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On $(2, k)$-connected graphs

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Abstract

A graph G is called $(2, k)$-connected if G is $2k$-edge-connected and $G - v$ is k-edge-connected for every vertex v. The study of $(2, 2)$-connected graphs is motivated by a theorem of Thomassen [J. Combin. Theory Ser. A 110 (2015), pp. 67–78] (that was a conjecture of Frank [SIAM J. Discrete Math. 5 (1992), no. 1, pp. 25–53]), which states that a graph has a 2-vertex-connected orientation if and only if it is (2,2)-connected. In this paper, we provide a construction of the family of $(2, k)$-connected graphs for k even, which generalizes the construction given by Jordán [J. Graph Theory 52 (2006), pp. 217–229] for (2,2)-connected graphs. We also solve the corresponding connectivity augmentation problem: given a graph G and an integer $k \geq 2$, what is the minimum number of edges to be added to make G $(2, k)$-connected. Both these results are based on a new splitting-off theorem for $(2, k)$-connected graphs.

KEYWORDS
connectivity, connectivity augmentation, orientation, splitting-off

1 | INTRODUCTION

Let $G = (V, E)$ be an undirected graph (in short, a graph), in which loops and parallel edges are allowed. A subset of V is called *nontrivial* if it is different from the empty set and the whole set V. For $U, W \subset V$, $d_G(U, W)$ denotes the number of edges with one end-vertex in $U \backslash W$ and the other end-vertex in $W \backslash U$. For the sake of convenience, $d_G(U, \overline{U})$ is denoted by $d_G(U)$. Given a set of edges $F \subseteq E$, we define $d_F(U) = d_{(V,F)}(U)$.

Let $H = (V + s, E)$ be a graph with a special vertex s such that no loop is incident to s. For convenience, in this paper, H will always denote a graph with such a special vertex s.

1.1 | Connectivity

In this paper, we will need the following mixed-connectivity concepts of graphs introduced by Kaneko and Ota [9]. Let ℓ and k be positive integers. The graph G is called *(ℓ, k)-connected* if $|V| > \ell$ and for all $U \subseteq V$, $F \subseteq E$ such that $k|U| + |F| < \ell k$, $G - U - F$ is connected. This notion contains both vertex-connectivity (for $k = 1$) and edge-connectivity (for $\ell = 1$). Indeed, G is *ℓ-vertex-connected* if and only if $|V| > \ell$ and for all $U \subset V$ such that $|U| < \ell$, $G - U$ is connected. Furthermore, G is *k-edge-connected* if and only if at least k edges enter all nontrivial sets of V. The graph $H = (V + s, E)$ is called *k-edge-connected in V* if at least k edges enter all nontrivial sets of V. In this paper, we consider $(2, k)$-connected graphs. Observe that G is *$(2, k)$-connected* if $|V| \geq 3$, G is $2k$-edge-connected and, for all $v \in V$, $G - v$ is k-edge-connected. Note that $(2, k)$-connectivity is stronger than $2k$-edge-connectivity but much weaker than $2k$-vertex-connectivity.

We will need some connectivity concepts in directed graphs as well. Let $D = (V, A)$ be a directed graph. We say that D is *strongly connected* if for every nontrivial vertex set X of V, there exists an arc entering X. The digraph D is called *ℓ-arc-connected* if, for all $F \subseteq A$ such that $|F| < \ell$, $D - F$ is strongly connected. Note that D is ℓ-arc-connected if and only if at least ℓ arcs enter all nontrivial sets of V. The digraph D is called *ℓ-vertex-connected* if $|V| > \ell$ and for all $X \subset V$ such that $|X| < \ell$, $D - X$ is strongly connected.

To motivate our problems, let us recall some results on orientations, constructions, splitting-off, and augmentations of graphs.

1.2 | Orientations

We start with the classic result on edge-connectivity.

Theorem 1.1 (Nash-Williams [12]). *An undirected graph has a k-arc-connected orientation if and only if it is $2k$-edge-connected.*

Inspired by Theorem 1.1, Frank [6] proposed a conjecture concerning vertex-connectivity.

Conjecture 1.1 (Frank [6]). *An undirected graph $G = (V, E)$ has a k-vertex-connected orientation if and only if G is $(k, 2)$-connected.*

Recently, some breakthroughs have been achieved on this conjecture. On the one hand, Durand de Gevigney [3] proved that Conjecture 1.1 is false for $k \geq 3$.

Theorem 1.2 (Durand de Gevigney [3]). *For every $k \geq 3$, there exist $(k, 2)$-connected undirected graphs that have no k-vertex-connected orientation. Moreover, for every $k \geq 3$, it is NP-complete to decide whether an undirected graph has a k-vertex-connected orientation.*

On the other hand, Thomassen [14] proved that Conjecture 1.1 is true for $k = 2$.

Theorem 1.3 (Thomassen [14]). *An undirected graph has a 2-vertex-connected orientation if and only if it is $(2, 2)$-connected.*

We mention that the special case of Theorem 1.3 when the graph is Eulerian was earlier proved by Berg and Jordán [2].

1.3 | Constructions

Theorem 1.1 can easily be proved by applying the following construction of Lovász [10] of $2k$-edge-connected graphs. Let K_2^{2k} be the graph on 2 vertices with $2k$ edges between them. The operation *pinching* k edges is defined as follows: subdivide each of the k edges by a new vertex and identify these new vertices.

Theorem 1.4 (Lovász [10]). *A graph is* $2k$*-edge-connected if and only if it can be obtained from* K_2^{2k} *by a sequence of the following two operations:*

(a) *adding a new edge,*
(b) *pinching* k *edges.*

Conjecture 1.1 drew attention on the family of (2, 2)-connected graphs. Jordán [8] gave the following construction of this family, similar to the above construction of 4-edge-connected graphs. For $k \geq 2$, let K_3^k be the graph on 3 vertices with k edges between each pair of vertices. Note that a (2, 2)-connected graph must contain at least 3 vertices, this is why the starting graph is different.

Theorem 1.5 (Jordán [8]). *A graph is* (2, 2)*-connected if and only if it can be obtained from* K_3^2 *by a sequence of the following two operations:*

(a) *adding a new edge,*
(b) *pinching* 2 *edges such that if one of them is a loop, then the other one is not adjacent to it.*

Unfortunately, this construction does not help prove Conjecture 1.1 for $k = 2$.

We will generalize Theorem 1.5 in Theorem 4.9.

We mention that concerning vertex-connectivity, a few results are known. Constructions are given only for 2- and 3-vertex-connected graphs, see Robbins [13], Barnette and Grünbaum [1], and also Tutte [15].

1.4 | Splitting-off

To prove Theorem 1.4, one has to consider the inverse operations: deleting an edge and complete splitting-off at a vertex of degree $2k$. Let us now introduce the operation of *complete splitting-off at a vertex* s of even degree. It consists of partitioning the set of edges incident to s into pairs, replacing each pair (su, sv) by a new edge uv and then deleting s. If the graph is minimally $2k$-edge-connected, that is, when no edge can be deleted without destroying $2k$-edge-connectivity, then the following result shows that there exists a vertex of degree $2k$.

