':K\cup\{0\}\to\mathbb{Z}_{>0}$ such that* (15) *and* (16) *hold. There exists an h-regular packing of k S_i-hyperforests B_i in $\mathcal{G}$ such that* (17) *and* (18) *hold if and only if* (26) *holds and*

$$\begin{aligned}\ell'(0)-\ell(K)+\ell_{|\mathcal{P}|}(K)+e_{\mathcal{E}}(\mathcal{P}) &\geq h|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V, &(34)\\ \ell'_{|\mathcal{P}|}(K)+e_{\mathcal{E}}(\mathcal{P}) &\geq h|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V. &(35)\end{aligned}$$

If $\mathcal{G}$ is a graph then Theorem 36 reduces to Theorem 28. For $h = k$, Theorem 36 reduces to Theorem 32. If $\ell(i) = \ell'(i)$ for $i = 1, \dots, k$ and $\ell(0) = \ell'(0) = \ell(K)$ then Theorem 36 reduces to Theorem 35. Note that Theorem 36 implies all the results of Section 5. Theorem 36 is proved in Subsection 6.5. It will follow from its graphic version, Theorem 28, by applying the operation trimming.

6 Proofs

In this section we prove our new results, Theorems 20, 23, 27, 28, and 36.

6.1 Proof of Theorems 20

Proof. To prove the **necessity**, let $\{B_1, \dots, B_k\}$ be an h-regular packing of k S_i-hyperbranchings such that (17) and (18) hold. Since the packing is h-regular, we have, by (18), that $h|V| \geq \sum_{i=1}^{k} |S_i| \geq \ell(0)$, thus (26) holds. To prove (29) and (30), let $\mathcal{P}$ be a subpartition of V.

For $i \in K$, we denote by $\mathcal{P}_i = \{X \in \mathcal{P} : X$ intersects the core of $B_i\}$. Since for every $X \in \mathcal{P}_i$ that contains no root of B_i, there exists a hyperarc of B_i that enters X, we get that

$$e_{\mathcal{A}(B_i)}(\mathcal{P}) \geq |\mathcal{P}_i| - |S_i|. \tag{36}$$

Then, by (36), $\max\{0, a - b\} = a - \min\{a, b\}$ and (17), we have

$$\begin{aligned} |S_i| - \ell(i) + e_{\mathcal{A}(B_i)}(\mathcal{P}) &\geq \max\{0, |\mathcal{P}_i| - \ell(i)\} \geq |\mathcal{P}_i| - \ell_{|\mathcal{P}|}(i), & (37) \\ e_{\mathcal{A}(B_i)}(\mathcal{P}) &\geq \max\{0, |\mathcal{P}_i| - \ell'(i)\} \geq |\mathcal{P}_i| - \ell'_{|\mathcal{P}|}(i). & (38) \end{aligned}$$

It follows, by (37), (18), and (38), that we have

$$\begin{aligned} e_{\mathcal{A}}(\mathcal{P}) &\geq \sum_{i=1}^{k} e_{\mathcal{A}(B_i)}(\mathcal{P}) \geq \sum_{i=1}^{k} |\mathcal{P}_i| - \ell_{|\mathcal{P}|}(K) + \ell(K) - \ell'(0), & (39) \\ e_{\mathcal{A}}(\mathcal{P}) &\geq \sum_{i=1}^{k} e_{\mathcal{A}(B_i)}(\mathcal{P}) \geq \sum_{i=1}^{k} |\mathcal{P}_i| - \ell'_{|\mathcal{P}|}(K). & (40) \end{aligned}$$

Further, we have

$$\sum_{i=1}^{k} |\mathcal{P}_i| = \sum_{i=1}^{k} \sum_{X \in \mathcal{P}_i} 1 = \sum_{X \in \mathcal{P}} \sum_{\substack{i=1 \\ X \in \mathcal{P}_i}}^{k} 1 \geq \sum_{X \in \mathcal{P}} h = h|\mathcal{P}|. \tag{41}$$

Finally, (39) and (41) imply (29); (40) and (41) imply (30).

Now let us prove the **sufficiency**. Suppose that (26), (29) and (30) hold. Note that, by (30) applied for $\mathcal{P} = \{V\}$, we obtain that $k \geq h$. Let $T = V$, $S = \{s_1, \dots, s_k\}$, $p(Y) = h - d^-_{\mathcal{A}}(Y)$ if $\emptyset \neq Y \subseteq V$ and 0 if $Y = \emptyset$. Then p is an intersecting supermodular set function on T. Let $f(v) = 0$ and $g(v) = h$ for every $v \in T$, $f(s_i) = \ell(i), g(s_i) = \ell'(i)$ for every $i \in K$, $\alpha = \ell(0)$ and $\beta = \ell'(0)$. Note that, by (15) and (16), we have $f \leq g$ and $\alpha \leq \beta$.

Claim 3. *For all $X \subseteq S, Y \subseteq T$ and subpartition $\mathcal{P}$ of $T - Y$, (7)– (10) hold.*

Proof. Since $f(v) = 0$ for every $v \in T$, the claim is equivalent to the following where $Z = T - Y$. For all $X \subseteq K, Z \subseteq V$ and subpartition $\mathcal{P}$ of Z, we have

$$
\begin{aligned}
h|\mathcal{P}| - e_{\mathcal{A}}(\mathcal{P}) - |X||\mathcal{P}| &\leq \ell'(\overline{X}), && (42)\\
\ell(X) + h|\mathcal{P}| - e_{\mathcal{A}}(\mathcal{P}) - |X||\mathcal{P}| &\leq h|Z| + |X||\overline{Z}|, && (43)\\
\ell(0) + h|\mathcal{P}| - e_{\mathcal{A}}(\mathcal{P}) - |X||\mathcal{P}| &\leq \ell'(\overline{X}) + h|Z| + |X||\overline{Z}|, && (44)\\
\ell(X) + h|\mathcal{P}| - e_{\mathcal{A}}(\mathcal{P}) - |X||\mathcal{P}| &\leq \ell'(0). && (45)
\end{aligned}
$$

Now, (42) is equivalent to (30) and (45) is equivalent to (29).

