Journal of Combinatorial Theory, Series B 105 (2014) 17–25

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Packing of rigid spanning subgraphs and spanning trees

Joseph Cheriyan [a], Olivier Durand de Gevigney [b], Zoltán Szigeti [b]

[a] Department of Combinatorics and Optimization, University of Waterloo, Canada
[b] Laboratoire G-SCOP, Grenoble-INP, UJF, CNRS, France

ARTICLE INFO

Article history:
Received 15 December 2011
Available online 17 December 2013

Keywords:
Packing
Matroid
Tree
Rigidity
Orientation

ABSTRACT

We prove that every $(6k+2\ell, 2k)$-connected simple graph contains k rigid and ℓ connected edge-disjoint spanning subgraphs.
This implies a theorem of Jackson and Jordán [7] providing a sufficient condition for the rigidity of a graph and a theorem of Jordán [8] on the packing of rigid spanning subgraphs. Both these results generalize the classic result of Lovász and Yemini [10] saying that every 6-connected graph is rigid. Our approach provides a transparent proof for this theorem.
Our result also gives two improved upper bounds on the connectivity of graphs that have interesting properties: (1) in every 8-connected graph there exists a packing of a spanning tree and a 2-connected spanning subgraph; (2) every 14-connected graph has a 2-connected orientation.

1. Introduction

In this paper, we consider sufficient conditions for the existence of a packing of spanning subgraphs in a given undirected graph $G=(V,E)$, where by a packing we mean a set of pairwise edge-disjoint subgraphs of G. Let us present a few examples in this area.

A first example is the existence of a packing of ℓ spanning trees in every 2ℓ-edge-connected graph. This result is an easy consequence of the classic theorem of Tutte [13] and Nash-Williams [11] that characterizes the existence of such a packing. It is well known that this characterization can be derived from matroid theory as follows. The spanning trees of G correspond to the bases of the graphic matroid $\mathcal{C}(G)$ of G. Hence, by matroid union [4], the packings of ℓ spanning trees of G correspond to

E-mail address: olivier.durand-de-gevigney@g-scop.inpg.fr (O. Durand de Gevigney).

0095-8956/$ – see front matter
http://dx.doi.org/10.1016/j.jctb.2013.11.003

the bases of the matroid $\mathcal{N}_{0,\ell}(G)$ defined as the union of ℓ copies of $\mathcal{C}(G)$. Thus, the existence of the required packing is characterized by the rank of E in $\mathcal{N}_{0,\ell}(G)$. Finally, using the formula of Edmonds [4] for the rank function of $\mathcal{N}_{0,\ell}(G)$ gives the theorem of Tutte and Nash-Williams.

A more recent example, due to Jordán [8], is the existence of a packing of k rigid spanning subgraphs in every $6k$-connected graph. The definition of rigidity is postponed to the next section but we mention here that the minimally rigid spanning subgraphs of G correspond to the bases of a matroid, namely the rigidity matroid $\mathcal{R}(G)$ of G. So, as in the previous argument, the existence of a packing of k rigid spanning subgraphs is characterized by the rank of E in the matroid $\mathcal{N}_{k,0}(G)$ defined as the union of k copies of $\mathcal{R}(G)$. Jordán [8] used the formula of Edmonds [4] for the rank function of $\mathcal{N}_{k,0}(G)$ to prove that $6k$-connectivity implies the desired lower bound on the rank of E.

Our main contribution is to provide a new example that gives a sufficient connectivity condition for the existence of a packing of k rigid spanning subgraphs and ℓ spanning trees. To prove this result, we naturally introduce the matroid $\mathcal{N}_{k,\ell}(G)$ defined as the union of k copies of the rigidity matroid $\mathcal{R}(G)$ and ℓ copies of the graphic matroid $\mathcal{C}(G)$.

As a packing of rigid spanning subgraphs turns out to be a packing of spanning 2-connected subgraphs, the packing result of Jordán [8] allowed him to settle the base case of a conjecture of Kriesell (see in [8]) on removable spanning trees and that of a conjecture of Thomassen [12] on orientation of graphs. Our result on the packing of rigid spanning subgraphs and spanning trees enables us to improve the results of Jordán on these conjectures.

2. Definitions

Let $G=(V,E)$ be a graph. For $X\subseteq V$, denote by $\boldsymbol{d_G(X)}$ the **degree** of X, that is, the number of edges of G with one end vertex in X and the other one in $V\setminus X$. We say that G is **Eulerian** if each vertex of G is of even degree.

A graph $G'=(V',E')$ is a **subgraph** of G if $V'\subseteq V$ and $E'\subseteq E$. The subgraph G' is called **spanning** if $V'=V$. A set of pairwise edge-disjoint subgraphs of G is called a **packing**.