Theorem 1.6 (Mader [11]). *Every minimally* $2k$*-edge-connected graph contains a vertex of degree* $2k$.

Then, the following splitting-off theorem of Lovász [10] implies the existence of a complete splitting-off at this vertex that preserves $2k$-edge-connectivity.

Theorem 1.7 (Lovász [10]). *Let $H = (V + s, E)$ be an ℓ-edge-connected graph for $\ell \geq 2$, where s is a vertex of even degree. Then, there exists a complete splitting-off at s such that the new graph is ℓ-edge-connected.*

We will also need the splitting-off result of Mader [11]. Let (su, sv) be a pair of (possibly parallel) edges in $H = (V + s, E)$. *Splitting-off* the pair (su, sv) at s in H consists in replacing the edges su, sv by a new edge uv. The graph arising from this splitting-off at s is denoted by $H_{u,v}$.

Theorem 1.8 (Mader [11]). *Let $H = (V + s, E)$ be an ℓ-edge-connected graph in V for $\ell \geq 2$ such that $d_H(s) \neq 3$ and $d_H(s) \geq 2$. Then, there exists a pair of edges (su, sv) in H such that $H_{u,v}$ is ℓ-edge-connected in V.*

For a pair (su, sv) of (possibly parallel) edges of H, if H and $H_{u,v}$ are $(2, k)$-connected in V, then the pair (su, sv) is called $(2, k)$-*admissible* (in short, *admissible* when k is clear from the context). A complete splitting-off is called *admissible* if the resulting graph is $(2, k)$-connected in V.

To get Theorem 1.5, one has to consider the inverse operations: deleting an edge and complete splitting-off at a vertex of degree 4. If the graph is minimally $(2, k)$-connected, that is, when no edge can be deleted without destroying $(2, k)$-connectivity, then the following result [9, Lemma 7] shows that there exists a vertex of degree $2k$. For the definitions of inner-set and tight bi-set, see Section 2.

Theorem 1.9 (Kaneko and Ota [9]). *Let $G = (V, E)$ be a minimally $(2, k)$-connected graph. Then, the inner-set of every tight bi-set contains a vertex of degree $2k$.*

We mention that Theorem 1.9 will be used in the proof of Theorem 4.9.

Jordán [8] proved a splitting-off theorem on (2, 2)-connected graphs. Here, it is possible that there exists no complete splitting-off preserving (2, 2)-connectivity, in this case a special kind of obstacle exists. Let $H = (V + s, E)$ be a graph with $d_H(s) = 4$, and $\{t, x, y, z\}$ the set of neighbors of s. The quadruple (t, X, Y, Z) is called a 2-*obstacle* at s if X, Y, and Z are pairwise disjoint vertex sets of $V - t$, $x \in X$, $y \in Y$, $z \in Z$ and $d_{H-t}(X) = d_{H-t}(Y) = d_{H-t}(Y) = 2$.

Theorem 1.10 (Jordán [8]). *Let $H = (V + s, E)$ be a (2, 2)-connected graph such that $|V| \geq 3$ and $d_H(s) = 4$. Then, there exists a (2, 2)-admissible complete splitting-off at s if and only if there exists no 2-obstacle at s.*

We will generalize Theorem 1.10 in Theorem 4.7.

1.5 | Augmentation

Theorem 1.7 has other applications, among others, it can be used to solve the ℓ-edge-connected augmentation problem (see Frank [5]).

Theorem 1.11 (Watanabe and Nakamura [16]). *Let $G = (V, E)$ be a graph and $\ell \geq 2$ an integer. The minimum cardinality of a set F of edges such that $(V, E \cup F)$ is ℓ-edge-connected is equal to*

$$\left\lceil \frac{1}{2} \max\left\{\sum_{X\in\mathcal{X}} (\ell - d_G(X))\right\}\right\rceil,$$

where $\mathcal{X}$ is a family of nontrivial pairwise disjoint sets of V.

The $(2, k)$-connectivity augmentation problem can be formulated as follows: what is the minimum number of edges whose addition results in a $(2, k)$-connected graph. The min-max theorem on this problem is presented in Theorem 4.12.

The ℓ-vertex-connectivity augmentation problem is still open. For fixed ℓ, Jackson and Jordán [7] provided a polynomial algorithm.

This paper is devoted to the study of $(2, k)$-connected graphs and is organized as follows. We give the necessary definitions in Section 2 and then some preliminary results in Section 3. The main results are presented in Section 4. First, we provide a new splitting-off theorem for $(2, k)$-connected graphs. As in the special case $k = 2$, the existence of a complete splitting-off preserving $(2, k)$-connectivity depends on the nonexistence of an obstacle. Second, we give a construction of the family of $(2, k)$-connected graphs for k even. These are the natural generalizations of the previous results of Jordán [8] on (2,2)-connected graphs. Finally, we solve the $(2, k)$-connectivity augmentation problem. We follow Frank's [5] approach: we find a minimal extension and then we apply our splitting-off theorem. This way we provide a new case for connectivity augmentation when a min-max formula exists.

2 | DEFINITIONS

Let Ω be a ground set. A subset of Ω is called *trivial* if it coincides with $\varnothing$ or Ω. The *complement* of a subset $U \subseteq \Omega$ is defined by $\overline{U} = \Omega \backslash U$. For $X_I \subseteq X_O \subseteq \Omega$, the pair of sets $\mathsf{X} = (X_O, X_I)$ is called *a bi-set* of Ω. The sets X_I, X_O, and $w^b(\mathsf{X}) = X_O \backslash X_I$ are the *inner-set*, the *outer-set*, and the *wall* of X, respectively[1]. If $X_I = \varnothing$ or $X_O = \Omega$, then the bi-set X is called *trivial*. The *intersection* and the *union* of two bi-sets $\mathsf{X} = (X_O, X_I)$ and $\mathsf{Y} = (Y_O, Y_I)$ are defined by $\mathsf{X} \sqcap \mathsf{Y} = (X_O \cap Y_O, X_I \cap Y_I)$ and $\mathsf{X} \sqcup \mathsf{Y} = (X_O \cup Y_O, X_I \cup Y_I)$, respectively. We encourage the readers to use figures like Figure 1 to check properties of bi-sets.

