Detachments preserving local edge-connectivity of graphs

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Abstract

Let $G = (V + s, E)$ be a graph with a designated vertex s of degree $d(s)$ and let $f(s) = (d_1, d_2, ..., d_p)$ be a partition of $d(s)$ into p positive integers. An $f(s)$-detachment of G is a graph G' obtained by 'splitting' s into p vertices, called the pieces of s, such that the degrees of the pieces of s in G' are given by $f(s)$. Thus every edge $sw \in E$ corresponds to an edge of G' connecting some piece of s to w. We give necessary and sufficient conditions for the existence of an $f(s)$-detachment of G in which the local edge-connectivities between pairs of vertices in V satisfy prespecified lower bounds.

Our result is a common generalization of a theorem of Mader on edge splittings preserving local edge-connectivities and a result of Fleiner on $f(s)$-detachments satisfying uniform lower bounds. It implies a conjecture of Fleiner on $f(s)$-detachments preserving local edge-connectivities. By using our characterization we extend a theorem of Frank on local edge-connectivity augmentation of graphs to the case when stars of given degrees are added, and we also solve the local edge-connectivity augmentation problem for 3-uniform hypergraphs.

1 Introduction

We consider loopless undirected graphs which may contain parallel edges. Let $G = (V + s, E)$ be a graph with a designated vertex s. A *degree specification* is a sequence $f(s) = (d_1, d_2, ..., d_p)$ of positive integers with $\sum_1^p d_i = d(s)$. An $f(s)$-*detachment* of G is a graph $G' = (V \cup \{s_1, s_2, ..., s_p\}, E')$ obtained by 'splitting' s into p vertices, $s_1, s_2, ..., s_p$, called the *pieces* of s in G', such that the degree of s_i is equal to d_i for $1 \leq i \leq p$. Thus every edge $sw \in E$ corresponds to an edge of $s_iw \in E'$, connecting some piece of s to w.

Let $H = (W, E)$ be a graph. For two vertices $u, v \in W$ the *local edge-connectivity* between u and v, denoted by $\lambda_H(u, v)$, is the maximum number of pairwise edge-disjoint paths from u to v. Let $U \subseteq W$ and let $r : U^2 \to Z_+$ be a symmetric integer valued function on pairs of vertices of U. We say that r is a *requirement function on* U and we call H r-*edge-connected in* U if

$$\lambda_H(x, y) \geq r(x, y) \text{ for all pairs } x, y \in U. \quad (1)$$

The main result of this paper is the following necessary and sufficient condition for the existence of an $f(s)$-detachment of G, which is r-edge-connected in V.

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Theorem 1.1 *Let $G = (V + s, E)$ be a graph such that there are no cut-edges incident to s. Let $f(s) = (d_1, d_2, ..., d_p)$ be a degree specification with $d_i \geq 2$, for all $1 \leq i \leq p$, and let r be a requirement function on V. Then there exists an $f(s)$-detachment of G, which is r-edge-connected in V, if and only if*

$$G \text{ is } r\text{-edge-connected in } V, \quad \text{and} \tag{2}$$

$$\lambda_{G-s}(u, v) \geq r(u, v) - \sum_{i=1}^{p} \lfloor d_i/2 \rfloor \tag{3}$$

holds for every pair $u, v \in V$.

The first results on the existence of highly connected detachments (with degree specifications) were obtained by Nash-Williams [8]. The version that we investigate, where the goal is to detach a single vertex s and the degree specification may be arbitrary, was first studied by Fleiner [3]. He proved the following theorem. For a positive integer k we call a graph H *k-edge-connected in U*, if H is r-edge-connected in U for the uniform requirement function $r(u, v) \equiv k$, $u, v \in U$.

Theorem 1.2 *[3] Let $G = (V + s, E)$ be k-edge-connected in V, for some $k \geq 2$, and let $f(s) = (d_1, d_2, ..., d_p)$ be a degree specification with $d_i \geq 2$, for all $1 \leq i \leq p$. Then there exists an $f(s)$-detachment of G which is k-edge-connected in V if and only if $\lambda_{G-s}(u, v) \geq k - \sum_{i=1}^{p} \lfloor d_i/2 \rfloor$ holds for every pair $u, v \in V$.*

Theorem 1.2 follows from Theorem 1.1 by setting $r(u, v) = k$ for every pair $u, v \in V$. Note that Fleiner's result is valid for a family of hypergraphs as well. We discuss this extension and show how it follows from Theorem 1.1 in Section 6.

In the same paper Fleiner conjectured that if there are no cut-edges incident to s in $G = (V + s, E)$ and $\lambda_{G-s}(u, v) \geq \lambda_G(u, v) - \sum_{i=1}^{p} \lfloor d_i/2 \rfloor$ holds for every pair $u, v \in V$, then there exists an $f(s)$-detachment G' of G for which $\lambda_{G'}(u, v) = \lambda_G(u, v)$ for every pair $u, v \in V$. Theorem 1.1 implies this conjecture by setting $r(u, v) = \lambda_G(u, v)$ for every $u, v \in V$.

Detachments are closely related to edge splittings. *Splitting off* two edges su, sv in a graph means replacing su, sv by a new edge uv. If $u = v$ then the resulting loop is deleted. Splitting off su, sv in a graph $G = (V + s, E)$ is called *λ-admissible*, if $\lambda_{G'}(u, v) = \lambda_G(u, v)$ holds for every pair $u, v \in V$, where G' is the graph obtained by splitting off the pair su, sv. This operation is a useful tool in proving theorems and designing algorithms for connectivity problems. We use, and in Theorem 1.1 we extend the following deep result of Mader on edge splittings preserving local edge-connectivity. The somewhat stronger form of this result that we state here, and a shorter proof, can be found in Frank [4].

Theorem 1.3 *[7] Let $G = (V + s, E)$ be a connected graph with $d(s) \neq 3$ such that there is no cut-edge incident to s. Then there is a λ-admissible splitting at s.*

Proof: We may assume that $d(s) \geq 4$. Let us define a degree specification for s and a requirement function on V by choosing $f(s) = (d_1, d_2)$ with $d_1 = 2$ and $d_2 = d(s) - 2$, and $r(u, v) = \lambda_G(u, v)$ for all pairs $u, v \in V$. We claim that $G, f(s)$ and r satify (2) and (3). Clearly, (2) must hold by the definition of r. Furthermore, we have $p = 2$ and $\sum_{i=1}^{p} \lfloor d_i/2 \rfloor = \lfloor d(s)/2 \rfloor$. It is easy to see that $\lambda_{G-s}(u, v) \geq \lambda_G(u, v) - \lfloor d(s)/2 \rfloor$. Thus (3) holds as well. Hence Theorem 1.1 implies that there exists an $f(s)$-detachment $G' = (V \cup \{s_1, s_2\}, E')$ of G which is r-edge-connected in V. This implies that splitting off the edges sx, sy in G, where s_1x, s_1y are the edges incident to s_1 in G', is λ-admissible. •

As a new application, we use Theorem 1.1 to solve the graph augmentation problem where stars of given degrees may be attached to a given graph in order to meet given local edge-connectivity requirements. This extends a theorem of Frank [5] on augmenting the local edge-connectivities by adding a smallest set of new edges. We also solve the local edge-connectivity augmentation problem for 3-uniform hypergraphs.

The organization of the paper is as follows. Section 2 contains further definitions and preliminary results. In Section 3 we describe the proof method of our main theorem and show how to reduce the problem to the case when $f(s) = (3, 3, ..., 3)$. In Section 4 we introduce the method of 'tight set contraction'. By using this method we complete the proof of the main result in Section 5. Applications are discussed in Section 6.

2 Preliminaries

Let $H = (W, E)$ be a graph. For $X, Y \subseteq W$ we use $d(X, Y)$ to denote the number of edges from $X - Y$ to $Y - X$. Let $d(X) = d(X, V - X)$ denote the *degree* of a set $X \subseteq W$. A singleton set $\{v\}$ is simply denoted by v. Thus $d(v)$ is the degree of vertex $v \in W$. The symbols $\subseteq$ and $\subset$ denote set containment and proper set containment respectively. Two sets X, Y are said to be *intersecting* if $X \cap Y$, $X - Y$, $Y - X$ are all non-empty.