If $|X| \geq h$ then, by (26), $\ell' \geq 0$ and (15), we have $h|\mathcal{P}| - e_{\mathcal{A}}(\mathcal{P}) - |X||\mathcal{P}| \leq 0 \leq h|V| - \ell(0) \leq h|Z| + |X||\overline{Z}| - \ell(0) \leq h|Z| + |X||\overline{Z}| - \ell(0) + \min\{\ell'(\overline{X}), \ell(0) - \ell(X)\}$, so (43) and (44) follow.

If $|X| \leq h$ then, by $|Z| \geq |\mathcal{P}|$, (16) and (15), we have $-e_{\mathcal{A}}(\mathcal{P}) \leq 0 \leq (h - |X|)(|Z| - |\mathcal{P}|) + (|X||V| - \ell'(X)) + \min\{\ell'(K) - \ell(0), \ell'(X) - \ell(X)\}$, so (43) and (44) follow. □

By Claim 3, Theorem 3 can be applied and hence there exists a simple bipartite graph $\boldsymbol{G} = (S, T, E)$ such that $|\Gamma_E(Y)| \geq p(Y)$ for every $Y \subseteq V$, $d_E(v) \leq h$ for every $v \in V$, $\ell(i) \leq d_E(s_i) \leq \ell'(i)$ for every $i \in K$ and $\ell(0) \leq |E| \leq \ell'(0)$.

Let $\boldsymbol{S_i}$ be the set of neighbors of s_i in G and $\boldsymbol{\mathcal{S}} = \{S_1, \ldots, S_k\}$. Since $|\mathcal{S}_Y| = |\Gamma_E(Y)| \geq h - d^-_{\mathcal{A}}(Y)$ for every $\emptyset \neq Y \subseteq V$ and $|\mathcal{S}_v| = d_E(v) \leq h$ for every $v \in V$, we may apply Theorem 4 to $\mathcal{D}$. Therefore there exists an h-regular packing of k S_i-hyperbranchings B_i in $\mathcal{D}$.

Furthermore, since $|S_i| = d_E(s_i)$ for every $i \in K$ and $\sum_{i=1}^k |S_i| = |E|$, we have that $\ell(i) \leq |S_i| \leq \ell'(i)$ for every $i \in K$ and $\ell(0) \leq \sum_{i=1}^k |S_i| \leq \ell'(0)$. This proves that the packing satisfies (17) and (18) which concludes the proof. □

6.2 Proof of Theorem 23

Proof. In this proof we use matroids so we consider spanning forests as edge sets. Let $\mathsf{M}_{\boldsymbol{G}} = (E, r_G)$ be the graphic matroid of $G = (V, E)$, that is the independent sets of M_G are the spanning forests of G and we have $r_G(F) = |V| - c(V, F)$ for every $F \subseteq E$. For $i = 1, \ldots, k$, by (13), we have $|V| - \ell(i) \geq 0$. We can hence define $\mathsf{M}_{\boldsymbol{i}} = (E, r_i)$ to be the truncated matroid of M_G at $|V| - \ell(i)$, that is the independent sets of M_i are the spanning forests in G with at most $|V| - \ell(i)$ edges (so at least $\ell(i)$ connected components) and, by the definition of r_G, we have

$$r_i(F) = \min\{|V| - c(V, F), |V| - \ell(i)\} = |V| - \max\{c(V, F), \ell(i)\} \text{ for every } F \subseteq E. \tag{46}$$

Let $\mathsf{M}^{\boldsymbol{*}} = (E, r^*)$ be the sum of the matroids $\mathsf{M}_1, \ldots, \mathsf{M}_k$. The next claim implies Theorem 23.

Claim 4. *The following statements are equivalent.*

(a) *There exists a packing of k spanning forests in G with $\ell(1), \ldots, \ell(k)$ connected components.*

(b) *There exists an independent set F in M^* of size $k|V| - \ell(K)$.*

(c) *$e_E(\mathcal{P}) \geq k|\mathcal{P}| - \ell_{|\mathcal{P}|}(K)$ for every partition $\mathcal{P}$ of V.*

Proof. (a) $\Leftrightarrow$ (b): First suppose that there exists a packing of spanning forests $F_1, \ldots, F_k$ in G such that $c(V, F_i) = \ell(i)$ for every $i \in K$. Then F_i is independent in M_i for every $i \in K$ and hence $F = \bigcup_{i=1}^k F_i$ is independent in M^*. Further, we have

$$|F| = \sum_{i=1}^k |F_i| = \sum_{i=1}^k (|V| - c(V, F_i)) = \sum_{i=1}^k (|V| - \ell(i)) = k|V| - \ell(K).$$

Now suppose that there exists an independent set F in M^* of size $k|V| - \ell(K)$. Then there exist pairwise disjoint edge sets $F_1, \ldots, F_k$ such that $\bigcup_{i=1}^k F_i = F$ and F_i is independent in M_i that is $r_i(F_i) = |F_i|$ for every $i \in K$. It follows, by (46), that

$$k|V| - \ell(K) = |F| = \sum_{i=1}^{k} |F_i| = \sum_{i=1}^{k} r_i(F_i) \leq \sum_{i=1}^{k} (|V| - \ell(i)) = k|V| - \ell(K),$$

so $r_i(F_i) = |V| - \ell(i)$ for every $i \in K$. Hence F_i is a spanning forest in G with $|V| - |F_i| = \ell(i)$ connected components for every $i \in K$.