Let $F\subseteq E$. We denote by $\boldsymbol{G_F}$ the spanning subgraph of G with edge set F, that is, $G_F=(V,F)$. Let us denote by $\boldsymbol{c(F)}$ the number of connected components of G_F and by $\boldsymbol{\mathcal{K}_F}$ the set of connected components of G_F of size 1.

Let $T\subseteq V$. We denote by $\boldsymbol{F(T)}$ the set of edges of G_F induced by T. We say that F is a $\boldsymbol{T}$**-join** if the set of odd degree vertices of G_F coincides with T. It is well known that if G_F is a connected graph and T is of even cardinality then G_F contains a T-join.

For a collection $\mathcal{G}$ of subsets of V, we say that $(V,\mathcal{G})$ is a **hypergraph**. We denote by $\boldsymbol{V(\mathcal{G})}$ the set of vertices that belong to at least one element of $\mathcal{G}$. We will use the following well-known fact:

$$\text{the sum of the sizes of the elements of } \mathcal{G} \text{ is equal to the sum,}$$
$$\text{for each vertex, of the number of elements of } \mathcal{G} \text{ containing it.} \tag{1}$$

A set X of vertices is called **connected** in $(V,\mathcal{G})$ if, for any partition of X into two non-empty parts, there exists an element of $\mathcal{G}$ intersecting both parts. In $(V,\mathcal{G})$ a **connected component** is a maximal connected vertex set. The number of connected components of this hypergraph is denoted by $\boldsymbol{c(\mathcal{G})}$. Let $\boldsymbol{\mathcal{K}_\mathcal{G}}$ be the set of connected components of $(V,\mathcal{G})$ of size 1.

For $X\in\mathcal{G}$, we define the **border** $\boldsymbol{X_B}$ of X as the set of vertices of X that belong to another element of $\mathcal{G}$, that is, $X_B=X\cap(\bigcup_{Y\in\mathcal{G}\setminus\{X\}}Y)$. We also define the **inner part** $\boldsymbol{X_I}$ of X as the set of vertices of X that belong to no other element of $\mathcal{G}$, that is, $X_I=X\setminus X_B$. Let $\boldsymbol{\mathcal{I}_\mathcal{G}}$ be the set of elements of $\mathcal{G}$ whose inner part is not empty, that is, $\mathcal{I}_\mathcal{G}=\{X\in\mathcal{G}\colon X_I\neq\emptyset\}$. Since every vertex of $V(\mathcal{G})$ is contained in at least two elements of $\mathcal{G}\cup\{X_I\colon X\in\mathcal{I}_\mathcal{G}\}$, we have, by (1),

$$\sum_{X\in\mathcal{G}}|X|+\sum_{X\in\mathcal{I}_\mathcal{G}}|X_I|\geqslant 2\big|V(\mathcal{G})\big|. \tag{2}$$

A graph $G=(V,E)$ is called **rigid** if $\sum_{X\in\mathcal{G}}(2|X|-3)\geqslant 2|V|-3$ for every collection $\mathcal{G}$ of sets of V such that $\{E(X), X\in\mathcal{G}\}$ partitions E. More details about rigid graphs will be given in Section 4.

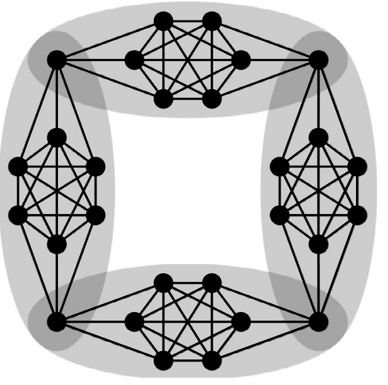

Fig. 1. A non-rigid $(6,3)$-connected simple graph $G=(V,E)$. The collection $\mathcal{G}$ of the four grey vertex-sets provides a partition of E. Hence, since $\sum_{X\in\mathcal{G}}(2|X|-3)=4(2\times 8-3)=52<53=2\times 28-3=2|V|-3$, G is not rigid. The reader can easily check that G is $(6,3)$-connected.

We will use the following connectivity concepts. The graph G is called ***p*-edge-connected** if $d_G(X)\geqslant p$ for every non-empty proper subset X of V. We say that G is ***p*-connected** if $|V|>p$ and $G-X$ is connected for all $X\subset V$ with $|X|\leqslant p-1$. As in [1], for a pair of positive integers (p,q), G is called **(p,q)-connected** if $|V|>\frac{p}{q}$ and $G-X$ is $(p-q|X|)$-edge-connected for all $X\subset V$, that is, if for every pair of disjoint subsets X and Y of V such that $Y\neq\emptyset$ and $X\cup Y\neq V$, we have

$$d_{G-X}(Y)\geqslant p-q|X|. \tag{3}$$

For a better understanding we mention that G is $(6,2)$-connected if G is 6-edge-connected, $G-v$ is 4-edge-connected for all $v\in V$ and $G-\{u,v\}$ is 2-edge-connected for all $u,v\in V$. It follows from the definitions that p-edge-connectivity is equivalent to (p,p)-connectivity. Moreover, since loops and parallel edges do not play any role in vertex connectivity, by the definition of (p,q)-connectivity, we have the following remark.