Note that

$$|w^b(\mathsf{X})| + |w^b(\mathsf{Y})| = |w^b(\mathsf{X} \sqcap \mathsf{Y})| + |w^b(\mathsf{X} \sqcup \mathsf{Y})|. \tag{1}$$

We say that Y *contains* X, denoted by $\mathsf{X} \sqsubseteq \mathsf{Y}$, if $X_O \subseteq Y_O$ and $X_I \subseteq Y_I$; while Y *strictly contains* X, denoted by $\mathsf{X} \sqsubset \mathsf{Y}$, if $\mathsf{X} \sqsubseteq \mathsf{Y}$ and $\mathsf{X} \neq \mathsf{Y}$. We say that X and Y are *innerly disjoint* if the inner-sets X_I and Y_I are disjoint. We extend the complement operation to bi-sets by defining the *complement* of X as $\overline{\mathsf{X}} = (\overline{X_I}, \overline{X_O})$. For a family $\mathcal{F}$ of bi-sets of Ω, we denote by $\Omega_I(\mathcal{F}) = \cup_{\mathsf{X}\in\mathcal{F}} X_I$ the union of the inner-sets of the members of $\mathcal{F}$. A bi-set function h^b is called *submodular* if, for all bi-sets X and Y,

$$h^b(\mathsf{X}) + h^b(\mathsf{Y}) \geq h^b(\mathsf{X} \sqcap \mathsf{Y}) + h^b(\mathsf{X} \sqcup \mathsf{Y}). \tag{2}$$

[1]In this study, we use a small letter b to differentiate bi-set functions from set functions. We also use a sans serif typeface (such as X) to differentiate bi-sets from sets.

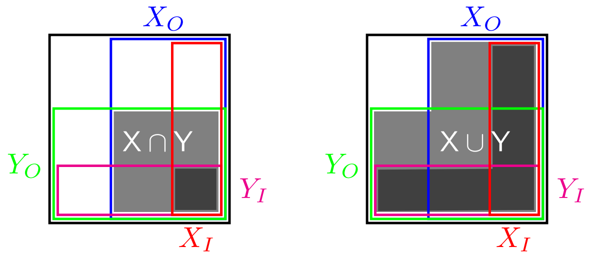

FIGURE 1 The intersection and the union of two bi-sets [Color figure can be viewed at wileyonlinelibrary.com]

Let $G = (V, E)$ be a graph. An edge e of G *enters* a bi-set $\mathsf{X} = (X_{\mathrm{O}}, X_{\mathrm{I}})$ of V, if one of the end-vertices of e belongs to $\overline{X_{\mathrm{O}}}$ and the other one to X_{I}. The *degree* of X, denoted by $d_G^{\mathrm{b}}(\mathsf{X})$, is the number of edges of G entering X. Note that the degree function of bi-sets is a generalization of the degree function of sets since $d_G(U) = d_G^{\mathrm{b}}((U, U))$ for any subset U of V. Observe that d_G^{b} is symmetric with respect to the complement operation of bi-sets and satisfies the following equation for all bi-sets X and Y of V.

$$d_G^{\mathrm{b}}(\mathsf{X}) + d_G^{\mathrm{b}}(\mathsf{Y}) = d_G^{\mathrm{b}}(\mathsf{X} \sqcap \mathsf{Y}) + d_G^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y}) + d_G(\overline{X_{\mathrm{O}}} \cap Y_{\mathrm{O}}, X_{\mathrm{I}} \cap \overline{Y_{\mathrm{I}}}) + d_G(\overline{Y_{\mathrm{O}}} \cap X_{\mathrm{O}}, Y_{\mathrm{I}} \cap \overline{X_{\mathrm{I}}}) \tag{3}$$

that can be established by checking that any edge contributes to the same amount on each side. It directly follows from 3 that d_G^{b} is submodular.

Let k be a positive integer. Recall that the graph G is $2k$-edge-connected if and only if $d_G(X) \geq 2k$ for all nontrivial sets X of V, that is, $d_G^{\mathrm{b}}(\mathsf{X}) \geq 2k$ for all nontrivial bi-sets X of V such that $w^{\mathrm{b}}(\mathsf{X})$ is empty. Moreover, for any vertex v, the graph $G - v$ is k-edge-connected if and only if $d_{G-v}(X) \geq k$ for all nontrivial set X of V, that is, $d_G^{\mathrm{b}}(\mathsf{X}) \geq k$ for all nontrivial bi-sets X of V such that $w^{\mathrm{b}}(\mathsf{X}) = \{v\}$. Note that if $|w^{\mathrm{b}}(\mathsf{X})| \geq 2$, then $k|w^{\mathrm{b}}(\mathsf{X})| \geq 2k$. These arguments show that $(2, k)$-connectivity can be reformulated using bi-sets as follows: the graph G is $(2, k)$-connected if and only if $|V| \geq 3$ and, for all nontrivial bi-sets X of V,

$$f_G^{\mathrm{b}}(\mathsf{X}) := d_G^{\mathrm{b}}(\mathsf{X}) + k|w^{\mathrm{b}}(\mathsf{X})| \geq 2k. \tag{4}$$

A bi-set X that satisfies 4 with equality is called *tight*. Equations 1 and 3 imply that, for all bi-sets X and Y of V, we have

$$f_G^{\mathrm{b}}(\mathsf{X}) + f_G^{\mathrm{b}}(\mathsf{Y}) = f_G^{\mathrm{b}}(\mathsf{X} \sqcap \mathsf{Y}) + f_G^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y}) + d_G(\overline{X_{\mathrm{O}}} \cap Y_{\mathrm{O}}, X_{\mathrm{I}} \cap \overline{Y_{\mathrm{I}}}) + d_G(\overline{Y_{\mathrm{O}}} \cap X_{\mathrm{O}}, Y_{\mathrm{I}} \cap \overline{X_{\mathrm{I}}}). \tag{5}$$

Let $H = (V + s, E)$ be a graph. We denote by $N_H(s)$ the set of neighbors of s in H. The graph H is called *$(2, k)$-connected in V* if $|V| \geq 3$, and 4 holds in H for all nontrivial bi-sets X of V. Note that, considering the graph H, for a set X (resp. a bi-set X), the complement $\overline{X}$ (resp. $\overline{\mathsf{X}}$) is taken with respect to the ground set $\Omega = V + s$. We will also need the complement X^{c} (resp. X^{c}) with respect to V, that is, $X^{\mathrm{c}} := V \backslash X$ and $\mathsf{X}^{\mathrm{c}} := (X_{\mathrm{I}}^{\mathrm{c}}, X_{\mathrm{O}}^{\mathrm{c}}) = (V \backslash X_{\mathrm{I}}, V \backslash X_{\mathrm{O}})$. Observe that

$$f_H^{\mathrm{b}}(\mathsf{X}) - d_H(s, X_{\mathrm{I}}) = d_H(X_{\mathrm{I}}, X \backslash X_{\mathrm{O}}) + k|w^{\mathrm{b}}(\mathsf{X})| = f_H^{\mathrm{b}}(\mathsf{X}^{\mathrm{c}}) - d_H(s, X_{\mathrm{O}}^{\mathrm{c}}). \tag{6}$$

By 5 and 4, we have immediately the following results.