(b) $\Leftrightarrow$ (c): First, (b) is equivalent to $r^*(E) \geq k|V| - \ell(K)$. This, by Theorem 1, is equivalent to $|E - F| + \sum_{i=1}^k r_i(F) \geq \sum_{i=1}^k (|V| - \ell(i))$ for every $F \subseteq E$, which, by (46), is equivalent to

$$|E - F| \quad \geq \quad \sum_{i=1}^{k} \max\{0, c(V, F) - \ell(i)\} \qquad \text{for every } F \subseteq E. \tag{47}$$

Secondly, since $-\ell_{|\mathcal{P}|}(i) = \max\{-\ell(i), -|\mathcal{P}|\}$ for $i = 1, \ldots, k$, (c) is equivalent to

$$e_E(\mathcal{P}) \quad \geq \quad \sum_{i=1}^{k} \max\{0, |\mathcal{P}| - \ell(i)\} \qquad \text{for every partition } \mathcal{P} \text{ of } V. \tag{48}$$

We hence need to prove that (47) and (48) are equivalent.

First suppose that (47) holds. Let $\mathcal{P}$ be a partition of V and $F \subseteq E$ the set of edges that do not cross $\mathcal{P}$. Then $e_E(\mathcal{P}) = |E - F|$ and $c(V, F) \geq |\mathcal{P}|$, and hence, (47) implies (48).

Now suppose that (48) holds. Let $F \subseteq E$ and $\mathcal{P} = \mathcal{P}(F)$. Then $|E - F| \geq e_E(\mathcal{P})$ and $c(V, F) = |\mathcal{P}|$, and hence, (48) implies (47). This concludes the proof of the claim. $\square$

As mentioned above Theorem 23 follows from Claim 4. $\square$

6.3 Proof of Theorem 27

Proof. The **necessity** of this theorem follows from the necessity of Theorem 36.

To prove the **sufficiency**, let us suppose that (13), (24) and (31) hold. We may suppose without loss of generality that

$$\ell(1) \geq \ldots \geq \ell(k). \tag{49}$$

By (31) applied for $\mathcal{P} = \{V\}$, we get that $k \geq h$. If $k = h$, Theorem 27 reduces to Theorem 23. We hence suppose that $k > h$. For a packing $\mathcal{F}$ of spanning forests, we denote by $\boldsymbol{c_{\min}(\mathcal{F})}$ the minimum number of connected components of a forest in $\mathcal{F}$.

Lemma 2. *There exist an index i and a packing $\mathcal{F} = \{F_1, \ldots, F_h\}$ of h spanning forests in G such that the following hold*

(i) $c(F_j) = \ell(j)$ *for* $j \in \{1, \ldots, i\}$,

(ii) $c_{\min}(\mathcal{F}) \geq \ell(i+1)$,

(iii) $\sum_{j=i+1}^{h} c(F_j) = \sum_{j=i+1}^{k} \ell(j)$.

Proof. Let $\boldsymbol{\ell(0)} = |V|$. Then, by (24), the following inequality holds for $i' = 0$. Let hence $\boldsymbol{i'}$ be the maximum integer such that

$$(h - i')\ell(i') \geq \sum_{j=i'+1}^{k} \ell(j). \tag{50}$$

By $k > h$, we have $i' < h$. Then, by the maximality of i', we have $(h-i'-1)\ell(i'+1) < \sum_{j=i'+2}^{k} \ell(j)$ that is

$$(h - i')\ell(i' + 1) < \sum_{j=i'+1}^{k} \ell(j). \tag{51}$$

Let the function $\boldsymbol{\ell'} : \{1, \dots, h\} \to \mathbb{Z}_{>0}$ be obtained from the function ℓ by evenly distributing $\ell(i'+1), \dots, \ell(k)$ over $\ell'(i'+1), \dots, \ell'(h)$ that is, for $\boldsymbol{\ell^*} = \lceil \frac{\sum_{j=i'+1}^{k} \ell(j)}{h-i'} \rceil$,

$$\begin{aligned} \ell'(j) &= \ell(j) && \text{for } 1 \leq j \leq i', && (52)\\ \ell'(j) &\in \{\ell^*, \ell^* - 1\} && \text{for } i' + 1 \leq j \leq h, && (53)\\ \sum_{j=i'+1}^{k} \ell(j) &= \sum_{j=i'+1}^{h} \ell'(j), && && (54) \end{aligned}$$

By (50) and (51), we have

$$\ell(i') \geq \ell^* > \ell(i' + 1). \tag{55}$$

Claim 5. *The conditions of Theorem 23 hold for ℓ', that is*

$$\begin{aligned} &|V| \geq \ell'(j) && \textit{for every } 1 \leq j \leq h, && (56)\\ &e_E(\mathcal{P}) \geq h|\mathcal{P}| - \sum_{j=1}^{h} \ell'_{|\mathcal{P}|}(j) && \textit{for every partition } \mathcal{P} \textit{ of } V. && (57) \end{aligned}$$

Proof. We first consider (56). For $1 \leq j \leq i'$, we have, by (13) and (52), that $|V| \geq \ell(j) = \ell'(j)$. For $i' + 1 \leq j \leq h$, we get, by (13), (55) and (53), that $|V| \geq \ell(i') \geq \ell^* \geq \ell'(j)$, so (56) holds. To show that (57) holds, let $\mathcal{P}$ be a partition of V.

If $|\mathcal{P}| \leq \ell^* - 1$ then, by (55), (49) and (52) for $1 \leq j \leq i'$, and, by (53) for $i' + 1 \leq j \leq h$, we get $|\mathcal{P}| \leq \ell^* - 1 \leq \ell'(j)$ for $1 \leq j \leq k$, thus $e_E(\mathcal{P}) \geq 0 = h|\mathcal{P}| - \sum_{j=1}^{h} \ell'_{|\mathcal{P}|}(j)$, so (57) holds.