Remark 1. Every p-connected graph contains a $(p,1)$-connected simple spanning subgraph and $(p,1)$-connectivity implies (p,q)-connectivity for all $q\geqslant 1$.

Let $D=(V,A)$ be a directed graph. We say that D is **strongly connected** if for every ordered pair $(u,v)\in V\times V$ of vertices there is a directed path from u to v in D. The digraph D is called ***p*-arc-connected** if $D-F$ is strongly connected for all $F\subseteq A$ with $|F|\leqslant p-1$. We say that D is ***p*-connected** if $|V|>p$ and $D-X$ is strongly connected for all $X\subset V$ with $|X|\leqslant p-1$.

3. Results

Lovász and Yemini proved the following sufficient condition for a graph to be rigid.

Theorem 1. *(See Lovász and Yemini [10].) Every* 6*-connected graph is rigid.*

The following result of Jackson and Jordán is, by Remark 1, a sharpening of Theorem 1.

Theorem 2. *(See Jackson and Jordán [7].) Every* $(6,2)$*-connected simple graph is rigid.*

Note that in Theorem 2 the connectivity condition is the best possible since there exist non-rigid $(5,2)$-connected simple graphs (see [10]) and non-rigid $(6,3)$-connected simple graphs, for an example see Fig. 1.

Jordán generalized Theorem 1 by giving the following sufficient condition for the existence of a packing of rigid spanning subgraphs.

Theorem 3. *(See Jordán [8].) Let $k \geqslant 1$ be an integer. In every $6k$-connected graph there exists a packing of k rigid spanning subgraphs.*

The main result of this paper (Theorem 4) contains a common generalization of Theorems 2 and 3. It provides a sufficient condition to have a packing of rigid spanning subgraphs and spanning trees. The proof of Theorem 4 will be given in Section 5.

Theorem 4. *Let $k \geqslant 1$ and $\ell \geqslant 0$ be integers. In every $(6k + 2\ell, 2k)$-connected simple graph there exists a packing of k rigid spanning subgraphs and ℓ spanning trees.*

Note that Theorem 4 applied for $k = 1$ and $\ell = 0$ provides Theorem 2. By Remark 1, every $6k$-connected graph contains a $(6k, 2k)$-connected simple spanning subgraph, thus Theorem 4 also implies Theorem 3. Let us see some corollaries of the previous results.

One can easily prove that rigid graphs with at least 3 vertices are 2-connected (see Lemma 2.6 in [6]) and so connected. Thus, Theorem 4 gives the following corollary.

Corollary 1. *Let $k \geqslant 1$ and $\ell \geqslant 0$ be integers. In every $(6k + 2\ell, 2k)$-connected simple graph there exists a packing of k 2-connected and ℓ connected spanning subgraphs.*

Corollary 1 allows us to improve two results of Jordán [8]. The first one deals with the following conjecture of Kriesell, see in [8].

Conjecture 1 *(Kriesell). For every positive integer p, there exists a (smallest) integer $f(p)$ such that every $f(p)$-connected graph G contains a spanning tree T for which $G - E(T)$ is p-connected.*

As Jordán [8] pointed out, Theorem 3 answers this conjecture for $p = 2$ by showing that $f(2) \leqslant 12$. Corollary 1 applied for $k = 1$ and $\ell = 1$ directly implies that $f(2) \leqslant 8$.

Corollary 2. *Every 8-connected graph G contains a spanning tree T such that $G - E(T)$ is 2-connected.*

The other improvement deals with the following conjecture of Thomassen.

Conjecture 2. *(See Thomassen [12].) For every positive integer p, there exists a (smallest) integer $g(p)$ such that every $g(p)$-connected graph G has a p-connected orientation.*

By applying Theorem 3 and an orientation result of Berg and Jordán [2], Jordán [8] proved the conjecture for $p = 2$ by showing that $g(2) \leqslant 18$. Applying the same approach, that is, using a packing theorem (Corollary 1) and an orientation theorem (Theorem 5), we can prove a more general result (Corollary 3) that, in turn, implies $g(2) \leqslant 14$.