Proposition 2.1. *Let $H = (V + s, E)$ be a $(2, k)$-connected graph in U, where $U = V$ or $U = V + s$, X and Y tight bi-sets of U.*

(a) *If $\mathsf{X} \sqcap \mathsf{Y}$ and $\mathsf{X} \sqcup \mathsf{Y}$ are nontrivial bi-sets of U, then $\mathsf{X} \sqcap \mathsf{Y}$ and $\mathsf{X} \sqcup \mathsf{Y}$ are tight and $d_H(\overline{X_O} \cap Y_O, X_I \cap \overline{Y_I}) = d_H(\overline{Y_O} \cap X_O, Y_I \cap \overline{X_I}) = 0$.*
(b) *If $\mathsf{X} \sqcap \overline{\mathsf{Y}}$ and $\overline{\mathsf{X}} \sqcap \mathsf{Y}$ are nontrivial bi-sets of U, then $\mathsf{X} \sqcap \overline{\mathsf{Y}}$ and $\overline{\mathsf{X}} \sqcap \mathsf{Y}$ are tight and $d_H(\overline{X_O} \cap \overline{Y_I}, X_I \cap Y_O) = d_H(Y_I \cap X_O, \overline{Y_O} \cap \overline{X_I}) = 0$.*

Proposition 2.2. *Let $H = (V + s, E)$ be a graph, X and Y bi-sets of $V + s$ such that $f_H^{\mathrm{b}}(\mathsf{X} \sqcap \mathsf{Y}) \ge 2k$ and $|w^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y})| \ge 2$. Then,*

$$(f_H^{\mathrm{b}}(\mathsf{X}) - 2k) + (f_H^{\mathrm{b}}(\mathsf{Y}) - 2k) \ge d_H^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y}) + d_H(\overline{X_O} \cap Y_O, X_I \cap \overline{Y_I}) + d_H(\overline{Y_O} \cap X_O, Y_I \cap \overline{X_I}). \tag{7}$$

3 | PRELIMINARIES

In this section, we provide the preliminary results that will be needed in the proofs of our main theorems.

3.1 | Blocking bi-sets

We introduce a special type of bi-sets that help characterize pairs of adjacent edges not to be admissible. Then, we provide a useful lemma about such bi-sets to be applied frequently in the later proofs.

Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V with a special vertex s and (su, sv) a pair of edges. A nontrivial bi-set X of V is called *a blocking bi-set* for the pair (su, sv) if either 8 or 9 is satisfied.

$$f_H^{\mathrm{b}}(\mathsf{X}) \le 2k + 1 \text{ and } X_I \text{ contains both } u \text{ and } v, \tag{8}$$

$$f_H^{\mathrm{b}}(\mathsf{X}) = 2k, X_I \text{ contains one of } u \text{ and } v, \text{ and } w^{\mathrm{b}}(\mathsf{X}) \text{ consists of the other one.} \tag{9}$$

Let X be a blocking bi-set for the pair (su, sv). Then, we say that X *blocks* (su, sv). If 8 occurs, then X is called *dangerous* and if 9 occurs, then X is called *critical*. Note that critical bi-sets are tight. The blocking bi-set X for the pair (su, sv) is called *maximal* if no blocking bi-set for (su, sv) contains strictly X. The term blocking is justified by the following lemma.

Lemma 3.1. *Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V. A pair (su, sv) is nonadmissible if and only if there exists a bi-set of V blocking (su, sv).*

Proof. The sufficiency is clear. Let us see the necessity. Since (su, sv) is nonadmissible, there exists a nontrivial bi-set X of V, which violates 4 in $H_{u,v}$. Since $f_H^b(\mathsf{X}) \geq 2k$, either $d_{H_{u,v}}^b(\mathsf{X}) = d_H^b(\mathsf{X}) - 2$, that is, $u, v \in X_I$ and $f_H^b(\mathsf{X}) \leq 2k + 1$, or $d_{H_{u,v}}^b(\mathsf{X}) = d_H^b(\mathsf{X}) - 1$, that is, $u \in X_I$ and $\{v\} = w^b(\mathsf{X})$ (or $v \in X_I$ and $\{u\} = w^b(\mathsf{X})$), and $f_H^b(\mathsf{X}) \leq 2k$. ■

Note that if a bi-set X blocks a pair (su, sv), then after any sequence of splitting-off of admissible pairs not containing su nor sv, X still blocks (su, sv). Hence, a nonadmissible pair in H remains nonadmissible in any graph arising from H by a sequence of splitting-off of admissible pairs. Note also that, by 8 and 9, for a blocking bi-set X,

$$|w^b(\mathsf{X})| \leq 1, \tag{10}$$

$$f_H^b(\mathsf{X}) - 2k \leq d_H(s, X_I) - 1. \tag{11}$$

Proposition 3.2. *Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V and X either a tight bi-set of V such that X_I contains a neighbor of s or a blocking bi-set. Then, $N_H(s)$ is not contained in X_O.*

Proof. By assumption, X satisfies 11 and X^c is a nontrivial bi-set of V, and hence, 6 and $(2, k)$-connectivity of H in V provide that $d_H(s, X_O^c) \geq 1$ and we are done. ■

Proposition 3.3. *Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V with $d_H(s)$ even.*

- **(i)** *For a nontrivial bi-set X of V, $d_H(s, X_I) \leq \lfloor \frac{1}{2}(d_H(s) - d_H(s, w^b(\mathsf{X})) + f_H^b(\mathsf{X}) - 2k) \rfloor$.*
- **(ii)** *If X is a dangerous bi-set of V, then $d_H(s, X_I) \leq \frac{1}{2} d_H(s)$.*
- **(iii)** *If X is a critical bi-set of V, then $d_H(s, X_I) \leq \frac{1}{2} d_H(s) - 1$.*
- **(iv)** *If X and Y are critical bi-sets of V with the same wall w and $d_H(s, w)$ is odd, then $N_H(s)$ is not contained in $X_O \cup Y_O$.*

Proof. (i) follows from $d_H(s, X_O^c) = d_H(s) - d_H(s, w^b(\mathsf{X})) - d_H(s, X_I)$, 6, $(2, k)$-connectivity of H in V and since $d_H(s, X_I)$ is integer.
(ii) and (iii) follow from (i) and from the conditions that X is dangerous (resp. X is critical) and $d_H(s)$ is even.
(iv) follows from $w^b(\mathsf{X}) = \{w\} = w^b(\mathsf{Y})$, (i), and from the facts that X and Y are critical and $d_H(s) - d_H(s, w)$ is odd, as follows: $d_H(s, X_O \cup Y_O) \leq d_H(s, X_I) + d_H(s, Y_I) + d_H(s, w) < \frac{1}{2}(d_H(s) - d_H(s, w)) + \frac{1}{2}(d_H(s) - d_H(s, w)) + d_H(s, w) = d_H(s)$. ■

We will heavily rely on the following lemma whose proof is quite technical.

Lemma 3.4. *Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V with $d_H(s)$ even. Let X be a maximal blocking bi-set for a pair (su, sv) with $u \in X_I$. Let $z \in N_H(s) \backslash X_I$ and Y a blocking bi-set for the pair (su, sz). Then, $w^b(\mathsf{X})$ and $w^b(\mathsf{Y})$ coincide and are a singleton.*

Proof. Note that

$$\text{if } \mathsf{Y} \text{ is dangerous or } w^{\mathrm{b}}(\mathsf{Y}) \cap X_{\mathrm{I}} = \emptyset, \ \text{ then } u \in X_{\mathrm{I}} \cap Y_{\mathrm{I}} \cap N_H(s). \tag{12}$$

We prove the lemma through the following claims.