If $|\mathcal{P}| \geq \ell^*$ then we have, by (55) and (49), that $|\mathcal{P}| \geq \ell^* > \ell(j)$ that is $\ell_{|\mathcal{P}|}(j) = \ell(j)$ for $i' + 1 \leq j \leq k$, and, by (53), we get $|\mathcal{P}| \geq \ell^* \geq \ell'(j)$ that is $\ell'_{|\mathcal{P}|}(j) = \ell'(j)$ for $i' + 1 \leq j \leq k$. Thus, by (52) and (54), we have

$$\sum_{j=1}^{k} \ell_{|\mathcal{P}|}(j) = \sum_{j=1}^{i'} \ell_{|\mathcal{P}|}(j) + \sum_{j=i'+1}^{k} \ell(j) = \sum_{j=1}^{i'} \ell'_{|\mathcal{P}|}(j) + \sum_{j=i'+1}^{h} \ell'(j) = \sum_{j=1}^{h} \ell'_{|\mathcal{P}|}(j). \tag{58}$$

By (31) and (58), we get that (57) holds: $e_E(\mathcal{P}) \geq h|\mathcal{P}| - \sum_{j=1}^{k} \ell_{|\mathcal{P}|}(j) = h|\mathcal{P}| - \sum_{j=1}^{h} \ell'_{|\mathcal{P}|}(j)$. □

According to Claim 5 and Theorem 23 applied for ℓ', there exists a packing $\mathcal{F}$ of h spanning forests in G with $\ell'(1), \dots, \ell'(h)$ connected components. By (52), (53), (54) and $c_{\min}(\mathcal{F}) \geq \ell^* - 1 \geq \ell(i + 1)$, the index i' and the packing $\mathcal{F}$ satisfy (i), (ii) and (iii), which completes the proof. □

By Lemma 2, there exists a packing of h spanning forests in G satisfying (i)–(iii). Let $\boldsymbol{\mathcal{F}} = \mathcal{F}_1 \cup \mathcal{F}_2$ be such a packing with maximum $i = |\mathcal{F}_1|$ and with minimum $c_{\min}(\mathcal{F}_2)$. Let $\boldsymbol{F_{\min}}$ be a member of $\mathcal{F}_2$ such that $c(F_{\min}) = c_{\min}(\mathcal{F}_2)$. By (ii), we have $c_{\min}(\mathcal{F}_2) \geq c_{\min}(\mathcal{F}) \geq \ell(i + 1)$. If

$c_{\min}(\mathcal{F}_2) = \ell(i+1)$ then by moving $F_{\min}$ from $\mathcal{F}_2$ to $\mathcal{F}_1$ we obtain a packing satisfying (i), (ii), and (iii) that contradicts the maximality of i. Therefore, we have

$$c_{\min}(\mathcal{F}_2) > \ell(i+1). \tag{59}$$

Let $\boldsymbol{\mathcal{T}}$ be the set of the connected components of the forests of $\mathcal{F}_2$.

Claim 6. *The vertex set of an element of $\mathcal{T}$ does not cross $\mathcal{P}(F_{\min})$.*

Proof. Suppose for a contradiction that there exists a connected component of a spanning forest $\boldsymbol{F}$ in $\mathcal{F}_2$ whose vertex set crosses $\mathcal{P}(F_{\min})$. Let $\boldsymbol{a}$ be an edge of F that connects two connected components of $F_{\min}$, $\boldsymbol{\mathcal{F}'_2} = \mathcal{F}_2 - \{F, F_{\min}\} + \{F - a, F_{\min} + a\}$ and $\mathcal{F}' = \mathcal{F}_1 \cup \mathcal{F}'_2$. Then $c(F-a) = c(F) + 1$ and, by (59), we have $c_{\min}(\mathcal{F}'_2) = c(F_{\min} + a) = c(F_{\min}) - 1 = c_{\min}(\mathcal{F}_2) - 1 \geq \ell(i+1)$. Then $c_{\min}(\mathcal{F}') = \min\{c_{\min}(\mathcal{F}_1), c_{\min}(\mathcal{F}'_2)\} \geq \ell(i+1)$. Hence $\mathcal{F}'$ is a packing of h spanning forests of G satisfying (i), (ii), (iii), and maximizing i; which contradicts the minimality of $c_{\min}(\mathcal{F})$. □

Let $\boldsymbol{G_j} = (V, E_j)$ where $\boldsymbol{E_j} = E(\mathcal{F}_j)$ for $j = 1, 2$.

Claim 7. *There exists an $(h-i)$-regular packing of $k-i$ forests in G_2 with $\ell(i+1), \dots, \ell(k)$ connected components.*