Theorem 5. *(See Király and Szigeti [9].) An Eulerian graph $G = (V, E)$ has an orientation D such that $D - v$ is p-arc-connected for all $v \in V$ if and only if $G - v$ is $2p$-edge-connected for all $v \in V$.*

Corollary 1 and Theorem 5 imply the following corollary which, specialized for $p = 1$, gives, by Remark 1, the claimed upper bound for $g(2)$.

Corollary 3. *Every $(12p + 2, 4p)$-connected simple graph G has an orientation D such that $D - v$ is p-arc-connected for all $v \in V$.*

Proof. Let $G=(V,E)$ be a $(12p+2,4p)$-connected simple graph. By Theorem 5 it suffices to prove that G contains an Eulerian spanning subgraph H such that $H-v$ is $2p$-edge-connected for all $v\in V$. By Corollary 1, in G there exists a packing of $2p$ 2-connected spanning subgraphs $H_i=(V,E_i)$ $(i=1,\dots,2p)$ and a spanning tree F. Define $H'=(V,\bigcup_{i=1}^{2p}E_i)$. For all $i=1,\dots,2p$, since H_i is 2-connected, H_i-v is connected; hence $H'-v$ is $2p$-edge-connected for all $v\in V$. Let T be the set of vertices of odd degree in H' and F' a T-join in the tree F. Now, adding the edges of this T-join F' to H' provides the required spanning subgraph of G. $\square$

Finally, we mention the following conjecture of Frank that would imply $g(2)=4$.

Conjecture 3. *(See Frank [5].) A graph has a 2-connected orientation if and only if it is $(4,2)$-connected.*

4. Preliminaries

Let $G=(V,E)$ be a graph. In this section we present some simple facts about the graphic matroid $\mathcal{C}(G)$, the rigidity matroid $\mathcal{R}(G)$ and the matroid $\mathcal{N}_{k,\ell}(G)$ introduced in the Introduction.

We will denote by $\boldsymbol{\mathcal{C}(G)}$ the **graphic matroid** of G on ground-set E, that is an edge set F of G is independent in $\mathcal{C}(G)$ if and only if G_F is a forest. Let $\boldsymbol{n}=|V|$ be the number of vertices in G. It is well known that the rank function $\boldsymbol{r_{\mathcal{C}}}$ of $\mathcal{C}(G)$ satisfies the following:

$$r_{\mathcal{C}}(F)=n-c(F). \tag{4}$$

We will denote by $\boldsymbol{\mathcal{R}(G)}$ the **rigidity matroid** of G on ground-set E with rank function $\boldsymbol{r_{\mathcal{R}}}$ (for a definition we refer the reader to [10]). For $F\subseteq E$, by a theorem of Lovász and Yemini [10], we have

$$r_{\mathcal{R}}(F)=\min\sum_{X\in\mathcal{G}}\bigl(2|X|-3\bigr), \tag{5}$$

where the minimum is taken over all collections $\mathcal{G}$ of subsets of V such that $\{F(X), X\in\mathcal{G}\}$ partitions F. Note that

$$r_{\mathcal{R}}(E)\leqslant 2|V|-3 \tag{6}$$

and equality holds if and only if G is rigid.

For a subset F of E, let $\mathcal{G}$ be a collection of subsets of V such that $\{F(X), X\in\mathcal{G}\}$ partitions F that minimizes the right hand side of (5). It is well known that each element of $\mathcal{G}$ induces a rigid subgraph of G_F. (For example, see the proof of Lemma 2.4 in [6].) Note also that, if G is simple, then every element of $\mathcal{G}$ of size 2 induces at most one (in fact exactly one) edge and contributes exactly one to the sum. So we have the following simple but very useful observation.

Remark 2. If G is simple, then

$$r_{\mathcal{R}}(F)=\min\sum_{X\in\mathcal{H}}\bigl(2|X|-3\bigr)+|F\setminus H|, \tag{7}$$

where the minimum is taken over all subsets $H\subseteq F$ and all collections $\mathcal{H}$ of subsets of V such that $\{F(X), X\in\mathcal{H}\}$ partitions H and each element of $\mathcal{H}$ induces a rigid subgraph of G_H of size at least 3.

The following claim provides insight into the structure of the minimizers of (7).

Claim 1. *Let $G=(V,E)$ be a simple graph and $F\subseteq E$. Let $H\subseteq F$ and $\mathcal{H}$ be a collection of subsets of V that minimize the right hand side of (7).*

(i) *For every $\mathcal{H}^*\subseteq\mathcal{H}$, $r_{\mathcal{R}}(\bigcup_{X\in\mathcal{H}^*}F(X))=\sum_{X\in\mathcal{H}^*}(2|X|-3)$.*
(ii) *For every non-empty $\mathcal{H}^*\subseteq\mathcal{H}$, there exists a vertex in $V(\mathcal{H}^*)$ that is contained in a single element of $\mathcal{H}^*$.*