The degree function of a graph satisfies the following well-known equalities.

Proposition 2.1 *Let $H = (W, E)$ be a graph. For arbitrary subsets $X, Y \subseteq W$,*

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y), \quad (4)$$
$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2d(X \cap Y, W - (X \cup Y)). \quad (5)$$

Let $H = (W, E)$ be a graph, let $U \subseteq W$ be a subset of vertices with $|U| \geq 2$ and let $r : U^2 \to Z_+$ be a requirement function on U. For set $\emptyset \neq X \subset U$ we define

$$R(X) = \max\{r(u, v) : u \in X, v \in U - X\}. \quad (6)$$

Clearly, R is symmetric on U, that is, $R(X) = R(U - X)$ holds for all $\emptyset \neq X \subset U$. The following property was verified by Frank.

Proposition 2.2 *[5, Proposition 5.4] For any two non-empty subsets $X, Y \subseteq U$ at least one of the following holds:*

$$R(X) + R(Y) \leq R(X \cap Y) + R(X \cup Y), \quad (7)$$
$$R(X) + R(Y) \leq R(X - Y) + R(Y - X). \quad (8)$$

If $X \cup Y = U$ then (8) always holds (with equality).

In the rest of this section let $G = (V + s, E)$ be a graph with a designated vertex s and let $r : V^2 \to Z_+$ be a requirement function on V. For sets $\emptyset \neq X \subset V$ we define

$$h(X) = d(X) - R(X). \quad (9)$$

Propositions 2.1 and 2.2 imply:

Proposition 2.3 *For any two non-empty subsets $X, Y \subset V$ at least one of the following holds:*

$$\begin{aligned} h(X) + h(Y) &\geq h(X \cap Y) + h(X \cup Y) + 2d(X, Y), &(10)\\ h(X) + h(Y) &\geq h(X - Y) + h(Y - X) + 2d(X \cap Y, V + s - (X \cup Y)). &(11)\end{aligned}$$

If $X \cup Y = V$ then (11) always holds (with equality).

Given a degree specification $f(s) = (d_1, d_2, ..., d_p)$, we shall use the notation:

$$\varphi = \sum_{i=1}^{p} \lfloor d_i/2 \rfloor. \tag{12}$$

It will be convenient to reformulate (2) and (3) in terms of the degree functions of G and $G - s$, functions h and R, and number φ. The following two lemmas are easy to deduce from Menger's theorem.

Lemma 2.4 *Condition (2) holds in $G = (V + s, E)$ if and only if*

$$h(X) \geq 0 \quad \text{for every } \emptyset \neq X \subset V. \tag{13}$$

Lemma 2.5 *Condition (3) holds in $G = (V + s, E)$ if and only if*

$$d_{G-s}(X) \geq R(X) - \varphi \quad \text{for every } \emptyset \neq X \subset V, \tag{14}$$

which is equivalent to

$$d_G(s, X) \leq h(X) + \varphi \quad \text{for every } \emptyset \neq X \subset V. \tag{15}$$

We shall need the following operations to construct the required $f(s)$-detachment of G. Let su, sv, sz be distinct edges in $G = (V + s, E)$. The operation 2-*split (on su, sv)* deletes the edges su, sv and adds a new vertex t and two new edges tu, tv. Similarly, operation 3-*split (on su, sv, sz)* deletes su, sv, sz and adds a new vertex t and three new edges tu, tv, tz. Note that (some of) the split edges may be parallel. Let $G = (V + s, E)$ be r-edge-connected in V. We say that a 2-split (or 3-split) is r-*admissible* in G if the resulting graph $G' = (V + s + t, E')$ is also r-edge-connected in V. Let $\emptyset \neq X \subset V$. We say that

$$X \text{ is } \begin{cases} \textit{tight} \\ \textit{dangerous} \\ \textit{bad} \end{cases} \quad \text{if} \quad h(X) = \begin{cases} = 0, \\ \leq 1, \\ \leq 2. \end{cases}$$

Note that, by definition, tight sets are dangerous, and dangerous sets are bad. For a pair (or triple) of edges su, sv (su, sv, sz) and a set $X \subset V$ we shall use $e(su, sv; X)$ ($e(su, sv, sz; X)$) to denote the number of those edges of the given pair (or triple, respectively) that enter X.

Lemma 2.6 *Let $G = (V + s, E)$ be r-edge-connected in V. Then*
(a) the 2-split on su, sv is r-admissible if and only if there is no dangerous set X with $e(su, sv; X) = 2$,
(b) the 3-split on su, sv, sz is r-admissible if and only if (i) there is no tight set X with $e(su, sv, sz; X) \geq 2$, and (ii) there is no bad set M with $e(su, sv, sz; M) = 3$.

Proof: We only prove (b). The proof of (a) is similar (but simpler). Let $G' = (V+s+t, E')$ denote the graph obtained from G by a 3-split on su, sv, sz. Observe that for proper subsets $X \subset V$ we have $d_{G'}(X+t) = d_G(X) - (2e(su, sv, sz; X) - 3)$.

To see necessity suppose that (i) or (ii) does not hold. Then there is a set $Y \subset V$, which is either tight or bad in G, and for which we have $d_{G'}(Y+t) \leq R(Y) - 1$. Thus, for some pair x, y with $x \in Y, y \in V - Y$ and $r(x, y) = R(Y)$ we must have $\lambda_{G'}(x, y) \leq r(x, y) - 1$. Hence the 3-split is not r-admissible.

To see sufficiency suppose that the 3-split on su, sv, sz is not r-admissible and let $x, y \in V$ with $\lambda_{G'}(x, y) \leq r(x, y) - 1$. Then there is a set $Y \subset V+s+t$ with $x \in Y, y \notin Y$ and $d_{G'}(Y) \leq r(x, y) - 1$. Since $d_{G'}$ and r are symmetric, we may assume that $s \notin Y$. Thus, since G is r-edge-connected in V and we have $d_G(N) = d_{G'}(N)$ for all $N \subseteq V$, we must have $t \in Y$. Let $X = Y - t$. Since $d_G(X) \geq R(X) \geq r(x, y)$, we can now deduce that (i) or (ii) does not hold. •

We close this section with two simple lemmas that we shall use in Sections 4 and 5. The proofs demonstrate the typical applications of inequalities (10) and (11).

Lemma 2.7 *Let $G = (V+s, E)$ be r-edge-connected in V and let X, Y be intersecting tight sets in G. Then either $X \cap Y$ and $X \cup Y$ are also tight, or $X - Y$ and $Y - X$ are tight and $d(s, X \cap Y) = 0$ holds.*

Proof: We apply Proposition 2.3 to the pair X, Y and use the fact that $h(Z) \geq 0$ for all $\emptyset \neq Z \subset V$ by Lemma 2.4. If (10) holds and $X \cup Y \neq V$ then we have $0 = h(X) + h(Y) \geq h(X \cap Y) + h(X \cup Y) \geq 0$. Thus equality holds everywhere, and hence $X \cap Y$ and $X \cup Y$ are also tight. If (11) holds then we have $0 = h(X) + h(Y) \geq h(X - Y) + h(Y - X) + 2d(X \cap Y, V + s - (X \cup Y)) \geq 2d(X \cap Y, s)$. This implies $d(s, X \cap Y) = 0$ and $h(X - Y) = h(Y - X) = 0$. Thus $X - Y$ and $Y - X$ are tight. •

We say that a triple su, sv, sz is *semi-admissible* if there is no tight set X with $e(su, sv, sz; X) \geq 2$ (that is, Lemma 2.6(b)(i) holds).

Lemma 2.8 *Let $G = (V+s, E)$ be r-edge-connected in V, let X be a tight set, let su, sv, sz be a semi-admissible triple, and let M be a maximal bad set with $e(su, sv, sz; M) = 3$. Then either $X \subseteq M$, or $d(s, X \cap M) = 0$, $M - X$ is bad, and $e(su, sv, sz; M - X) = 3$.*

Proof: If $X \subseteq M$ or $X \cap M = \emptyset$ then we are done. $M \subseteq X$ cannot hold, since $e(su, sv, sz; M) = 3$ and the triple is semi-admissible. Thus we may assume that X, M is an intersecting pair.