Proof. Let $\{\boldsymbol{V_1}, \dots, \boldsymbol{V_p}\} = \mathcal{P}(F_{\min})$. For $T \in \mathcal{T}$, let $\boldsymbol{\alpha(T)}$ and $\boldsymbol{\beta(T)}$ be the indices such that $V(T) \in \mathcal{P}(F_{\alpha(T)})$ and $V(T) \subseteq V_{\beta(T)}$. Note that $\beta(T)$ is well defined because, by Claim 6, $V(T)$ does not cross $\mathcal{P}(F_{\min})$. We introduce an order on $\mathcal{T}$. For $T, T' \in \mathcal{T}$, we say T precedes T' if either $\alpha(T) < \alpha(T')$ or $\alpha(T) = \alpha(T')$ and $\beta(T) \leq \beta(T')$; in case of equality we can arbitrarily choose the order. Note that if $T, T' \in \mathcal{T}$ intersect then $\alpha(T) \neq \alpha(T')$ and $\beta(T) = \beta(T')$, so there are at least $p-1$ elements of $\mathcal{T}$ between T and T' in the order. For $j \in \{i+1, \dots, k\}$, let $\boldsymbol{F^*_j}$ be the union of the first $\ell(j)$ consecutive elements of $\mathcal{T}$ in the order that have not been taken yet. By (49) and (59), we have $\ell(j) \leq \ell(i+1) < c(F_{\min}) = |\mathcal{P}(F_{\min})| = p$. Hence each F^*_j contains pairwise disjoint elements of $\mathcal{T}$, implying that each F^*_j is a forest. By (iii), each element of $\mathcal{T}$ is used exactly once in $\boldsymbol{\mathcal{F}^*} = \{F^*_{i+1}, \dots, F^*_k\}$. Since $\mathcal{F}_2$ is an $(h-i)$-regular packing, $\mathcal{F}^*$ is an $(h-i)$-regular packing of $k-i$ forests in G_2 with $\ell(i+1), \dots, \ell(k)$ connected components. □

By Claim 7, there exists an $(h-i)$-regular packing $\mathcal{F}^*$ of $k-i$ forests in G_2 with $\ell(i+1), \dots, \ell(k)$ connected components. Recall that $\mathcal{F}_1$ is an i-regular packing of i forests in G_1 with $\ell(1), \dots, \ell(i)$ connected components. Further, $E_1 \cap E_2 = \emptyset$ and $E(\mathcal{F}^*) \subseteq E_2$. Therefore, $\mathcal{F}_1 \cup \mathcal{F}^*$ is an h-regular packing of k forests in G with $\ell(1), \dots, \ell(k)$ connected components that completes the proof of Theorem 27. □

6.4 Proof of Theorem 28

Proof. The **necessity** of this theorem follows from the necessity of Theorem 36.

Let us now prove the **sufficiency**. Suppose that (26), (32) and (33) hold. If $\ell'(0) \geq h|V|$ then, by (15) and (26), we have $\ell'(K) \geq \ell'(0) \geq h|V| \geq \ell(0) \geq \ell(K)$. Further, by (16), we have $\ell'(i) \geq \ell(i)$ for every $1 \leq i \leq k$. Then there exists $\ell^* : K \to \mathbb{Z}_{\geq 0}$ such that $\ell(i) \leq \ell^*(i) \leq \ell'(i)$ for every $1 \leq i \leq k$ and $\ell^*(K) = h|V|$. By applying the proof technic of Claim 7 to the h spanning forests, each consisting of $|V|$ vertices, and to $\ell^*(1), \dots, \ell^*(k)$, we obtain that there exists an h-regular packing of k forests consisting of respectively $\ell^*(1), \dots, \ell^*(k)$ isolated vertices. This packing satisfies the requirements.

From now on we suppose that $\ell'(0) < h|V|$. Let $\boldsymbol{\ell^*} : K \to \mathbb{Z}_{\geq 0}$ be a function that satisfies

$$\ell'(0) \geq \ell^*(K), \tag{60}$$

$$\ell'(i) \geq \ell^*(i) \geq \ell(i) \qquad \text{for every } 1 \leq i \leq k, \tag{61}$$

$$\ell'(0) - \ell^*(K) + \ell^*_{|\mathcal{P}|}(K) \geq h|\mathcal{P}| - e_E(\mathcal{P}) \qquad \text{for every partition } \mathcal{P} \text{ of } V, \tag{62}$$

$$\ell^*(K) \text{ is maximum.} \tag{63}$$

Note that such ℓ^* exists because, by (15), (16) and (32), ℓ satisfies (60), (61) and (62).

Lemma 3. $\ell^*(K) = \ell'(0)$.

Proof. Suppose that it is not the case. Then, by (60), $\ell^*(K) < \ell'(0)$.

Claim 8. *There exists* $p^* \in \{0, \dots, |V|\}$ *and* $j \in K$ *such that*

$$\ell'(0) - \ell^*(K) + \ell^*_p(K) > \ell'_p(K) \qquad \textit{for every } 0 \leq p \leq p^*, \tag{64}$$

$$\ell^*(j) \leq p^* < \ell'(j). \tag{65}$$

Proof. Note that, by $\ell^*(K) < \ell'(0)$, we have $\ell'(0) - \ell^*(K) + \ell^*_0(K) = \ell'(0) - \ell^*(K) > 0 = \ell'_0(K)$. Further, by (16) and (15), we have $\ell'(0) - \ell^*(K) + \ell^*_{|V|}(K) = \ell'(0) \leq \ell'(K) = \ell'_{|V|}(K)$. It follows that there exists a maximum $p^* \in \{0, \dots, |V|-1\}$ satisfying (64). By the maximality of p^* and (64), we have $\ell'(0) - \ell^*(K) + \ell^*_{p^*+1}(K) \leq \ell'_{p^*+1}(K)$ and $\ell'(0) - \ell^*(K) + \ell^*_{p^*}(K) > \ell'_{p^*}(K)$. This implies that there exists $j \in K$ such that $\min\{p^*+1, \ell'(j)\} = \min\{p^*, \ell'(j)\} + 1$ and $\min\{p^*+1, \ell^*(j)\} = \min\{p^*, \ell^*(j)\}$. Hence $\ell'(j) \geq p^*+1$ and $\ell^*(j) \leq p^*$, that is (65) holds. □

By Claim 8, there exists $\boldsymbol{p^*} \in \mathbb{Z}$ and $\boldsymbol{j} \in K$ satisfying (64) and (65). Let $\boldsymbol{\ell^+} : K \to \mathbb{Z}_{>0}$ be defined as follows: $\ell^+(i) = \ell^*(i)$ for all $i \in K - \{j\}$ and $\ell^+(j) = \ell^*(j) + 1$.