(iii) $|\mathcal{I}_{\mathcal{H}}| + |\mathcal{K}_{\mathcal{H}}| \geqslant c(\mathcal{H})$.
(iv) *The connected components of* $(V, \mathcal{H})$ *and those of* G_H *coincide.*

Proof. (i) Since $\{F(X), X \in \mathcal{H}\}$ partitions H, we have, by (7) and subadditivity of $r_{\mathcal{R}}$,

$$\begin{aligned}\sum_{X\in\mathcal{H}} \bigl(2|X|-3\bigr) + |F\setminus H| = r_{\mathcal{R}}(F)\\
&\leqslant r_{\mathcal{R}}\Bigl(\bigcup_{X\in\mathcal{H}^*} F(X)\Bigr) + r_{\mathcal{R}}\Bigl(\bigcup_{X\in\mathcal{H}\setminus\mathcal{H}^*} F(X)\Bigr) + r_{\mathcal{R}}(F\setminus H)\\
&\leqslant \sum_{X\in\mathcal{H}^*} r_{\mathcal{R}}\bigl(F(X)\bigr) + \sum_{X\in\mathcal{H}\setminus\mathcal{H}^*} r_{\mathcal{R}}\bigl(F(X)\bigr) + |F\setminus H|\\
&\leqslant \sum_{X\in\mathcal{H}^*} \bigl(2|X|-3\bigr) + \sum_{X\in\mathcal{H}\setminus\mathcal{H}^*} \bigl(2|X|-3\bigr) + |F\setminus H|.\end{aligned}$$

So equality holds everywhere and (i) follows.

(ii) By contradiction, suppose that every vertex of $V(\mathcal{H}^*)$ is contained in at least two elements of $\mathcal{H}^*$. Hence, by (5), (i), since the size of each element of $\mathcal{H}^*$ is at least 3 and by (1), we have $2|V(\mathcal{H}^*)|-3 \geqslant r_{\mathcal{R}}(\bigcup_{X\in\mathcal{H}^*} F(X)) = \sum_{X\in\mathcal{H}^*}(2|X|-3) = \sum_{X\in\mathcal{H}^*}|X| + \sum_{X\in\mathcal{H}^*}(|X|-3) \geqslant 2|V(\mathcal{H}^*)|+0$, a contradiction.

(iii) Let C be a connected component of $(V, \mathcal{H})$ that is not in $\mathcal{K}_{\mathcal{H}}$ and $\mathcal{H}^*$ the elements of $\mathcal{H}$ contained in C. By (ii), there exists in C a vertex v contained in a single element X of $\mathcal{H}^*$. Hence, by definition of $\mathcal{H}^*$, $v \in X_I$ and so $X \in \mathcal{I}_{\mathcal{H}}$. Thus we proved that C contains an element of $\mathcal{I}_{\mathcal{H}}$. Since the connected components of $(V, \mathcal{H})$ are disjoint, (iii) follows.

(iv) Let U be a connected component of G_H and $\emptyset \neq W \subset U$. Then, there exists an edge of H with one end in W and the other end in $U \setminus W$. Since $\{F(X), X \in \mathcal{H}\}$ partitions H, this edge is contained in an element of $\mathcal{H}$ that intersects both W and $U \setminus W$. So U is connected in $(V, \mathcal{H})$.

Let U be a connected component of $(V, \mathcal{H})$ and $\emptyset \neq W \subset U$. Then, there exists an element X of $\mathcal{H}$ intersecting both W and $U \setminus W$. Since $X \subseteq U$ and X induces a rigid, and so connected, subgraph of G_H, there exists an edge of H with one end in $X \cap W \subseteq W$ and the other in $X \setminus W \subseteq U \setminus W$. So U is connected in G_H. This ends the proof of (iv). □

As we mentioned in the Introduction, to have a packing of k rigid spanning subgraphs and ℓ spanning trees in G, we must find k bases in the rigidity matroid $\mathcal{R}(G)$ and ℓ bases in the graphic matroid $\mathcal{C}(G)$ all pairwise disjoint. To do that we will need the following matroid. For $k \geqslant 0$ and $\ell \geqslant 0$, define $\boldsymbol{\mathcal{N}_{k,\ell}(G)}$ as the matroid on ground-set E, obtained by taking the matroid union of k copies of the rigidity matroid $\mathcal{R}(G)$ and ℓ copies of the graphic matroid $\mathcal{C}(G)$. Let $\boldsymbol{r_{k,\ell}}$ be the rank function of $\mathcal{N}_{k,\ell}(G)$. By a theorem of Edmonds [4], for the rank of matroid unions,

$$r_{k,\ell}(E) = \min_{F\subseteq E} k r_{\mathcal{R}}(F) + \ell r_{\mathcal{C}}(F) + |E\setminus F|. \tag{8}$$

Observe that

$$r_{k,\ell}(E) \leqslant k r_{\mathcal{R}}(E) + \ell r_{\mathcal{C}}(E) \leqslant k(2n-3) + \ell(n-1). \tag{9}$$

Jordán [8] used the matroid $\mathcal{N}_{k,0}(G)$ to prove Theorem 3 and pointed out that using $\mathcal{N}_{k,\ell}(G)$ one could prove a theorem on the packing of rigid spanning subgraphs and spanning trees. We tried to fulfill this gap by following the proof of [8] but we failed. To achieve this aim we had to find a new proof technique.