Claim 3.5. The bi-sets X and Y satisfy the following:

(a) If $w^{\mathrm{b}}(\mathsf{Y}) \cap X_{\mathrm{I}}$ is empty, then $f_H^{\mathrm{b}}(\mathsf{X} \sqcap \mathsf{Y}) \geq 2k$.
(b) If $w^{\mathrm{b}}(\overline{\mathsf{X}} \sqcap \mathsf{Y})$ is empty, then $f_H^{\mathrm{b}}(\overline{\mathsf{X}} \sqcap \mathsf{Y}) \geq 2k$.
(c) If $w^{\mathrm{b}}(\mathsf{X} \sqcap \overline{\mathsf{Y}})$ is empty, then $f_H^{\mathrm{b}}(\mathsf{X} \sqcap \overline{\mathsf{Y}}) \geq 2k$.
(d) If $w^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y})$ is empty, then $f_H^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y}) \geq 2k + 2$.

Proof. By the $(2, k)$-connectivity of H in V and since none of X_{O} and Y_{O} contains V, proving (a), (b), or (c) reduces to check that the inner-set of the bi-set resulting from the intersection is nonempty.

(a) By $w^{\mathrm{b}}(\mathsf{Y}) \cap X_{\mathrm{I}} = \emptyset$ and $u \in X_{\mathrm{I}} \cap Y_{\mathrm{O}} = X_{\mathrm{I}} \cap Y_{\mathrm{I}}$.
(b) By $w^{\mathrm{b}}(\overline{\mathsf{X}} \sqcap \mathsf{Y}) = \emptyset$ and $z \in \overline{X_{\mathrm{I}}} \cap Y_{\mathrm{O}} = (\overline{X_{\mathrm{O}}} \cap Y_{\mathrm{I}}) \cup w^{\mathrm{b}}(\overline{\mathsf{X}} \sqcap \mathsf{Y}) = \overline{X_{\mathrm{O}}} \cap Y_{\mathrm{I}}$.
(c) If $X_{\mathrm{I}} \cap \overline{Y_{\mathrm{O}}} = \emptyset$, then $X_{\mathrm{O}} \cap \overline{Y_{\mathrm{I}}} = w^{\mathrm{b}}(\mathsf{X} \sqcap \overline{\mathsf{Y}}) \cup (X_{\mathrm{I}} \cap \overline{Y_{\mathrm{O}}}) = \emptyset$, that is, $X_{\mathrm{O}} \subseteq Y_{\mathrm{I}}$. So, by 8 or 9, $u, v \in Y_{\mathrm{I}}$, thus Y blocks (su, sv). Since $z \in Y_{\mathrm{O}} \backslash X_{\mathrm{I}}$, we have either $z \in Y_{\mathrm{I}} \backslash X_{\mathrm{I}}$ or $z \in Y_{\mathrm{O}} \backslash Y_{\mathrm{I}}$. In the first case, $X_{\mathrm{I}} \subsetneq Y_{\mathrm{I}}$ and in the latter case, $X_{\mathrm{O}} \subsetneq Y_{\mathrm{O}}$. It follows that Y strictly contains X that contradicts the maximality of X.
(d) Suppose that $w^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y}) = \emptyset$. Then, $u, v \in X_{\mathrm{O}} \cup Y_{\mathrm{O}} = X_{\mathrm{I}} \cup Y_{\mathrm{I}}$. Thus, by $z \in Y_{\mathrm{O}} \backslash X_{\mathrm{I}} = Y_{\mathrm{I}} \backslash X_{\mathrm{I}}$, $\mathsf{X} \sqcup \mathsf{Y}$ strictly contains X and $X_{\mathrm{I}} \cup Y_{\mathrm{I}} \neq \emptyset$. Since X and Y are blocking bi-sets, by Proposition 3.3 and 12, we have $d_H(s, X_{\mathrm{I}} \cup Y_{\mathrm{I}}) = d_H(s, X_{\mathrm{I}}) + d_H(s, Y_{\mathrm{I}}) - d_H(s, X_{\mathrm{I}} \cap Y_{\mathrm{I}}) \leq \frac{1}{2} d_H(s) + \frac{1}{2} d_H(s) - 1 = d_H(s) - 1$, that is, there exists a neighbor of s in $V \backslash (X_{\mathrm{I}} \cup Y_{\mathrm{I}})$, and hence $V \neq X_{\mathrm{O}} \cup Y_{\mathrm{O}}$. It follows that $\mathsf{X} \sqcup \mathsf{Y}$ is a nontrivial bi-set of V containing u and v in its inner-set. Hence, by the maximality of X, $\mathsf{X} \sqcup \mathsf{Y}$ does not block (su, sv), and then, $f_H^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y}) \geq 2k + 2$. ■

Claim 3.6. At least one of $w^{\mathrm{b}}(\mathsf{X})$ and $w^{\mathrm{b}}(\mathsf{Y})$ is not empty.

Proof. Suppose that $w^{\mathrm{b}}(\mathsf{X}) = \emptyset = w^{\mathrm{b}}(\mathsf{Y})$. Then, the conditions of Claim 3.5 are satisfied and $f_H^{\mathrm{b}}(\mathsf{X} \sqcap \mathsf{Y}) = d_H(X_{\mathrm{I}} \cap Y_{\mathrm{I}})$, $f_H^{\mathrm{b}}(\overline{\mathsf{X}} \sqcap \mathsf{Y}) = d_H(Y_{\mathrm{I}} \backslash X_{\mathrm{I}})$, $f_H^{\mathrm{b}}(\mathsf{X} \sqcap \overline{\mathsf{Y}}) = d_H(X_{\mathrm{I}} \backslash Y_{\mathrm{I}})$, and $f_H^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y}) = d_H(X_{\mathrm{I}} \cup Y_{\mathrm{I}})$. Since X and Y are blocking bi-sets, by 12 and Claim 3.5, we have $4k + 2 = (2k + 1) + (2k + 1) \geq d_H(X_{\mathrm{I}}) + d_H(Y_{\mathrm{I}}) = d_H(\overline{X_{\mathrm{I}} \cup Y_{\mathrm{I}}}, X_{\mathrm{I}} \cap Y_{\mathrm{I}}) + d_H(X_{\mathrm{I}} \backslash Y_{\mathrm{I}}, Y_{\mathrm{I}} \backslash X_{\mathrm{I}}) + \frac{1}{2}(d_H(X_{\mathrm{I}} \cap Y_{\mathrm{I}}) + d_H(X_{\mathrm{I}} \backslash Y_{\mathrm{I}}) + d_H(Y_{\mathrm{I}} \backslash X_{\mathrm{I}}) + d_H(X_{\mathrm{I}} \cup Y_{\mathrm{I}})) \geq 1 + 0 + \frac{1}{2}(2k + 2k + 2k + (2k + 2)) = 4k + 2$. Thus, equality holds everywhere, in particular, $d_H(X_{\mathrm{I}})$ is odd and $d_H(X_{\mathrm{I}} \cap Y_{\mathrm{I}})$ and $d_H(X_{\mathrm{I}} \backslash Y_{\mathrm{I}})$ are even. This contradicts $d_H(X_{\mathrm{I}}) = d_H(X_{\mathrm{I}} \cap Y_{\mathrm{I}}) + d_H(X_{\mathrm{I}} \backslash Y_{\mathrm{I}}) - 2 d_H(X_{\mathrm{I}} \cap Y_{\mathrm{I}}, X_{\mathrm{I}} \backslash Y_{\mathrm{I}})$. ■

Claim 3.7. None of $w^{\mathrm{b}}(\mathsf{X})$ and $w^{\mathrm{b}}(\mathsf{Y})$ is empty.