Claim 9. ℓ^+ *satisfies* (60)*,* (61) *and* (62)*.*

Proof. By $\ell^*(K) < \ell'(0)$, we have $\ell^+(K) = \ell^*(K) + 1 \leq \ell'(0)$, so (60) holds for ℓ^+.

By (61), we have $\ell(i) \leq \ell^*(i) = \ell^+(i) = \ell^*(i) \leq \ell'(i)$ for every $i \in K - \{j\}$ and, by (65), we have $\ell(j) \leq \ell^*(j) \leq \ell^+(j) = \ell^*(j) + 1 \leq \ell'(j)$, so (61) holds for ℓ^+.

Let $\mathcal{P}$ be a partition of V. If $|\mathcal{P}| > p^*$ then, by $\ell^*(j) \leq p^*$ and (62), we have $\ell'(0) - \ell^+(K) + \ell^+_{|\mathcal{P}|}(K) = \ell'(0) - (\ell^*(K)+1) + (\ell^*_{|\mathcal{P}|}(K)+1) \geq h|\mathcal{P}| - e_E(\mathcal{P})$. If $|\mathcal{P}| \leq p^*$ then, by (64) and (33), we have $\ell'(0) - \ell^+(K) + \ell^+_{|\mathcal{P}|}(K) = \ell'(0) - (\ell^*(K)+1) + \ell^*_{|\mathcal{P}|}(K) \geq \ell'_{|\mathcal{P}|}(K) \geq h|\mathcal{P}| - e_E(\mathcal{P})$. Therefore, (62) holds for ℓ^+, that completes the proof of Claim 9. □

Claim 9 and $\ell^+(K) > \ell^*(K)$ contradict the maximality of $\ell^*(K)$, completing the proof of Lemma 3. □

By Lemma 3 and (62), (31) holds for ℓ^*. By (16) and (61), (13) holds for ℓ^*. Finally, by Lemma 3, we have $h|V| > \ell'(0) = \ell^*(K)$, so (24) holds for ℓ^*. Hence, by Theorem 27, there exists an h-regular packing of k forests $F_1, \dots, F_k$ with respectively $\ell^*(1), \dots, \ell^*(k)$ connected components. We can choose a vertex set S_i such that $|S_i| = \ell^*(i)$ and F_i becomes an S_i-forest for every $1 \leq i \leq k$. By (61), Lemma 3 and (15), this packing satisfies (17) and (18), that completes the proof of Theorem 28. □

6.5 Proof of Theorem 36

Proof. We first show the **necessity**. Suppose that there exists an h-regular packing of k S_i-hyperforests in $\mathcal{G}$ such that (17) and (18) hold. Then we can orient $\mathcal{G}$ to get a dypergraph $\mathcal{D}$ that has an h-regular packing of k S_i-hyperbranchings in $\mathcal{D}$ such that (17) and (18) hold. Then the necessity of Theorem 20 implies the necessity of Theorem 36.

To show the **sufficiency**, suppose that (26), (34) and (35) hold in $\mathcal{G}$.

Lemma 4. *The hypergraph $\mathcal{G}$ can be trimmed to a graph G that satisfies* (26)*,* (32) *and* (33)*.*

Proof. We prove the lemma by induction on $\sum_{X\in\mathcal{E}}|X|$. If for every $X \in \mathcal{E}$, $|X| = 2$ then $\mathcal{G}$ is a graph and, (34) and (35) coincide with (32) and (33). Otherwise, there exists $X \in \mathcal{E}$ such that $|X| \geq 3$. We show that we can always remove a vertex from X without violating (26), (34) or (35). Note that the removal of a vertex from a hyperedge does not effect (26). Suppose that no vertex of X can be removed from X without violating the conditions, that is, for every $x \in X$, one of (34) and (35) is violated after the removal of x from X. By $|X| \geq 3$, there must be at least two vertices violating the same condition. Let $a, b \in X$ such that their removals violate the same condition. Since this condition was satisfied before the removal, there exist partitions $\mathcal{P}_a$ and $\mathcal{P}_b$ of V, such that either (66) or (67) hold, $e_{\mathcal{E}}(\mathcal{P}_a)$ decreases when removing a from X and $e_{\mathcal{E}}(\mathcal{P}_b)$ decreases when removing b from X.

$$\ell_{|\mathcal{P}_a|}(K) + e_{\mathcal{E}}(\mathcal{P}_a) - h|\mathcal{P}_a| = \ell(K) - \ell'(0) = \ell_{|\mathcal{P}_b|}(K) + e_{\mathcal{E}}(\mathcal{P}_b) - h|\mathcal{P}_b|, \tag{66}$$
$$\ell'_{|\mathcal{P}_a|}(K) + e_{\mathcal{E}}(\mathcal{P}_a) - h|\mathcal{P}_a| = \quad 0 \quad = \ell'_{|\mathcal{P}_b|}(K) + e_{\mathcal{E}}(\mathcal{P}_b) - h|\mathcal{P}_b|. \tag{67}$$