We apply Proposition 2.3 to the pair X, M and use the fact that $h(Z) \geq 0$ for all $\emptyset \neq Z \subset V$ by Lemma 2.4. If (10) holds and $X \cup M \neq V$, then we have $0 + 2 \geq h(X) + h(M) \geq h(X \cap M) + h(X \cup M) \geq 0 + 3$, a contradiction. Here $h(X \cup M) \geq 3$ follows from the maximality of M.

Thus (11) holds. Then we have $0 + 2 \geq h(X) + h(M) \geq h(X - M) + h(M - X) + 2d(X \cap M, V + s - (X \cup M)) \geq 2d(s, X \cap M)$. If $d(s, X \cap M) \geq 1$ then this implies $d(s, X \cap M) = 1$ and $h(M - X) = 0$. Hence $M - X$ is tight, and we must have $e(su, sv, sz; M - X) \geq 2$. This contradicts the fact that su, sv, sz is a semi-admissible triple. Thus $d(s, X \cap M) = 0$ must hold. In this case we get $h(M - X) \leq 2$ and $e(su, sv, sz; M - X) = 3$, as required. •

3 Detachments by admissible and feasible 2-splits

We start this section by an informal description of the main steps of the proof of Theorem 1.1. It will not be difficult to reduce the problem to the case when each term d_i in $f(s)$ is either two or three. Thus either there is a term, say d_p, for which $d_p = 2$ or each term in $f(s)$ is equal to three (in particular, $d_p = 3$). The proof is then by induction on $d(s)$. We shall perform a d_p-split at vertex s (hence creating a new vertex t, and decreasing the degree of s), extend the requirement function to $V + t$, reduce the degree specification by deleting the last term, and then apply induction to show that the resulting graph G' has a good detachment G'' (with respect to the modified requirements and degree specification). From this detachment of G' we can obtain a good $f(s)$-detachment of G by simply adding vertex t to the pieces of s. In order to make this proof work we need to make sure that (a) the d_p-split that we perform preserves r-edge-connectivity in V, and (b) the resulting graph satisfies (2) and (3) on vertex set $V+t$ with respect to the extended requirement function and the reduced degree specification. In this section we make this precise and show that the required d_p-split exists when $d_p = 2$. (The other case, when $d_p = 3$, will be settled in Section 5.)

In the rest of this section let $G = (V + s, E)$ be a graph satisfying (2) and (3) with respect to a requirement function $r : V^2 \to Z_+$ and degree specification $f(s) = (d_1, d_2, ..., d_p)$ with $d_p \in \{2, 3\}$. Furthermore, suppose that r is *smooth*, that is, $r(u, v) \geq 2$ for every pair $u, v \in V$. Note that since r is smooth and (2) holds, it follows that there are no cut-edges in G. Recall that (2) and (3) are equivalent to (13) and (14), respectively, by Lemmas 2.4 and 2.5.

Let $G' = (V + t + s, E')$ be obtained from G by a d_p-split. Here t is the new vertex created by the split. The *extended requirement function* r' on $V + t$ in G' is defined as follows.

$$r'(u,v) \quad = \quad \begin{cases} r(u,v) & \text{if } u,v \in V, \\ 2 & \text{if } u = t, v \in V (\text{ or } v = t, u \in V) \end{cases} \tag{16}$$

The *reduced degree specification* $f'(s)$ is obtained from $f(s)$ by deleting the last term, i.e. $f'(s) = (d_1, ..., d_{p-1})$. With the extended requirement function r' we also define function R', defined on the proper subsets X of $V + t$ by $R'(X) = \max\{r'(u, v) : u \in X, v \in (V + t) - X\}$, which satisfies:

$$R'(X) \quad = \quad \begin{cases} R(X) & \text{if } \emptyset \neq X \subset V, \\ 2 & \text{if } X = \{t\}, V, \\ R(X - t) & \text{otherwise} \end{cases} \tag{17}$$

With the reduced degree specification $f'(s)$ we also define the corresponding number $\varphi' = \sum_1^{p-1} \lfloor d_i/2 \rfloor$, which satisfies:

$$\varphi' = \varphi - 1. \tag{18}$$

The proof of the next lemma is simple.

Lemma 3.1 *Suppose that $G' = (V+t+s, E')$ is obtained from G by an r-admissible d_p-split. Then G' is r'-edge-connected in $V + t$.*

Thus by performing an r-admissible d_p-split we can make sure that G' satisfies (2) with respect to r' and $V + t$. To guarantee that G' satisfies (3) as well, we need to find an r-admissible d_p-split satisfying some additional conditions, described in the next lemma.

First observe that condition (3) for G', r', and $f'(s)$ is equivalent to

$$d_{G'-s}(X) \geq R'(X) - \varphi' \text{ for all } \emptyset \neq X \subset V + t. \tag{19}$$

This form of (3) and (18) show that, in order to meet (19), the degree of certain subsets $X \subset V$ in $G' - s$ must be strictly larger than their degree in $G - s$. Let

$$\mathcal{C}' = \{X \subset V : d_{G-s}(X) = R(X) - \varphi\}, \tag{20}$$

and let $\mathcal{C}$ denote the inclusionwise minimal members of $\mathcal{C}'$. Sets in $\mathcal{C}$ are called *cores*.

Lemma 3.2 *Suppose that $G' = (V+t+s, E')$ is obtained from G by an r-admissible 2-split on su, sv (or 3-split on su, sv, sz). Then G' satisfies (19) if and only if $e(su, sv; C) \geq 1$ ($e(su, sv, sz; C) \geq 1$, respectively) for every $C \in \mathcal{C}$.*

Proof: Since the split is r-admissible, r is smooth, and by using (17), we can deduce that (19) holds for $X = \{t\}, V$. For the remaining proper subsets of $V + t$ it follows from the symmetry of R' that (19) is equivalent to

$$d_{G'-s}(X) \geq R(X) - \varphi + 1 \text{ for every } \emptyset \neq X \subset V. \tag{21}$$

The definition of $\mathcal{C}$ shows that (21) holds if and only if there is at least one split edge from s to each of the cores. This proves the lemma. •

We shall also use the fact that $X \in \mathcal{C}'$ holds if and only if

$$d(s, X) = h(X) + \varphi. \tag{22}$$

We say that a 2-split on su, sv is *feasible* if $e(su, sv; C) \geq 1$ for every $C \in \mathcal{C}$. Similarly, a 3-split on su, sv, sz is *feasible* if $e(su, sv, sz; C) \geq 1$ holds for every $C \in \mathcal{C}$. To identify feasible splits the following properties of cores will be useful. We say that a degree specification $f(s) = (d_1, d_2, ..., d_p)$ is 3-*regular* if $d_i = 3$ for $1 \leq i \leq p$. Note that $d(s) \leq 3\varphi$ and equality holds if and only if $f(s)$ is 3-regular.

Lemma 3.3 *The set of cores satisfies the following*
(a) the cores are pairwise disjoint,
(b) $d(s, C) \geq \varphi$ for each $C \in \mathcal{C}$,
(c) $|\mathcal{C}| \in \{0, 2, 3\}$,
(d) if $|\mathcal{C}| = 3$ then $f(s)$ is 3-regular, each core C is tight, and satisfies $d(s, C) = \varphi$.

Proof: Since G satisfies (3), $G - s$ satisfies (13) with respect to $R - \varphi$, and each core is tight in $G - s$ (also with respect to $R - \varphi$). Thus the minimality of the cores and Lemma 2.7 (applied to $G - s$) imply that (a) holds. Property (b) follows from (22), (2), and Lemma 2.4. To see (c) let C be a core. The symmetry of the degree function of $G - s$ and R in V implies that $V - C \in \mathcal{C}'$. Hence we have $|\mathcal{C}| \geq 2$. It follows from (a),(b), and $d(s) \leq 3\varphi$, that $|\mathcal{C}| \leq 3$. Moreover, $|\mathcal{C}| = 3$ may hold only if $f(s)$ is 3-regular, and $d(s, C) = \varphi$ for each core C. In this case $h(C) = 0$ follows from (22). Thus C is tight in G. •

Lemma 3.4 *Suppose that $\mathcal{C} = \{C_1, C_2\}$, $\varphi \geq 2$, and let sc_1, sc_2 be a pair of edges with $c_i \in C_i$, for $1 \leq i \leq 2$. Then the 2-split on sc_1, sc_2 is r-admissible and feasible.*

Proof: By the choice of c_1, c_2, the 2-split on sc_1, sc_2 is feasible. We shall prove that it is r-admissible as well. Let $\alpha_i = d(s, C_i)$, $1 \leq i \leq 2$. By Lemma 3.3(b) we have $\alpha_i \geq \varphi$, and by (22) we have $h(C_i) = \alpha_i - \varphi$, for $1 \leq i \leq 2$.