5. Proofs

In this section we provide the proofs of our results. Let us first demonstrate our proof technique by giving a transparent proof for Theorems 1 and 2. We emphasize that in the first two proofs we use only Remark 2 from the Preliminaries.

Proof of Theorem 1. By Remark 1, we may assume that G is simple. Then, by (7), there exist a subset $H \subseteq E$ and a collection $\mathcal{H}$ of subsets of V of sizes at least 3 such that $\{E(X), X \in \mathcal{H}\}$ partitions H and $r_{\mathcal{R}}(E) = \sum_{X \in \mathcal{H}}(2|X| - 3) + |E \setminus H|$. If $V \in \mathcal{H}$, then $r_{\mathcal{R}}(E) \geqslant 2|V| - 3$, hence, by (6), G is rigid. So in the following we assume that $V \notin \mathcal{H}$ and find a contradiction.

Recall that, for $X \in \mathcal{H}$, $X_B = X \cap (\bigcup_{Y \in \mathcal{H}-X} Y)$, $X_I = X \setminus X_B$ and $\mathcal{I}_{\mathcal{H}} = \{X \in \mathcal{H}\colon X_I \neq \emptyset\}$.

Each edge of H being induced by an element of $\mathcal{H}$, it contributes neither to $d_{G-X_B}(X_I)$ for $X \in \mathcal{I}_{\mathcal{H}}$ nor to $d_G(v)$ for $v \in V \setminus V(\mathcal{H})$. Thus, since for $X \in \mathcal{I}_{\mathcal{H}}$, $\emptyset \neq X_I \neq V \setminus X_B$, we have, by 6-connectivity of G,

$$\begin{aligned} |E \setminus H| &\geqslant \frac{1}{2}\Bigg(\sum_{X \in \mathcal{I}_{\mathcal{H}}} d_{G-X_B}(X_I) + \sum_{v \in V \setminus V(\mathcal{H})} d_G(v)\Bigg) \\ &\geqslant \frac{1}{2}\Bigg(\sum_{X \in \mathcal{I}_{\mathcal{H}}} \big(6 - |X_B|\big) + \sum_{v \in V \setminus V(\mathcal{H})} 6\Bigg) && (\star) \\ &\geqslant \sum_{X \in \mathcal{I}_{\mathcal{H}}} \big(3 - |X_B|\big) + 2\big(|V| - \big|V(\mathcal{H})\big|\big). && (10) \end{aligned}$$

By $|X| \geqslant 3$ for $X \in \mathcal{H} \setminus \mathcal{I}_{\mathcal{H}}$, (10) and (2), we have

$$\begin{aligned} r_{\mathcal{R}}(E) &= \sum_{X \in \mathcal{H}} \big(2|X| - 3\big) + |E \setminus H| \\ &\geqslant \Bigg(\sum_{X \in \mathcal{H}} |X| + \sum_{X \in \mathcal{I}_{\mathcal{H}}} \big(|X| - 3\big)\Bigg) + \Bigg(\sum_{X \in \mathcal{I}_{\mathcal{H}}} \big(3 - |X_B|\big) + 2\big(|V| - \big|V(\mathcal{H})\big|\big)\Bigg) \\ &\geqslant \sum_{X \in \mathcal{H}} |X| + \sum_{X \in \mathcal{I}_{\mathcal{H}}} |X_I| + 2\big(|V| - \big|V(\mathcal{H})\big|\big) \\ &\geqslant 2|V|. \end{aligned}$$

Hence, by (6), we have $2|V| - 3 \geqslant r_{\mathcal{R}}(E) \geqslant 2|V|$, a contradiction. $\square$

Proof of Theorem 2. The proof of Theorem 2 is obtained from the proof of Theorem 1 by replacing $d_{G-X_B}(X_I) \geqslant 6 - |X_B|$ by $d_{G-X_B}(X_I) \geqslant 6 - 2|X_B|$ in the inequality ($\star$). This means that in the proof of Theorem 1 we used $(6, 2)$-connectivity instead of 6-connectivity. $\square$

Here comes the proof of the main result.