As removing a from X decreases $e_{\mathcal{E}}(\mathcal{P}_a)$, there exist $X_a, X_{\overline{a}} \in \mathcal{P}_a$ such that $a \in X_a$ and $X - a \subseteq X_{\overline{a}}$. The same is also true for b and $\mathcal{P}_b$, i.e. there exist $X_b, X_{\overline{b}} \in \mathcal{P}_b$ such that $b \in X_b$, and $X - b \subseteq X_{\overline{b}}$. Let $\boldsymbol{\mathcal{P}}_{\sqcup} = \mathcal{P}_a \sqcup \mathcal{P}_b$ and $\boldsymbol{\mathcal{P}}_{\sqcap} = \mathcal{P}_a \sqcap \mathcal{P}_b$. Since $X_{\overline{a}} \cap X_{\overline{b}} \supseteq X - a - b \neq \emptyset$, $X_a \cap X_{\overline{b}} \supseteq \{a\} \neq \emptyset$ and $X_{\overline{a}} \cap X_b \supseteq \{b\} \neq \emptyset$, we get that $X \subseteq X_a \cup X_b \cup X_{\overline{a}} \cup X_{\overline{b}}$ is contained in a member of $\mathcal{P}_{\sqcup}$. Thus X does not cross $\mathcal{P}_{\sqcup}$, i.e.

$$e_{\{X\}}(\mathcal{P}_a) + e_{\{X\}}(\mathcal{P}_b) > e_{\{X\}}(\mathcal{P}_{\sqcap}) + e_{\{X\}}(\mathcal{P}_{\sqcup}).$$

Then, according to Lemma 1, we obtain that

$$e_{\mathcal{E}}(\mathcal{P}_a) + e_{\mathcal{E}}(\mathcal{P}_b) > e_{\mathcal{E}}(\mathcal{P}_{\sqcap}) + e_{\mathcal{E}}(\mathcal{P}_{\sqcup}). \tag{68}$$

By (4) and (5), we can apply Claim 1 for $\ell^* \in \{\ell, \ell'\}$ and every $i \in K$, and we get

$$\min\{\ell^*(i), |\mathcal{P}_a|\} + \min\{\ell^*(i), |\mathcal{P}_b|\} \geq \min\{\ell^*(i), |\mathcal{P}_{\sqcap}|\} + \min\{\ell^*(i), |\mathcal{P}_{\sqcup}|\}.$$

By summing up these inequalities for $i \in K$, it follows that

$$\ell^*_{|\mathcal{P}_a|}(K) + \ell^*_{|\mathcal{P}_b|}(K) \geq \ell^*_{|\mathcal{P}_{\sqcap}|}(K) + \ell^*_{|\mathcal{P}_{\sqcup}|}(K). \tag{69}$$

If the violated condition was (34) then, by (66), (69) for ℓ, (68), (34) and (4), we have

$$\begin{aligned} h(|\mathcal{P}_a| + |\mathcal{P}_b|) &= 2(\ell'(0) - \ell(K)) + \ell_{|\mathcal{P}_a|}(K) + \ell_{|\mathcal{P}_b|}(K) + e_{\mathcal{E}}(\mathcal{P}_a) + e_{\mathcal{E}}(\mathcal{P}_b) \\ &> 2(\ell'(0) - \ell(K)) + \ell_{|\mathcal{P}_{\sqcap}|}(K) + \ell_{|\mathcal{P}_{\sqcup}|}(K) + e_{\mathcal{E}}(\mathcal{P}_{\sqcap}) + e_{\mathcal{E}}(\mathcal{P}_{\sqcup}) \\ &\geq h(|\mathcal{P}_{\sqcap}| + |\mathcal{P}_{\sqcup}|) \\ &= h(|\mathcal{P}_a| + |\mathcal{P}_b|), \end{aligned}$$

a contradiction.

If the violated condition was (35) then, by (67), (69) for ℓ', (68), (35), and (4), we have

$$\begin{aligned} h(|\mathcal{P}_a| + |\mathcal{P}_b|) &= \ell'_{|\mathcal{P}_a|}(K) + \ell'_{|\mathcal{P}_b|}(K) + e_{\mathcal{E}}(\mathcal{P}_a) + e_{\mathcal{E}}(\mathcal{P}_b) \\ &> \ell'_{|\mathcal{P}_\sqcap|}(K) + \ell'_{|\mathcal{P}_\sqcup|}(K) + e_{\mathcal{E}}(\mathcal{P}_\sqcap) + e_{\mathcal{E}}(\mathcal{P}_\sqcup) \\ &\geq h(|\mathcal{P}_\sqcap| + |\mathcal{P}_\sqcup|) \\ &= h(|\mathcal{P}_a| + |\mathcal{P}_b|), \end{aligned}$$

a contradiction.

The proof of Lemma 4 is complete. □

By Lemma 4, the hypergraph $\mathcal{G}$ can be trimmed to a graph G satisfying (26), (32) and (33). Therefore, according to Theorem 28, there exists an h-regular packing of k S_i-forests in G such that (17) and (18) hold. We can orient every S_i-forest in the packing to get an S_i-branching, which can be obtained by trimming from an S_i-hyperbranching in an orientation of an S_i-hyperforest of $\mathcal{G}$. This way we obtained the required packing which completes the proof of the theorem. □

7 Conclusion

We have presented results on packing hyperbranchings and the corresponding results on packing hyperforests. It is a natural question whether these results also hold for mixed hyperbranchings.

For the most general problem where each root set is of size ℓ, the answer is affirmative.

Theorem 37 ([33])**.** *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, and $h, k, \ell \in \mathbb{Z}_{>0}$. There exists an h-regular packing of k mixed hyperbranchings in $\mathcal{F}$ each with ℓ roots if and only if* (21) *and* (22) *hold and*

$$e_{\mathcal{E}}(\mathcal{P}) + e_{\mathcal{A}}(\mathcal{P}) \quad \geq \quad h|\mathcal{P}| - k\ell \qquad \text{for every subpartition } \mathcal{P} \text{ of } V.$$

For the simplest problem where the root sets are of size $\ell(1), \ldots, \ell(k)$, we propose the following natural conjecture.