For a contradiction suppose that the 2-split is not r-admissible. Then Lemma 2.6 implies that there exists a dangerous set $X \subset V$ with $e(sc_1, sc_2; X) = 2$. Let $\beta_i = d(s, X \cap C_i)$, $1 \leq i \leq 2$. Since $c_i \in X$, we have $\beta_i \geq 1$ for $1 \leq i \leq 2$. If $C_1 \cup C_2 \subseteq X$ then (14) implies $R(X) - \varphi \leq d_{G-s}(X) \leq d(X) - \alpha_1 - \alpha_2 \leq d(X) - 2\varphi \leq R(X) + 1 - 2\varphi$, contradicting the assumption $\varphi \geq 2$. Thus we may assume, without loss of generality, that $C_1 - X \neq \emptyset$.

Claim 3.5 *(a) If $X \cup C_1 \neq V$, then $h(X \cup C_1) \geq \alpha_1 - \varphi + 2$, and*
(b) $h(C_1 - X) \geq \alpha_1 - \varphi + 1 - \beta_1$.

Proof: Since there exist no cores other than C_1 and C_2, each set in $\mathcal{C}'$ contains C_1 or C_2. Thus $V - (X \cup C_1) \notin \mathcal{C}'$ must hold, and by the symmetry of $\mathcal{C}'$, we also have $X \cup C_1 \notin \mathcal{C}'$. Hence (14) implies $R(X \cup C_1) - \varphi + 1 \leq d_{G-s}(X \cup C_1)$. Now, since $\beta_2 \geq 1$, it follows from Lemma 3.3(a) that $d(X \cup C_1) \geq d_{G-s}(X \cup C_1) + \alpha_1 + \beta_2 \geq R(X \cup C_1) + \alpha_1 - \varphi + 2$. Thus (a) holds.

Since C_1 is a core, we have $R(C_1 - X) - \varphi + 1 \leq d_{G-s}(C_1 - X)$. This implies $d(C_1 - X) = d_{G-s}(C_1 - X) + \alpha_1 - \beta_1 \geq R(C_1 - X) - \varphi + 1 + \alpha_1 - \beta_1$. Thus (b) holds. •

Now we apply Proposition 2.3 to X and C_1. If (10) holds and $C_1 \cup X \neq V$ then, by using Lemma 2.4, the fact that X is dangerous, (22), and Claim 3.5(a), we obtain $1 + (\alpha_1 - \varphi) \geq h(X) + h(C_1) \geq h(X \cap C_1) + h(X \cup C_1) \geq \alpha_1 - \varphi + 2$, a contradiction. Thus (11) must hold for X and C_1. Then by Lemma 2.4 and Claim 3.5(b) we have $1 + (\alpha_1 - \varphi) \geq h(X) + h(C_1) \geq h(X - C_1) + h(C_1 - X) + 2d(s, X \cap C_1) \geq \alpha_1 - \varphi + 1 + \beta_1$, a contradiction, since $\beta_1 \geq 1$. This final contradiction shows that the 2-split is r-admissible, as claimed. •

Lemma 3.6 *Suppose that $\varphi \geq 2$ and $f(s)$ is not 3-regular. Then there exists an r-admissible and feasible 2-split in G.*

Proof: By Lemma 3.3(c),(d) we have $|\mathcal{C}| \in \{0, 2\}$. First suppose $\mathcal{C} = \emptyset$. Since $d(s) \geq 4$ and there are no cut-edges incident to s in G, Theorem 1.3 implies that there is a λ-admissible splitting su, sv at s in G. Since G is r-edge-connected in V, the 2-split on su, sv is r-admissible. Since there are no cores, this 2-split is feasible as well.

Now suppose $|\mathcal{C}| = \{C_1, C_2\}$. In this case, by Lemma 3.3(b), we can choose a pair of edges sc_1, sc_2 with $c_i \in C_i$, for $1 \leq i \leq 2$. By Lemma 3.4 the 2-split on sc_1, sc_2 is r-admissible and feasible. •

4 Contracting tight sets

An important step in the proof of Theorem 1.3 is the contraction of non-singleton tight sets, see [4, 7]. We shall also follow this approach in Section 5. In this section we describe the details of this method and prove that, roughly speaking, contracting non-singleton tight sets does not change the problem.

Let $G = (V + s, E)$ be a graph and let $r : V^2 \to Z_+$ be a requirement function on V. Suppose that G is r-edge-connected in V and let $T \subset V$ be a tight set in G (with respect to r). Let $\hat{G} = (\hat{V} + s, \hat{E})$ be obtained from G by contracting T into a single vertex t. Each set $\hat{X} \subset \hat{V}$ will correspond to a unique subset of V, that we shall denote by X, defined as follows.

$$X = \begin{cases} \hat{X} & \text{if } t \notin \hat{X}, \\ (\hat{X} - t) \cup T & \text{if } t \in \hat{X}. \end{cases} \tag{23}$$

We shall use $\hat{d}$ and $\hat{\lambda}$ to denote the degree function and the local edge-connectivity function in $\hat{G}$, respectively, and we define a requirement function $\hat{r} : \hat{V}^2 \to Z_+$ for $\hat{G}$ as follows:

$$\hat{r}(u, v) = \begin{cases} r(u, v) & \text{if } u, v \in \hat{V} - t, \\ \max\{r(u, x) : x \in T\} & \text{if } v = t, \\ \max\{r(x, v) : x \in T\} & \text{if } u = t. \end{cases} \tag{24}$$

With $\hat{d}$ and $\hat{r}$ we also define functions $\hat{R}$ and $\hat{h}$ on sets $\hat{X} \subset \hat{V}$ by $\hat{R}(\hat{X}) = \max\{\hat{r}(u,v) : u \in \hat{X}, v \in \hat{V} - \hat{X}\}$ and $\hat{h}(\hat{X}) = \hat{d}(\hat{X}) - \hat{R}(\hat{X})$. The next lemma follows directly from the definition of $\hat{r}$ and the fact that $\hat{d}(\hat{X}) = d_G(X)$ for all $\hat{X} \subset \hat{V}$ (without using the fact that T is tight).

Lemma 4.1 *For $\hat{X} \subset \hat{V}$ we have*
(a) $\hat{R}(\hat{X}) = R(X)$, and
(b) $\hat{h}(\hat{X}) = h(X)$.

It follows that $\hat{G}$ is $\hat{r}$-edge-connected in $\hat{V}$. We also obtain the following easy but useful corollaries.

Lemma 4.2 *Suppose that the 2-split on su, sv is r-admissible in G. Then the 2-split on su', sv' is $\hat{r}$-admissible in $\hat{G}$, where su' and sv' denote the edges of $\hat{G}$ corresponding to su and sv.*

Proof: Suppose that su', sv' is not $\hat{r}$-admissible in $\hat{G}$. Then by Lemma 2.6(a) there exists a dangerous set $\hat{X} \subset \hat{V}$ with $\hat{h}(\hat{X}) \leq 1$ and $e(su', sv'; \hat{X}) = 2$. By Lemma 4.1(b) it follows that X is a dangerous set in G with $e(su, sv; X) = 2$. By Lemma 2.6(a) it contradicts the fact that the 2-split on su, sv is r-admissible in G. •

Lemma 4.3 *Suppose that G satisfies (14) with respect to r and some degree specification $f(s)$. Then $\hat{G}$ satisfies (14) with respect to $\hat{r}$ and $f(s)$.*

Proof: Suppose (14) fails in $\hat{G}$ and let $\hat{W} \subset \hat{V}$ be a set with $d_{\hat{G}-s}(\hat{W}) < \hat{R}(\hat{W}) - \varphi$. Since $d_{G-s}(W) = d_{\hat{G}-s}(\hat{W})$, and by Lemma 4.1(a), we have $d_{G-s}(W) < R(W) - \varphi$, contradicting the fact that (14) holds in G. •

Mader [7] and Frank [4, Claim 3.2] showed that if a splitting is $\hat{\lambda}$-admissible in $\hat{G}$ then the corresponding splitting is λ-admissible in G. We need a similar fact for 3-splits and arbitrary requirement functions.