Proof of Theorem 4. Let $k \geqslant 1$ and $\ell \geqslant 0$ be integers and $G = (V, E)$ a $(6k + 2\ell, 2k)$-connected simple graph. To prove the theorem we use the matroid $\mathcal{N}_{k,\ell}(G)$ defined in Section 4 and show that

$$r_{k,\ell}(E) = k(2n - 3) + \ell(n - 1). \tag{11}$$

Choose F a smallest-size set of edges that gives the rank of E in $\mathcal{N}_{k,\ell}(G)$, that is, which minimizes the right hand side of (8). By (7), there exist a subset $H \subseteq F$ and a collection $\mathcal{H}$ of subsets of V of sizes at least 3 such that $\{F(X), X \in \mathcal{H}\}$ partitions H and

$$r_{\mathcal{R}}(F) = \sum_{X \in \mathcal{H}} \big(2|X| - 3\big) + |F \setminus H|. \tag{12}$$

Claim 2. $H = F$.

Proof. Since $\mathcal{H}$ is a collection of subsets of V of sizes at least 3 such that $\{H(X), X \in \mathcal{H}\}$ partitions H, we have, by (12), $r_{\mathcal{R}}(H) \leqslant \sum_{X \in \mathcal{H}}(2|X| - 3) = r_{\mathcal{R}}(F) - |F \setminus H|$. Hence, since the rank function $r_{\mathcal{C}}$ is non-decreasing and $k \geqslant 1$, we have

$$\begin{aligned} kr_{\mathcal{R}}(H) + \ell r_{\mathcal{C}}(H) + |E \setminus H| &\leqslant kr_{\mathcal{R}}(F) + \ell r_{\mathcal{C}}(F) + |E \setminus H| - k|F \setminus H| \\ &\leqslant kr_{\mathcal{R}}(F) + \ell r_{\mathcal{C}}(F) + |E \setminus F|. \end{aligned}$$

Thus H also minimizes the right hand side of (8) and, by $H \subseteq F$ and the minimality of F, $H = F$. $\square$

If $V \in \mathcal{H}$, then, by (12), $r_{\mathcal{R}}(F) \geqslant \sum_{X \in \mathcal{H}}(2|X| - 3) \geqslant 2n - 3$ and, by Claim 2 and Remark 2, G_F is connected, that is, $r_{\mathcal{C}}(F) = n - 1$. Hence, by (9), we have (11) and the theorem is proved. From now on, we assume that $V \notin \mathcal{H}$ and we will show a contradiction.

Recall the definitions of the border $X_B = X \cap (\bigcup_{Y \in \mathcal{H} - X} Y)$, the inner part $X_I = X \setminus X_B$ for $X \in \mathcal{H}$, $\mathcal{I}_{\mathcal{H}} = \{X \in \mathcal{H}\colon X_I \neq \emptyset\}$ and the sets $\mathcal{K}_F$ and $\mathcal{K}_{\mathcal{H}}$ of connected components of G_F and $(V, \mathcal{H})$ of size 1. By Claim 1 (iv), $\mathcal{K}_F = \mathcal{K}_{\mathcal{H}}$.

Let us use the connectivity condition on G to show a lower bound on $|E \setminus F|$.

Claim 3. $|E \setminus F| \geqslant k(\sum_{X \in \mathcal{I}_{\mathcal{H}}}(3 - |X_B|) + 3|\mathcal{K}_F|) + \ell(|\mathcal{I}_{\mathcal{H}}| + |\mathcal{K}_F|)$.

Proof. By $V \notin \mathcal{H}$, for $X \in \mathcal{I}_{\mathcal{H}}$, $\emptyset \neq X_I \neq V \setminus X_B$. Then, for $X \in \mathcal{I}_{\mathcal{H}}$ and for $v \in \mathcal{K}_F$, we have, by $(6k + 2\ell, 2k)$-connectivity of G,

$$d_{G - X_B}(X_I) \geqslant (6k + 2\ell) - 2k|X_B|, \tag{13}$$

$$d_G(v) \geqslant 6k + 2\ell. \tag{14}$$

Since, by Claim 2, every edge of F is induced by an element of $\mathcal{H}$ and by definition of X_I, for $X \in \mathcal{I}_{\mathcal{H}}$, no edge of F contributes to $d_{G - X_B}(X_I)$. Each $v \in \mathcal{K}_F$ is a connected component of the graph G_F, thus no edge of F contributes to $d_G(v)$. Hence, by (13), (14) and $\ell \geqslant 0$, we obtain the required lower bound on $|E \setminus F|$,

$$\begin{aligned} |E \setminus F| &\geqslant \frac{1}{2}\Bigl(\sum_{X \in \mathcal{I}_{\mathcal{H}}} d_{G - X_B}(X_I) + \sum_{v \in \mathcal{K}_F} d_G(v)\Bigr) \\ &\geqslant \frac{1}{2}\Bigl((6k + 2\ell)|\mathcal{I}_{\mathcal{H}}| - 2k \sum_{X \in \mathcal{I}_{\mathcal{H}}} |X_B| + (6k + 2\ell)|\mathcal{K}_F|\Bigr) \\ &\geqslant k\Bigl(\sum_{X \in \mathcal{I}_{\mathcal{H}}} (3 - |X_B|) + 3|\mathcal{K}_F|\Bigr) + \ell\bigl(|\mathcal{I}_{\mathcal{H}}| + |\mathcal{K}_F|\bigr). \qquad \square \end{aligned}$$