Conjecture 1. *Let $F = (V, E \cup A)$ be a mixed graph, $k \in \mathbb{Z}_{>0}$, and $\ell : K \to \mathbb{Z}_{>0}$. There exists a packing of k spanning mixed branchings in F with $\ell(1), \ldots, \ell(k)$ roots if and only if* (13) *holds and*

$$\ell_{|\mathcal{P}|}(K) + e_E(\mathcal{P}) + e_A(\mathcal{P}) \quad \geq \quad k|\mathcal{P}| \qquad \text{for every subpartition } \mathcal{P} \text{ of } V.$$

References

[1] K. Bérczi, A. Frank, *Variations for Lovász' submodular ideas,* in Building Bridges, Springer, (2008) 137–164.

[2] K. Bérczi, A. Frank, *Packing arborescences,* in: S. Iwata (Ed.), RIMS Kokyuroku Bessatsu B23: Combinatorial Optimization and Discrete Algorithms, Lecture Notes, (2010) 1–31.

[3] K. Bérczi, A. Frank, *Supermodularity in Unweighted Graph Optimization I: Branchings and Matchings,* Math. Oper. Res. 43(3) (2018) 726–753.

[4] M. C. Cai, *Common root function of two graphs*, J. Graph Theory, 13 (1989) 249–256.

[5] O. Durand de Gevigney, V. H. Nguyen, Z. Szigeti, *Matroid-Based Packing of Arborescences*, SIAM J. Discret. Math. 27(1) (2013) 567–574.

[6] J. Edmonds, *Edge-disjoint branchings*, in Combinatorial Algorithms, B. Rustin ed., Academic Press, New York, (1973) 91–96.

[7] J. Edmonds and D. R. Fulkerson, *Transversals and matroid partition*, J. Res. Nat. Bur. Standards, B69 (1965) 147–53.

[8] Q. Fortier, Cs. Király, M. Léonard, Z. Szigeti, A. Talon, *Old and new results on packing arborescences*, Discret. Appl. Math. 242 (2018) 26–33.

[9] Q. Fortier, Cs. Király, Z. Szigeti, S. Tanigawa, *On packing spanning arborescences with matroid constraint*, J. Graph Theory 93(2) (2020) 230–252.

[10] A. Frank, *On disjoint trees and arborescences,* In Algebraic Methods in Graph Theory, 25, Colloquia Mathematica Soc. J. Bolyai, North-Holland, (1978) 59–169.

[11] A. Frank, *Rooted k-connections in digraphs*, Discret. Appl. Math. 157 (2009) 1242–1254.

[12] A. Frank, Connections in Combinatorial Optimization, Oxford University Press, 2011.

[13] A. Frank, T. Király, Z. Király, *On the orientation of graphs and hypergraphs,* Discret. Appl. Math. 131(2) (2003) 385–400.

[14] A. Frank, T. Király, M. Kriesell, *On decomposing a hypergraph into k connected sub-hypergraphs,* Discret. Appl. Math. 131 (2) (2003) 373–383.

[15] A. Frank, É. Tardos, *An application of submodular flows*, Linear Algebra and its Applications, 114/115, (1989) 329–348.

[16] H. Gao, D. Yang, *Packing of maximal independent mixed arborescences*, Discret. Appl. Math. 289 (2021) 313–319.

[17] H. Gao, D. Yang, *Packing of spanning mixed arborescences*, J. Graph Theory, Volume 98, Issue 2 (2021) 367–377.

[18] M. R. Garey, D. S. Johnson, *Computers and In-tractability. A Guide to the Theory of NP-Completeness*, W. H. Freeman, 1979.

[19] P. Hoppenot, *Packing forests,* Master Thesis, UGA-Grenoble INP, (2023).

[20] F. Hörsch, Z. Szigeti, *Reachability in arborescence packings*, Discret. Appl. Math. 320 (2022) 170–183.

[21] F. Hörsch, Z. Szigeti, *Packing of mixed hyperarborescences with flexible roots via matroid intersection*, Electronic Journal of Combinatorics 28 (3) (2021) P3.29.

[22] N. Kamiyama, N. Katoh, A. Takizawa, *Arc-disjoint in-trees in directed graphs*, Comb. 29 (2009) 197–214.

[23] N. Katoh, S. Tanigawa, *Rooted-tree decomposition with matroid constrains and the infinitesimal rigidity of frameworks with boundaries*, SIAM J. Discret. Math. 27(1), (2013) 155–185.

[24] Cs. Király, *On maximal independent arborescence packing*, SIAM J. Discret. Math. 30(4) (2016) 2107–2114.

[25] Cs. Király, Z. Szigeti, S. Tanigawa, *Packing of arborescences with matroid constraints via matroid intersection*, Math. Program. 181 (2020) 85–117.

[26] L. Lovász, *On two minimax theorems in graphs*, J. Comb. Theory, Ser. B 21 (1976) 96–103.

[27] M. Martin, *Packing arborescences*, Master Thesis, UGA-Grenoble INP, (2022).

[28] T. Matsuoka, Z. Szigeti, *Polymatroid-Based Capacitated Packing of Branchings,* Discret. Appl. Math. 270 (2019) 190–203.

[29] C. St. J. A. Nash-Williams, *Edge-disjoints spanning trees of finite graphs*, Journal of the London Mathematical Society, 36 (1961) 445–450.

[30] Y. H. Peng, C. C. Chen, K. M. Koh, *On edge-disjoint spanning forests of multigraphs*, Soochow Journal of Mathematics 17 (1991) 317–326.

[31] Z. Szigeti, *Packing mixed hyperarborescences*, submitted.

[32] Z. Szigeti, *Matroid-rooted packing of arborescences*, submitted.

[33] Z. Szigeti, *A survey on packing arborescences*, in preparation.

[34] W.T. Tutte, *On the problem of decomposing a graph into n connected factors*, Journal of the London Mathematical Society, 36 (1961) 221–230.