Lemma 4.4 *Suppose that the 3-split on su', sv', sz' is $\hat{r}$-admissible in $\hat{G}$. Then the 3-split on su, sv, sz is r-admissible in G, where su, sv, sz denote the edges corresponding to su', sv', sz' in G.*

Proof: For a contradiction suppose that the 3-split on su, sv, sz is not r-admissible in G. By Lemma 2.6(b) this implies that there is set $Y \subset V$ in G for which either (i) Y is tight and $e(su, sv, sz; Y) \geq 2$, or (ii) Y is bad and $e(su, sv, sz; Y) = 3$.

Claim 4.5 *Y and T are intersecting sets.*

Proof: Since T is tight in G, it follows from Lemma 4.1 that t is a singleton tight set in $\hat{G}$ (with respect to $\hat{r}$). Since the 3-split on su', sv', sz' is $\hat{r}$-admissible in $\hat{G}$, it follows from Lemma 2.6(b) that $e(su', sv', sz'; \{t\}) \leq 1$. Thus $Y - T \neq \emptyset$. If $Y \cap T = \emptyset$ or $T \subseteq Y$ then there is a set $\hat{Y}$ in $\hat{G}$ corresponding to Y. By Lemma 4.1(b) we have $h(Y) = \hat{h}(\hat{Y})$, and hence, by Lemma 2.6(b), $\hat{Y}$ shows that the 3-split on su', sv', sz' is not $\hat{r}$-admissible in $\hat{G}$, a contradiction. •

First suppose that Y is tight. In this case Claim 4.5 and Lemma 2.7, applied to Y and T, imply that either $Y \cup T$ is tight, or $Y - T$ is tight and $d(s, Y \cap T) = 0$. In each of these cases we obtain a tight set ($Y \cup T$ or $Y - T$) satisfying (i) which is not intersecting with T. This contradicts Claim 4.5. Thus the 3-split on su, sv, sz is semi-admissible.

Next suppose that Y is bad. We may assume that Y is a maximal bad set. Now Claim 4.5 and Lemma 2.8 imply that $Y - T$ is bad and $e(su, sv, sz; Y - T) = 3$. Thus $Y - T$ satisfies (ii) and is not intersecting with T. This contradicts Claim 4.5, and completes the proof of the lemma. •

5 The proof of Theorem 1.1

In this section first we consider the case of 3-regular degree specifications and prove that, if the necessary conditions hold, there exists an r-admissible and feasible 3-split in G. This will enable us to complete the proof of our main theorem.

In this section (until the proof of Theorem 1.1) let $G = (V + s, E)$ be a graph satisfying (2) and (3) with respect to a smooth requirement function $r : V^2 \to Z_+$ and 3-regular degree specification $f(s) = (d_1, d_2, ..., d_p)$ with $p \geq 2$. These conditions imply that there are no cut-edges in G and $\varphi \geq 2$. Note that (2) and (3) are equivalent to (13) and (14), respectively, by Lemmas 2.4 and 2.5, and recall the definition of cores from Section 3, and the fact that G has at most three cores.

First we consider the case when G has three cores. In this case we can find an r-admissible and feasible 3-split without tight set contractions.

Lemma 5.1 *Suppose that $\mathcal{C} = \{C_1, C_2, C_3\}$ and let sc_1, sc_2, sc_3 be a triple of edges with $c_i \in C_i$, for $1 \leq i \leq 3$. Then the 3-split on sc_1, sc_2, sc_3 is r-admissible and feasible.*

Proof: By the choice of c_1, c_2, c_3, the 3-split on sc_1, sc_2, sc_3 is feasible. We shall prove that it is r-admissible as well by showing that conditions (i) and (ii) of Lemma 2.6(b) hold.

For a contradiction first suppose, without loss of generality, that there is a maximal tight set X with $e(sc_1, sc_2; X) = 2$. By Lemma 3.3(d) each core is tight. Thus Lemma 2.7, and the maximality of X imply that $C_1 \cup C_2 \subseteq X$. Now we can use Lemmas 2.5, 3.3(b), and (15) to deduce $\varphi = \varphi + h(X) \geq d(s, X) \geq 2\varphi$. This contradicts the fact that $\varphi \geq 2$. Thus Lemma 2.6(b)(i) holds, and hence the triple is semi-admissible.

Now suppose that there is a maximal bad set M with $e(sc_1, sc_2, sc_3; M) = 3$. Since the triple is semi-admissible, and each core is tight, Lemma 2.8 implies that $C_1 \cup C_2 \cup C_3 \subseteq M$. By Lemmas 2.5, 3.3(b), and (15), this gives $\varphi + 2 \geq \varphi + h(M) \geq d(s, M) \geq 3\varphi$. Again, this contradicts the fact that $\varphi \geq 2$. Thus Lemma 2.6(b)(ii) holds as well. •

Now we turn to the case when $|\mathcal{C}| \in \{0, 2\}$. In this case we prove the existence of an r-admissible and feasible 3-split by using the method of tight set contractions. To this end, in addition to the assumptions listed at the beginning of this section, in the next three lemmas we shall also assume that

$$\text{every tight set in } G \text{ is a singleton} \tag{25}$$

Lemma 5.2 *Suppose that (25) holds and the 2-split on su, sv is r-admissible in G. Then there exists an edge sz for which su, sv, sz is a semi-admissible triple.*

Proof: Let sa be an edge different from su, sv and suppose that the triple su, sv, sa is not semi-admissible. Then there exists a tight set X with $e(su, sv, sa; X) \geq 2$. Since the 2-split on su, sv is r-admissible, Lemma 2.6(a) implies that we must have, without loss of generality, $e(sa, su; X) = 2$. Now (25) implies that $u = a$ and $\{u\}$ is tight. By Lemma 2.5 we have $d(s, u) \leq \varphi$, and hence there exists an edge sb, different from sv, with $b \neq u$. If the triple su, sv, sb is not semi-admissible either, then we must have $b = v$ and $\{v\}$ is tight. Moreover, $d(s, v) \leq \varphi$ holds, and hence, since $f(s)$ is 3-regular, and hence $d(s) = 3\varphi$, there exists an edge sz with $z \neq u, v$. We conclude that the triple su, sv, sz is semi-admissible, as required. •

We shall strengthen Lemma 5.2 and prove that every r-admissible 2-split is part of an r-admissible 3-split unless G has the following special structure. We say that a graph $H = (W + s, F)$ with $d(s) = 6$ is an *obstacle* (with respect to a given requirement function r on W), if H is r-edge-connected in W and there exist two sets $M_1, M_2 \subset W$ such that
(i) M_1 and M_2 are bad,
(ii) $M_1 - M_2$ and $M_2 - M_1$ are tight, and
(iii) $d_H(s, M_1 - M_2) = d_H(s, M_1 \cap M_2) = d_H(s, M_2 - M_1) = 2$.

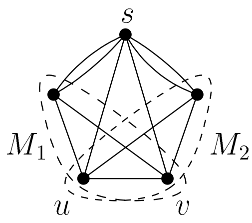

Figure 1: An obstacle with respect to the requirement function $r \equiv 4$.

An obstacle is shown in Figure 5. It follows from Lemma 2.6(b) that if H is an obstacle, then there is no r-admissible 3-split on su, sv, sz, for any choice of the edge sz, where su, sv is the pair of edges from s to $M_1 \cap M_2$. On the other hand, an r-admissible 3-split exists.