Thus, by (12), Claims 2, 3, $|X| \geqslant 3$ ($X \in \mathcal{H} \setminus \mathcal{I}_{\mathcal{H}}$), Claim 1 (iv), (iii) and (2), we get

$$\begin{aligned} r_{k,\ell}(E) &= k \sum_{X \in \mathcal{H}} \bigl(2|X| - 3\bigr) + |E \setminus F| + \ell\bigl(n - c(F)\bigr) \\ &\geqslant k\Bigl(\sum_{X \in \mathcal{H}} |X| + \sum_{X \in \mathcal{I}_{\mathcal{H}}} \bigl(|X| - 3\bigr)\Bigr) + k\Bigl(\sum_{X \in \mathcal{I}_{\mathcal{H}}} (3 - |X_B|) + 3|\mathcal{K}_F|\Bigr) \\ &\quad + \ell\bigl(|\mathcal{I}_{\mathcal{H}}| + |\mathcal{K}_F|\bigr) + \ell\bigl(n - c(F)\bigr) \\ &\geqslant k\Bigl(\sum_{X \in \mathcal{H}} |X| + \sum_{X \in \mathcal{I}_{\mathcal{H}}} |X_I| + 2|\mathcal{K}_{\mathcal{H}}|\Bigr) + \ell\bigl(c(\mathcal{H}) + n - c(F)\bigr) \\ &\geqslant 2kn + \ell n. \end{aligned}$$

By $k \geqslant 1$ and $\ell \geqslant 0$, this contradicts (9). $\square$

Remark that the proof actually shows that if G is simple and $(6k + 2\ell, 2k)$-connected and if $F \subseteq E$ is such that $|F| \leqslant 3k + \ell$, then in $G' = (V, E \setminus F)$ there exists a packing of k rigid spanning subgraphs and ℓ spanning trees.

We mention that Theorem 4 was slightly generalized by Durand de Gevigney and Nguyen [3] for finding bases of a particular count matroid and spanning trees pairwise edge-disjoint. Their proof applies the discharging method.

References

[1] A.R. Berg, T. Jordán, Sparse certificates and removable cycles in l-mixed p-connected graphs, Oper. Res. Lett. 33 (2) (2005) 111–114.
[2] A.R. Berg, T. Jordán, Two-connected orientations of Eulerian graphs, J. Graph Theory 52 (3) (2006) 230–242.
[3] O. Durand de Gevigney, Orientations of graphs: structures and algorithms, PhD Thesis, 2013.
[4] J. Edmonds, Matroid partition, in: Mathematics of the Decision Sciences: Part 1, in: Lect. Appl. Math., vol. 11, AMS, Providence, RI, 1968, pp. 335–345.
[5] A. Frank, Connectivity and network flows, in: Handbook of Combinatorics, Elsevier, Amsterdam, 1995, pp. 117–177.
[6] B. Jackson, T. Jordán, Connected rigidity matroids and unique realizations of graphs, J. Combin. Theory Ser. B 94 (1) (2005) 1–29.
[7] B. Jackson, T. Jordán, A sufficient connectivity condition for generic rigidity in the plane, Discrete Appl. Math. 157 (8) (2009) 1965–1968.
[8] T. Jordán, On the existence of k edge-disjoint 2-connected spanning subgraphs, J. Combin. Theory Ser. B 95 (2) (2005) 257–262.
[9] Z. Király, Z. Szigeti, Simultaneous well-balanced orientations of graphs, J. Combin. Theory Ser. B 96 (5) (2006) 684–692.
[10] L. Lovász, Y. Yemini, On generic rigidity in the plane, SIAM J. Algebr. Discrete Methods 3 (1) (1982) 91–98.
[11] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. Lond. Math. Soc. 36 (1961) 445–450.
[12] C. Thomassen, Configurations in graphs of large minimum degree, connectivity, or chromatic number, Ann. N.Y. Acad. Sci. 555 (1) (1989) 402–412.
[13] W.T. Tutte, On the problem of decomposing a graph into n connected factors, J. Lond. Math. Soc. 36 (1) (1961) 221–230.