Proof. By contradiction suppose that the claim is false. Then, by Claim 3.6 and 10, one of X and Y has an empty wall, call it A, and the other one has a wall of size one, call it B. Suppose that $w^{\mathrm{b}}(\mathsf{B}) \cap A_{\mathrm{I}} = \emptyset$. By Claim 3.5(a), $f_H^{\mathrm{b}}(\mathsf{A} \sqcap \mathsf{B}) \geq 2k$. If $\mathsf{A} = \mathsf{X}$, then, by Claim 3.5(c), $f_H^{\mathrm{b}}(\mathsf{X} \sqcap \overline{\mathsf{Y}}) \geq 2k$, otherwise $\mathsf{A} = \mathsf{Y}$ and then, by Claim 3.5(b), $f_H^{\mathrm{b}}(\overline{\mathsf{X}} \sqcap \mathsf{Y}) \geq 2k$; in both cases, $f_H^{\mathrm{b}}(\mathsf{A} \sqcap \overline{\mathsf{B}}) \geq 2k$. Since B is a blocking bi-set and $w^{\mathrm{b}}(\mathsf{B})$ is a singleton, we have, by 11,

$$d_H^{\mathrm{b}}(\mathsf{B}) - d_H(s, B_{\mathrm{I}}) = (f_H^{\mathrm{b}}(\mathsf{B}) - k|w^{\mathrm{b}}(\mathsf{B})|) - d_H(s, B_{\mathrm{I}}) \leq k - 1. \qquad (\bigstar)$$

Then, by $w^{\mathrm{b}}(\mathsf{A}) = \emptyset$, 5 applied for $\mathsf{A} \sqcap \mathsf{B}$ and $\mathsf{A} \sqcap \overline{\mathsf{B}}$, since the edges between $A_{\mathrm{I}} \backslash B_{\mathrm{I}}$ and $A_{\mathrm{I}} \cap B_{\mathrm{I}}$ enter B but not s, A is a blocking bi-set and by $\bigstar$, we have the following contradiction: $2k + 2k \leq f_H^{\mathrm{b}}(\mathsf{A} \sqcap \mathsf{B}) + f_H^{\mathrm{b}}(\mathsf{A} \sqcap \overline{\mathsf{B}}) = f_H^{\mathrm{b}}(\mathsf{A}) + 2d_H(A_{\mathrm{I}} \backslash B_{\mathrm{I}}, A_{\mathrm{I}} \cap B_{\mathrm{I}}) \leq f_H^{\mathrm{b}}(\mathsf{A}) + 2(d_H^{\mathrm{b}}(\mathsf{B}) - d_H(s, B_{\mathrm{I}})) \leq (2k + 1) + 2(k - 1)$.

From now on we suppose that $w^{\mathrm{b}}(\mathsf{B}) \cap A_{\mathrm{I}} \neq \emptyset$. Since $w^{\mathrm{b}}(\mathsf{B})$ is a singleton, it follows that $w^{\mathrm{b}}(\mathsf{B}) \cap \overline{A_{\mathrm{I}}} = \emptyset$. Then, by Claim 3.5(d), $f_H^{\mathrm{b}}(\mathsf{A} \sqcup \mathsf{B}) \geq 2k + 2$. If $\mathsf{A} = \mathsf{X}$, then, by Claim 3.5(b), $f_H^{\mathrm{b}}(\overline{\mathsf{X}} \sqcap \mathsf{Y}) \geq 2k$, otherwise $\mathsf{A} = \mathsf{Y}$ and then, by Claim 3.5(c), $f_H^{\mathrm{b}}(\mathsf{X} \sqcap \overline{\mathsf{Y}}) \geq 2k$; in both cases, $f_H^{\mathrm{b}}(\overline{\mathsf{A}} \sqcap \mathsf{B}) \geq 2k$. Recall that B is a blocking bi-set and $w^{\mathrm{b}}(\mathsf{B})$ is a singleton. Then, by 12, we have

$$d_H^{\mathrm{b}}(\mathsf{B}) - d_H(s, A_{\mathrm{I}} \cap B_{\mathrm{I}}) = (f_H^{\mathrm{b}}(\mathsf{B}) - k|w^{\mathrm{b}}(\mathsf{B})|) - d_H(s, A_{\mathrm{I}} \cap B_{\mathrm{I}}) \leq k. \qquad (\bigstar\bigstar)$$

Then, by the symmetry of f_H^{b}, by 5 applied for $\mathsf{A} \sqcup \mathsf{B}$ and $\mathsf{A} \sqcup \overline{\mathsf{B}}$, since the edges between $\overline{A_{\mathrm{I}} \cup B_{\mathrm{I}}}$ and $B_{\mathrm{I}} \backslash A_{\mathrm{I}}$ enter B but not $A_{\mathrm{I}} \cap B_{\mathrm{I}}$, since A is a blocking bi-set and by $\bigstar\bigstar$, we have the following contradiction: $(2k + 2) + 2k \leq f_H^{\mathrm{b}}(\mathsf{A} \sqcup \mathsf{B}) + f_H^{\mathrm{b}}(\overline{\mathsf{A}} \sqcap \mathsf{B}) = f_H^{\mathrm{b}}(\mathsf{A} \sqcup \mathsf{B}) + f_H^{\mathrm{b}}(\mathsf{A} \sqcup \overline{\mathsf{B}}) = f_H^{\mathrm{b}}(\mathsf{A}) + 2d_H(\overline{A_{\mathrm{I}} \cup B_{\mathrm{I}}}, B_{\mathrm{I}} \backslash A_{\mathrm{I}}) \leq f_H^{\mathrm{b}}(\mathsf{A}) + 2(d_H^{\mathrm{b}}(\mathsf{B}) - d_H(s, A_{\mathrm{I}} \cap B_{\mathrm{I}})) \leq (2k + 1) + 2k$. ■

Claim 3.8. The bi-sets X and Y have the same wall.

Proof. By Claim 3.7 and 10, both $w^{\mathrm{b}}(\mathsf{X})$ and $w^{\mathrm{b}}(\mathsf{Y})$ are singletons. For a contradiction suppose that $w^{\mathrm{b}}(\mathsf{X}) \neq w^{\mathrm{b}}(\mathsf{Y})$, that is, $w^{\mathrm{b}}(\mathsf{X}) \cap w^{\mathrm{b}}(\mathsf{Y}) = \emptyset$. We have three cases.