Lemma 5.3 *Suppose that G is an obstacle and (25) holds. Then there is an r-admissible 3-split in G.*

Proof: Since G is an obstacle and $f(s)$ is 3-regular, we have $d(s) = 6$, $f(s) = (3, 3)$, $\varphi = 2$, and there exist sets $M_1, M_2 \subset V$ satisfying properties (i),(ii), and (iii) above. It follows from (ii) and (25), that $\{a\} = M_1 - M_2$ and $\{b\} = M_2 - M_1$ are singleton tight sets in G. We claim that a 3-split on sa, sb, su is r-admissible, where $u \in M_1 \cap M_2$. To see this observe that the triple is semi-admissible, since a, b, u are distinct vertices and (25) holds. Furthermore, a bad set M with $e(sa, sb, su; M) = 3$ would satisfy $d(s, M) \geq 5$. This is impossible, since $d(s, M) \leq h(M) + \varphi \leq 4$, by (15). Thus conditions (i) and (ii) of Lemma 2.6(b) hold, and hence the 3-split is r-admissible, as claimed. •

Lemma 5.4 *Suppose that G is not an obstacle, (25) holds, and the 2-split on su, sv is r-admissible. Then there exists an edge sz such that the 3-split on su, sv, sz is r-admissible.*

Proof: By Lemma 5.2 the triple su, sv, sa is semi-admissible for some edge sa. If the 3-split on su, sv, sa is r-admissible then we are done. Otherwise there exists a maximal bad set M with $e(su, sv, sa; M) = 3$, by Lemma 2.6. By (15) we have $d(s, M) \leq h(M) + \varphi \leq 2 + \varphi$. Thus, since $d(s) = 3\varphi$ and $\varphi \geq 2$, it follows that there exists an edge sb with $b \in V - M$. Observe that the triple su, sv, sb is semi-admissible by (25), using the fact that the 2-split on su, sv is r-admissible. Thus either the 3-split on su, sv, sb is r-admissible, or there exists a maximal bad set M' with $e(su, sv, sb; M') = 3$. Clearly, M and M' are intersecting sets.

Apply Proposition 2.3 to the pair M, M'. If (10) holds (and $M \cup M' \neq V$) then we have $2 + 2 \geq h(M) + h(M') \geq h(M \cap M') + h(M \cup M') \geq 2 + 3$, a contradiction. Here we used the maximality of M and the fact that the 2-split on su, sv is r-admissible (and hence, by Lemma 2.6(a), $h(M \cap M') \geq 2$ holds). Now suppose that (11) holds, and let $N = V - (M \cup M')$. The existence of the edges su, sv implies that

$$d(M \cap M', N + s) \geq 2. \tag{26}$$

By (11) we have $2 + 2 \geq h(M) + h(M') \geq h(M - M') + h(M' - M) + 2d(M \cap M', V + s - (M \cup M')) \geq 2d(M \cap M', N + s)$. By (26) this implies that $h(M - M') = h(M' - M) = 0$ and $d(M \cap M', N + s) = 2$. Therefore $M - M'$ and $M' - M$ are both tight and by Lemma 2.5 we have $d(s, M - M'), d(s, M' - M) \leq \varphi$. Suppose that, in addition, we have $d(s, N) = 0$. Then we obtain, by using $\varphi \geq 2$, that $3\varphi \geq 2\varphi + 2 \geq d(s, M - M') + d(s, M' - M) + d(s, M \cap M') + d(s, N) = d(s) = 3\varphi$, which implies that $\varphi = 2$, and G is an obstacle (by choosing $M_1 = M$ and $M_2 = M'$), contradicting our assumption.

Thus $d(s, N) \geq 1$ holds. Let sc be an edge with $c \in N$. As above, it can be seen that su, sv, sc is a semi-admissible triple and either the 3-split on su, sv, sc is r-admissible or there is a maximal bad set M'' with $e(su, sv, sc; M'') = 3$. In the former case we are done. In the latter case we apply Proposition 2.3 to the pairs M, M'' and M', M'', as above, and conclude that $M - M''$, $M'' - M$, $M' - M''$, and $M'' - M'$ are all tight. Since (25) holds and the sets M, M', M'' are pairwise distinct maximal bad sets, it follows that $M \cap M' = M \cap M'' = M' \cap M''$ and $M \cup M' \cup M'' = M \cap M' \cap M'' \cup \{a, b, c\}$.

Claim 5.5 *Let X, Y be disjoint non-empty subsets of V with $d(X, Y) = 0$ and suppose that $X \cup Y$ is bad. Then X and Y are both tight.*

Proof: Clearly, we have $R(X \cup Y) \leq \max\{R(X), R(Y)\}$, so we may suppose, without loss of generality, that $R(X \cup Y) \leq R(X)$ holds. Since G is r-edge-connected in V and r is smooth, Lemma 2.4 implies that $R(X) \leq d(X)$ and $2 \leq R(Y) \leq d(Y)$. Since $d(X, Y) = 0$, we have $d(X) = d(X \cup Y) - d(Y)$. Thus $R(X \cup Y) \leq R(X) \leq d(X) = d(X \cup Y) - d(Y) \leq R(X \cup Y) + 2 - 2$. Hence equality must hold in each of these inequalities, which implies that X and Y are both tight. •

Now we can deduce that $d(c, M'' - c) \geq 1$ as follows. For a contradiction suppose that $d(c, M'' - c) = 0$ holds. Then, by applying Claim 5.5 with $X = \{c\}$ and $Y = M'' - c$, we obtain that the set $M'' - c$ is tight. Since $e(su, sv; M'' - c) = 2$, this contradicts the fact, by Lemma 2.6(a), that the 2-split on su, sv is r-admissible. This proves $d(c, M'' - c) \geq 1$. Thus there is an edge cr with $r \in M'' - c$. We have already shown that $M'' - c = M \cap M'$, and that equality holds in (26). Since the edges su, sv also enter the set $M \cap M'$, we obtain $2 = d(N + s, M \cap M') \geq 3$, a contradiction. This proves the lemma. •

Lemma 5.6 *There exists an r-admissible and feasible 3-split in G.*

Proof: By Lemma 3.3(b),(c) we have $|\mathcal{C}| \in \{0, 2, 3\}$, and there is an edge from s to each of the cores. If $|\mathcal{C}| = 3$ then the lemma follows from Lemma 5.1, so we may assume that $|\mathcal{C}| \in \{0, 2\}$ holds. In this case, starting from G, let us contract non-singleton tight sets (and update the requirement function) as long as possible, and let $\hat{G} = (\hat{V} + s, \hat{E})$ and $\hat{r}$ denote the resulting graph and requirement function, respectively. By Lemmas 4.1 and 4.3 $\hat{G}$ satisfies (13) and (14) with respect to $\hat{r}$ and $f(s)$, and by construction (25) holds. Note that $\hat{r}$ is smooth, and the degree specification for G and $\hat{G}$ are the same (in particular, $\hat{d}(s) = 3\varphi \geq 6$ holds).

Let us choose a pair su, sv of edges in G in such a way that $e(su, sv; C) \geq 1$ for each core C (if there is any), and the 2-split on su, sv is r-admissible in G. By Theorem 1.3 and Lemma 3.4 this can be done. Lemma 4.2 shows that the 2-split on su', sv' is $\hat{r}$-admissible in $\hat{G}$, where su', sv' are the edges of $\hat{G}$ corresponding to su, sv. Thus, by Lemmas 5.3 and 5.4, either there exists an $\hat{r}$-admissible 3-split su', sv', sz' in $\hat{G}$, or $\hat{G}$ is an obstacle with respect to $\hat{r}$ and there is an $\hat{r}$-admissible 3-split in $\hat{G}$. It follows from Lemma 4.4 that the corresponding 3-splits in G are r-admissible. In the former case the choice of the pair su, sv implies that this 3-split in G is feasible as well. In the latter case, when $\hat{G}$ is an obstacle, (19) can be verified directly by using the fact that the 3-split is r-admissible and $f(s) = (3, 3)$. Thus Lemma 3.2 implies that the 3-split is also feasible. •

We are ready to prove the main result of this paper.

Proof: (of Theorem 1.1) First we prove necessity. Suppose that $G' = (V + \{s_1, s_2, ..., s_p\}, E')$ is an $f(s)$-detachment of G, which is r-edge-connected in V. Since contracting the pieces of s into a single vertex does not decrease the local edge-connectivities between pairs of vertices of V, it follows that G is also r-edge-connected in V. Thus (2) holds. Consider a pair $u, v \in V$ and let $P_1, ..., P_l$ be edge-disjoint paths from u to v in G', where $l = \lambda_{G'}(u, v)$. For each piece s_i, $1 \leq i \leq p$, the number of those edges incident to s_i that belong to some of these paths is even. Thus at most $\varphi = \sum_{i=1}^{p} \lfloor d_i/2 \rfloor$ paths can go through the set of pieces of s, and hence at least $l - \varphi$ paths lie entirely in $G' - \{s_1, ..., s_p\} = G - s$. Since G' is r-edge-connected in V, this implies $\lambda_{G-s}(u, v) \geq l - \varphi = \lambda_{G'}(u, v) - \varphi \geq r(u, v) - \varphi$, which gives (3).

From now on we prove sufficiency. Suppose that $G = (V + s, E)$, $r : V^2 \to Z_+$, and $f(s) = (d_1, d_2, ..., d_p)$ satisfy the hypotheses of the theorem and that (2) and (3) hold. We shall prove that there is an $f(s)$-detachment of G which is r-edge-connected in V. We claim that it is sufficient to consider degree specifications for which $d_i \leq 3$ for $1 \leq i \leq p$. To see this suppose that, without loss of generality, $d_1 \geq 4$, and replace the degree specification $f(s)$ by $f^*(s) = (2, d_1 - 2, d_2, ..., d_p)$. The claim follows from the fact that (a) conditions (2) and (3) remain valid (since φ remains the same), and (b) an $f^*(s)$-detachment G^* of G which is r-edge-connected in V gives rise to an $f(s)$-detachment of G, which is also r-edge-connected in V, by identifying two pieces of s in G^* with degree two and degree $d_1 - 2$, respectively. Thus, by iteratively applying this reduction to the degree specification, as long as it is necessary, we may assume that each term in $f(s)$ is equal to two or three. So we can also assume that either $d_p = 2$, or $d_p = 3$ and $f(s)$ is 3-regular.

Our proof is by induction on $d(s)$. If $d(s) \leq 3$ then, since $d_i \geq 2$ for all $1 \leq i \leq p$, we must have $p = 1$, and the theorem is trivial. Thus we may assume $d(s) \geq 4$. Since $d_i \leq 3$ for all $1 \leq i \leq p$, this implies $p \geq 2$ and $\varphi \geq 2$. By focusing on the 2-edge-connected component of G containing s, we may also assume that G is 2-edge-connected in $V + s$. With this assumption (and since $\varphi \geq 2$) we can increase some of the requirements up to 2, if necessary, without violating conditions (2) and (3). Thus we may also assume $r \geq 2$.

Note that by Lemmas 2.4 and 2.5 (2) and (3) hold if and only if (13) and (14) hold. If $d_p = 2$ then Lemma 3.6 implies that there is an r-admissible and feasible 2-split in G. If $d_p = 3$ (and hence $f(s)$ is 3-regular) then Lemma 5.6 implies that there is an r-admissible and feasible 3-split in G. Let $G' = (V + t + s, E')$ be the graph obtained from G by an r-admissible and feasible 2- or 3-split, and let r' and $f'(s)$ be the extended requirement function and reduced degree specification, respectively. Now the choice of the split and Lemmas 3.1 and 3.2 imply that G' satisfies (2) and (3) with respect to $V + t$, r', and φ'. Clearly, $d'(s) < d(s)$. Thus, by induction, G' has an $f'(s)$-detachment G'' which is r-edge-connected in V. Since G'' is an $f(s)$-detachment of G, the theorem follows. •

We close this section by extending Theorem 1.1 to arbitrary degree specifications. We need the following definitions. Let $H = (V, E)$ be a graph and let $r : V^2 \to Z_+$ be a requirement function. As before, let $R(X) = \max\{r(u, v) : u \in X, v \in V - X\}$ be defined on sets $X \subset V$, and let $q_H(X) = R(X) - d_H(X)$. A connected component D of H is called a *marginal component* (with respect to r) if $q_H(W) \leq 0$ for every $W \subset D$ and $q_H(D) \leq 1$ hold. Note that if r is smooth then there are no marginal components in H. We also need a lemma due to Frank.

Lemma 5.7 *[5, Lemma 5.6] Suppose that $H = (V, E)$ has no marginal components and let $H' = (V + s, E)$ be a graph obtained from H by adding a new vertex s and α new edges between V and s so that H' is r-edge-connected in V. In addition, suppose that γ is an integer satisfying*

$$\sum_{i=1}^{t} q_H(X_i) \leq \gamma \tag{27}$$

for every subpartition $\{X_1, ..., X_t\}$ of V. Then it is possible to delete $\alpha - \gamma$ edges incident to s in H' so that none of the remaining new edges is a cut-edge in the resulting graph H'', and that H'' is r-edge-connected in V.

Theorem 5.8 *Let $G = (V + s, E)$ be a graph, let r be a requirement function on V, and suppose that $G - s$ has no marginal components with respect to r. Let $f(s) = (d_1, d_2, ..., d_p)$ be an arbitrary degree specification, where the number of $d_i = 1$ terms equals ψ. Then there exists an $f(s)$-detachment of G which is r-edge-connected in V if and only if (2) and (3) hold, and*

$$\sum_{i=1}^{t} q_{G-s}(X_i) \leq d(s) - \psi \tag{28}$$

holds for every subpartition $\{X_1, ..., X_t\}$ of V.

Proof: First we prove the only if direction. Suppose the desired $f(s)$-detachment G' exists. As in the proof of Theorem 1.1, this implies that (2) and (3) hold. Let S be the set of pieces of s in G' and let $L \subseteq S$ denote the set of pieces with degree one. It is easy to see that G' is r-edge-connected in V if and only if $G' - L$ is r-edge-connected in V. Since $G' - L$ is r-edge-connected in V, we have $d_{G'-L}(X) \geq R(X)$ for all $\emptyset \neq X \subset V$. Thus $d_{G'-L}(S - L, X) \geq R(X) - d_{G-s}(X) = q_{G-s}(X)$ for all $\emptyset \neq X \subset V$, which implies (28), since $d_{G'-L}(S - L, V) = d(s) - \psi$.

To see the other direction we can use Lemma 5.7 to show that it is possible to delete ψ edges incident to s in G so that the resulting graph G^* is still r-edge-connected in V and has no cut-edges incident to s. Let $f^*(s)$ be the degree specification obtained from $f(s)$ by deleting each term with $d_i = 1$. It is easy to see that G^* satisfies (3) with respect to r and $f^*(s)$. Now Theorem 1.1 implies that G^* has an $f^*(s)$-detachment $G'' = (V + \{s_1, s_2, ..., s_{p-\psi}\}, E'')$ which is r-edge-connected in V. By adding ψ new vertices to G'' and an edge wv_w, for some $v_w \in V$, for each new vertex w, we obtain the desired $f(s)$-detachment of G. •

6 Applications and corollaries

In this section we apply Theorem 1.1 to deduce new results on graph and hypergraph augmentation problems. In the local edge-connectivity augmentation problem we are given a graph (or hypergraph) $G = (V, E)$ and a requirement function $r : V^2 \to Z_+$, and the goal is to find a smallest set F of new edges (or hyperedges of given size) for which the augmented graph (or hypergraph) $G' = (V, E + F)$ is r-edge-connected in V. For graphs this problem was solved, in terms of a min-max equality and a polynomial algorithm, by Frank [5]. For hypergraphs it is NP-hard to find a smallest augmentation consisting of hyperedges of size two [2].

First we consider the extension of the graph problem, where, instead of adding edges, we *attach* stars of given degrees. (A star of degree t is the graph $K_{1,t}$.) In other words, we add (an independent set of) new vertices, and connect each new vertex s_i to V by d_i edges, where the d_i's are given positive integers. For simplicity, we shall assume that G has no marginal components with respect to r. The following result can be extended to the case when marginal components may exist, following [5]. We leave these details to the interested reader.