Case 1. $|w^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y})| = 2$. Then, $w^{\mathrm{b}}(\mathsf{X} \sqcap \mathsf{Y}) = \emptyset$. By Claim 3.5(a), $f_H^{\mathrm{b}}(\mathsf{X} \sqcap \mathsf{Y}) \geq 2k$. Hence, by 7, 11 applied for X, and by the facts that Y is a blocking bi-set and if Y is dangerous, then $z \in (Y_{\mathrm{I}} \backslash X_{\mathrm{I}}) \cap N_H(s)$, we have the following contradiction: $d_H^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y}) \leq (f_H^{\mathrm{b}}(\mathsf{X}) - 2k) + (f_H^{\mathrm{b}}(\mathsf{Y}) - 2k) < d_H(s, X_{\mathrm{I}}) + d_H(s, Y_{\mathrm{I}} \backslash X_{\mathrm{I}}) = d_H(s, X_{\mathrm{I}} \cup Y_{\mathrm{I}}) \leq d_H^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y})$.

Case 2. $|w^{\mathrm{b}}(\mathsf{X} \sqcup \mathsf{Y})| = 1$. Then, we may call X and Y as A and B such that $w^{\mathrm{b}}(\mathsf{A} \sqcap \overline{\mathsf{B}}) = \emptyset$ and $|w^{\mathrm{b}}(\mathsf{A} \sqcup \overline{\mathsf{B}})| = 2$. If $\mathsf{A} = \mathsf{X}$, then, by Claim 3.5(c), $f_H^{\mathrm{b}}(\mathsf{X} \sqcap \overline{\mathsf{Y}}) \geq 2k$, otherwise $\mathsf{A} = \mathsf{Y}$ and then, by Claim 3.5(b), $f_H^{\mathrm{b}}(\overline{\mathsf{X}} \sqcap \mathsf{Y}) \geq 2k$; in both cases, $f_H^{\mathrm{b}}(\mathsf{A} \sqcap \overline{\mathsf{B}}) \geq 2k$. Since A is a blocking bi-set, we have, by 12, $f_H^{\mathrm{b}}(\mathsf{A}) - 2k \leq d_H(s, A_{\mathrm{I}} \cap B_{\mathrm{I}})$. By symmetry of f_H^{b} and 11, $f_H^{\mathrm{b}}(\overline{\mathsf{B}}) - 2k =$

$f_H^b(\mathsf{B}) - 2k < d_H(s, B_I)$. Then, 7 applied for A and $\bar{\mathsf{B}}$ contradicts the following: $(f_H^b(\mathsf{A}) - 2k) + (f_H^b(\bar{\mathsf{B}}) - 2k) < d_H(s, A_I \cap B_I) + d_H(s, B_I) \leq d_H(\overline{A_O} \cap \overline{B_I}, A_I \cap B_O) + (d_H^b(\mathsf{A} \sqcup \bar{\mathsf{B}}) + d_H(B_I \cap A_O, \overline{B_O} \cap \overline{A_I}))$.

Case 3. $|w^b(\mathsf{X} \sqcup \mathsf{Y})| = 0$. Then, $|w^b(\mathsf{X} \sqcap \mathsf{Y})| = 2$. By Claim 3.5(d), since X is a blocking bi-set, f_H^b is submodular, Y is a blocking bi-set and by 12, we have the following contradiction: $1 = (2k + 2) - (2k + 1) \leq f_H^b(\mathsf{X} \sqcup \mathsf{Y}) - f_H^b(\mathsf{X}) \leq f_H^b(\mathsf{Y}) - f_H^b(\mathsf{X} \sqcap \mathsf{Y}) \leq (2k + d_H(s, X_I \cap Y_I)) - (d_H(s, X_I \cap Y_I) + k|w^b(\mathsf{X} \sqcap \mathsf{Y})|) = 0$.

Claims 3.7 and 3.8 and 10 prove Lemma 3.4. ■

Proposition 3.9. *Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V with $d_H(s) \geq 4$ even. If there exists no admissible pair incident to s, then $d_H(s, u) < \frac{1}{2} d_H(s)$ for each neighbor u of s.*

Proof. Since any pair incident to s is nonadmissible, by Lemma 3.1, there exists a bi-set that blocks it. By contradiction, suppose that $d_H(s, u) \geq \frac{1}{2} d_H(s) \geq 2$ for some $u \in N_H(s)$. Let X be a maximal blocking bi-set for (su, su). Clearly, we have $u \in X_I$. By Proposition 3.2, there exists a vertex v in $N_H(s) \backslash X_O$. Let Y be a blocking bi-set for the pair (su, sv). By Lemma 3.4, X and Y have the same wall and thus $u, v \in Y_O \backslash w^b(\mathsf{X}) = Y_I$. This gives $d_H(s, Y_I) \geq d_H(s, u) + d_H(s, v) \geq \frac{d_H(s)}{2} + 1$ that contradicts Proposition 3.3. ■

3.2 | Obstacles

Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V such that $d_H(s)$ is even. We extend the definition of 2-obstacle (defined in Section 1.4) as follows. The pair $(t, \mathcal{C})$ is called a *t-star k-obstacle* at s (in short, an *obstacle*) if

$$t \text{ is a neighbor of } s \text{ with } d_H(s, t) \text{ odd}, \tag{13}$$

$$\mathcal{C} \text{ is a collection of critical bi-sets}, \tag{14}$$

$$\text{each element of } \mathcal{C} \text{ has wall } \{t\}, \tag{15}$$

$$\text{the elements of } \mathcal{C} \text{ are pairwise innerly disjoint}, \tag{16}$$

$$N_H(s) \backslash \{t\} \subseteq V_I(\mathcal{C}). \tag{17}$$

Note that a t-star k-obstacle for $k = 2$ is a 2-obstacle. Note also that if $(t, \mathcal{C})$ is an obstacle at s, then, by Lemma 3.1, no pair (st, su) with $u \in N_H(s) \backslash \{t\}$ is admissible. Some basic properties of obstacles are proven in the following proposition.

Proposition 3.10. *Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V with $d_H(s)$ even and $(t, \mathcal{C})$ an obstacle at s. Then,*

$$|\mathcal{C}| \geq 3, \tag{18}$$

$$H - st \text{ is } (2, k)\text{-connected in } V. \tag{19}$$

Proof. 18: By 17, 13 and $d_H(s)$ even, $|\mathcal{C}| \geq 1$. Let X and Y be two (not necessarily distinct) elements of $\mathcal{C}$. By 14, 15, 13, and Proposition 3.3(iv), $N_H(s)\backslash(X_O \cup Y_O)$ is nonempty. Thus, by 17, there exists an element in $\mathcal{C}\backslash\{\mathsf{X}, \mathsf{Y}\}$.