Theorem 6.1 *Let $G = (V, E)$ be a graph and let $r : V^2 \to Z_+$ be a requirement function, such that G has no marginal components with respect to r. Then G can be made r-edge-connected in V by attaching p stars with degrees $d_1, ..., d_p$, where $d_i \geq 2$ for all $1 \leq i \leq p$, if and only if*

$$\sum_{i=1}^{t} q_G(X_i) \leq \sum_{i=1}^{p} d_i \tag{29}$$

holds for every subpartition $\{X_1, ..., X_t\}$ of V and

$$\lambda_G(u, v) \geq r(u, v) - \sum_{i=1}^{p} \lfloor d_i/2 \rfloor \tag{30}$$

for every pair $u, v \in V$.

Proof: Necessity is easy to see. To prove sufficiency suppose that (29) and (30) hold for G. Observe that the required attachment exists if and only if G can be extended to a graph $G' = (V + s, E')$ by adding a new vertex s such that $d_{G'}(s) = \sum_{i=1}^{p} d_i$, and G' has an $f(s)$-detachment which is r-edge-connected in V, for $f(s) = (d_1, d_2, ..., d_p)$. Condition (29) and Lemma 5.7 imply that an extension G' which is r-edge-connected in V and which satisfies (2) and $d_{G'}(s) = \sum_{i=1}^{p} d_i$ exists. Condition (30) shows that G' satisfies (3) with respect to r and $f(s)$. Thus Theorem 1.1 implies that G' has an $f(s)$-detachment which is r-edge-connected in V. This proves the theorem. •

With Theorem 6.1 we can also solve the following optimization problem: given G,r, and an integer w, determine the smallest number γ_w for which G can be made r-edge-connected by attaching γ_w stars of degree w each. When we augment the local edge-connectivities, attaching stars of degree two is equivalent to adding new edges. Thus for $w = 2$ we obtain Frank's theorem as a corollary. (Note that (29) implies (30) if $d_i = 2$ for $1 \leq i \leq p$.)

Theorem 6.2 *[5] Let $G = (V, E)$ be a graph and let $r : V^2 \to Z_+$ be a requirement function, such that G has no marginal components with respect to r. Then G can be made r-edge-connected by adding γ new edges if and only if $\sum_{i=1}^{t} q(X_i) \leq 2\gamma$ holds for every subpartition $\{X_1, ..., X_t\}$ of V.*

A 3-*hypergraph* is a hypergraph with hyperedges of size at most three. The next problem we consider is the local edge-connectivity augmentation problem for 3-hypergraphs, where the new hyperedges that we add are also of size at most three.

Theorem 6.3 *Let $G = (V, E)$ be a 3-hypergraph and let $r : V^2 \to Z_+$ be a smooth requirement function. Then G can be made r-edge-connected by adding γ new hyperedges of size two and β new hyperedges of size three if and only if*

$$\sum_{i=1}^{t} q_G(X_i) \leq 2\gamma + 3\beta \tag{31}$$

holds for every subpartition $\{X_1, ..., X_t\}$ of V, and

$$\lambda_G(u,v) \geq r(u,v) - (\gamma + \beta) \tag{32}$$

for every pair $u, v \in V$.

Proof: Necessity is easy to see. To see sufficiency, suppose that G satisfies (31) and (32). Let us construct a graph $G' = (V', E')$ by replacing every edge $e = uvw$ of size three in G by a special vertex v_e and three graph edges v_eu, v_ev, v_ew. The key observation is that replacing hyperedges of size three by stars of degree three (and vice versa) does not change the local edge-connectivities in V. Thus (32) remains valid in G'. Define a requirement function r' on pairs of vertices of G' by putting $r'(u,v) = r(u,v)$ for all pairs $u, v \in V$ and $r'(x,y) = 0$ otherwise. Since r is smooth, and by the definition of r', it follows that G' has no marginal components with respect to r'. Furthermore, since (31) and (32) hold, and by the definition of G' and r', it can be seen that G' satisfies (29) and (30) with respect to r' and degree sequence $d_1, d_2, ..., d_p$, where $p = \gamma + \beta$, $d_i = 2$ for $1 \leq i \leq \gamma$, and $d_i = 3$ for $\gamma + 1 \leq i \leq p$. Thus we can use Theorem 6.1 to deduce that G' can be made r-edge-connected in V by attaching γ stars of degree two and β stars of degree three. Note that, using the fact that $r'(v_e, y) = 0$ if v_e is a special vertex, the proof of Theorem 6.1 shows that the stars can be attached to G' so that no star is attached to any of the special vertices. By replacing each star tu, tv of degree two by an edge uv, and each star tu, tv, tw of degree three by a hyperedge uvw, as well as each star v_eu, v_ev, v_ew by the hyperedge uvw, we obtain an r-edge-connected 3-hypergraph on V, as required. •

Fleiner [3] proved that Theorem 1.2 is valid for 3-hypergraphs, as well, provided no hyperedge of size three contains the designated vertex s. In fact, his proof works only in the more general 3-hypergraph setting. He also proved Theorem 6.3 in the special case of uniform requirements. Here we note that the 3-hypergraph version of Theorem 1.2 (and Theorem 1.1) is easy to deduce directly from Theorem 1.1 by the 3-hyperedge/star replacement method that we used in the proof of Theorem 6.3.

The detachment problem considered by Nash-Williams [8, 9] was different in the following sense. Given a graph $G = (V, E)$, a *global degree specification* f for G assigns a degree specification $f(v) = (d_1^v, ..., d_{p_v}^v)$ to each vertex $v \in V$. We say that G' is a *global f-detachment of* G if G' can be obtained from G by simultaneously detaching each vertex v into p_v pieces such that the degrees of the pieces are given by $f(v)$. Thus every edge uv in G corresponds to an edge connecting some piece of u to some piece of v in G'. Nash-Williams proved the following theorem. (See [1] for a somewhat shorter proof.)

Theorem 6.4 *[9] Let $G = (V, E)$ be a graph, let f be a global degree specification for G, and let $k \geq 2$ be an integer. Then G has a k-edge-connected global f-detachment if and only if G is k-edge-connected, $d_i^v \geq k$ for every $v \in V$, $1 \leq i \leq p_v$, and neither of the following two cases occurs:*
(a) k is odd and G has a cut-vertex v with $f(v) = (k, k)$,
(b) k is odd, $V = \{u, v\}$, and $f(u) = f(v) = (k, k)$.

Fleiner [3] showed that Theorem 6.4 can be deduced from Theorem 1.2. It might be interesting to prove some kind of local edge-connectivity version of Theorem 6.4, using Theorem 1.1.

We do not discuss the algorithmic aspects of our results in detail but note that the proofs of Theorem 1.1 and its applications are algorithmic, and give rise to polynomial algorithms for constructing an r-edge-connected $f(s)$-detachment, if exists, and augmenting the local edge-connectivity of a graph or 3-hypergraph optimally by attaching stars or adding hyperedges of size at most three. This follows from the fact that checking whether a 2-split or 3-split is r-admissible and feasible can be done by max-flow computations. An efficient algorithm for the special case of finding λ-admissible splittings is given in [6].

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References

[1] A. Berg, B. Jackson, T. Jordán, Highly edge-connected detachments of graphs and digraphs, to appear in *J. Graph Theory.*

[2] B. Cosh, B. Jackson, Z. Király, Local connectivity augmentation in hypergraphs is NP-complete, manuscript, 2002.

[3] B. Fleiner, Detachments of vertices of graphs preserving edge-connectivity, submitted to *SIAM J. Discrete Math.*

[4] A. Frank, On a theorem of Mader, *Discrete Mathematics* 101 (1992) 49-57.

[5] A. Frank, Augmenting graphs to meet edge-connectivity requirements, *SIAM J. Discrete Math.* **5**, 25–53 (1992).

[6] H.N. Gabow, Efficient splitting off algorithms for graphs, in Proc. 26th Annual ACM Symposium on the Theory of Computing, ACM, New-York, 1994, pp. 696-705.

[7] W. Mader, A reduction method for edge-connectivity in graphs, *Ann. Discrete Math.* 3 (1978) 145-164.

[8] C. St. J. A. Nash-Williams, Detachments of graphs and generalised Euler trails, Surveys in combinatorics 1985, 137–151, London Math. Soc. Lecture Note Ser., 103, Cambridge Univ. Press, 1985.

[9] C. St. J. A. Nash-Williams, Connected detachments of graphs and generalized Euler trails, *J. London Math. Soc.*, Vol. 31, 1985, pp. 17-29.