19: Suppose that $H - st$ is not $(2, k)$-connected in V, that is, by $(2, k)$-connectivity of H, there exists in H a nontrivial tight bi-set X of V such that $t \in X_I$. By 14, every $\mathsf{Y} \in \mathcal{C}$ is a tight bi-set of V. Hence, by Proposition 2.1(b) and $d_H(X_I \cap Y_O, \overline{X_O} \cap \overline{Y_I}) \geq d_H(s, t) \geq 1$, $\overline{\mathsf{X}} \sqcap \mathsf{Y}$ or $\mathsf{X} \sqcap \overline{\mathsf{Y}}$ is trivial, that is, since X and Y are nontrivial, $Y_I \subseteq X_O$ or $X_I \subseteq Y_O$. If $Y_I \subseteq X_O$ for all $\mathsf{Y} \in \mathcal{C}$, then, by 17 and $t \in X_I$, we have $N_H(s) \subseteq X_O$ and, by the tightness of X, this contradicts Proposition 3.2. So there exists $\mathsf{Y}^* \in \mathcal{C}$ such that $X_I \subseteq Y^*_O$. For all $\mathsf{Y} \in \mathcal{C}$, since H is $(2, k)$-connected in V and Y is critical, $d_H(t, Y_I) = d_H(Y_I) - (f_H^{\mathrm{b}}(\mathsf{Y}) - k|w^{\mathrm{b}}(\mathsf{Y})|) \geq 2k - (2k - k) = k$. By tightness of X, $t \in X_I$, 13, 16, 18, and $X_I \subseteq Y^*_O$, we have the following contradiction, $2k - k|w^{\mathrm{b}}(\mathsf{X})| = f_H^{\mathrm{b}}(\mathsf{X}) - k|w^{\mathrm{b}}(\mathsf{X})| = d_H^{\mathrm{b}}(\mathsf{X}) = d_H(X_I) - d_H(X_I, w^{\mathrm{b}}(\mathsf{X})) \geq d_H(t, s) + \sum_{\mathsf{Y} \in \mathcal{C}\backslash\{Y^*\}, w^{\mathrm{b}}(\mathsf{X}) \cap Y_I = \emptyset} d_H(t, Y_I) \geq 1 + (2 - |w^{\mathrm{b}}(\mathsf{X})|)k$. ■

The following lemma shows that to find an obstacle one does not have to focus on the disjointness of the inner-sets.

Lemma 3.11. *Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V with $d_H(s)$ even. If there exists a pair $(t, \mathcal{F})$ satisfying 13-15 and 17, then there exists a t-star k-obstacle at s.*

Proof. The proof applies the uncrossing method. Choose a pair $(t, \mathcal{C})$ satisfying 13-15 and 17 such that $\sum_{\mathsf{X} \in \mathcal{C}} |X_I|$ is minimum. Suppose there exist two distinct elements X and Y in $\mathcal{C}$ such that $X_I \cap Y_I \neq \emptyset$, that is, $\mathsf{X} \sqcap \mathsf{Y}$ is a nontrivial bi-set of V. By choice of $\mathcal{C}$, none of the bi-sets X and Y contains the other. Hence, $\mathsf{X} \sqcap \overline{\mathsf{Y}}$ and $\overline{\mathsf{X}} \sqcap \mathsf{Y}$ are nontrivial bi-sets of V. By 13-15, we can apply Proposition 3.3(iv), and we get that $\mathsf{X} \sqcup \mathsf{Y}$ is a nontrivial bi-set of V. Note that critical bi-sets are tight nontrivial bi-sets of V. Hence, by Proposition 2.1(a) and (b), $\mathsf{X} \sqcap \mathsf{Y}$, $\mathsf{X} \sqcap \overline{\mathsf{Y}}$, and $\overline{\mathsf{X}} \sqcap \mathsf{Y}$ are tight. The bi-sets among them, which contain a neighbor of s, are critical bi-sets with wall t. Hence, they can replace X and Y in $\mathcal{C}$ contradicting the minimality of $\sum_{\mathsf{X} \in \mathcal{C}} |X_I|$. ■

4 | RESULTS

4.1 | A new splitting-off theorem

The first result of this section shows the existence of an obstacle when no pair of edges incident to the special vertex is admissible.

Theorem 4.1. *Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V with $d_H(s) \geq 2$ even and $k \geq 2$. If there exists no admissible pair at s, then $d_H(s) = 4$ and there exists an obstacle at s.*

Proof. Suppose that there exists no admissible splitting-off at s.

Claim 4.2. There exists a vertex t and a family $\mathcal{F}$ of dangerous blocking bi-sets such that 15 holds for $\mathcal{F}$ and every pair of edges incident to s but not to t is blocked by an element of $\mathcal{F}$.

Proof. By Lemma 3.1, for each pair of edges incident to s, there exists a bi-set that blocks it. Let X be a maximal blocking bi-set for a pair (su, sv) with $u \in X_I$. By Proposition 3.2, there exists a neighbor z of s in $\overline{X_O}$. Let Y be a maximal blocking bi-set for the pair (su, sz). By Lemma 3.4, the wall of X and the wall of Y coincide and are reduced to a singleton, say $\{t\}$. By $u \in X_I$ and $z \in \overline{X_O}$, t is different from u and from z. Thus, Y is a dangerous blocking bi-set.

For the same reasons, every maximal blocking bi-set for a pair (sa, sb) with $a \in Y_I$ and $b \in \overline{Y_O}$ is a dangerous blocking bi-set with wall $\{t\}$. By repeating this argument once more, we have that every pair (sa, sb) with $a, b \notin \{t\}$ is blocked by a dangerous blocking bi-set with wall $\{t\}$. This proves the claim. ∎

Let t and $\mathcal{F}$ be, respectively, the vertex and the family that exist by Claim 4.2.

Claim 4.3. The degree of s in $H' = H - t$ is 3.

Proof. By $(2, k)$-connectivity in V of H, H' is k-edge-connected in $V' = V - t$. For every pair (su', sv') of edges in H', by the definition of $\mathcal{F}$, there exists $\mathsf{Z} \in \mathcal{F}$ for u', v'. Then, by $w^b(\mathsf{Z}) = \{t\}$ and since Z is a dangerous bi-set, $d_{H'}(Z_I) = d_H^b(\mathsf{Z}) = f_H^b(\mathsf{Z}) - k|w^b(\mathsf{Z})| \leq k + 1$, that is, by $u', v' \in Z_I$, the splitting-off the pair (su', sv') destroys the k-edge-connectivity in V' of H'. Hence, by $k \geq 2$ and Theorem 1.8, the claim follows. ∎

By $d_H(s)$ even and Claim 4.3 and Proposition 3.9, $d_H(s, t)$ is odd and smaller than $\frac{1}{2}d_H(s)$, that is, $d_H(s, t) = 1$ and $d_H(s) = 4$. Hence, by Proposition 3.2, the inner-set of each element of $\mathcal{F}$ contains exactly two neighbors of s and $|\mathcal{F}| = 3$. So, for $\mathsf{X} \in \mathcal{F}$, $\mathsf{X}^c = (X_I{}^c, X_O^c)$ is a nontrivial bi-set of V and X_O^c contains exactly one neighbor of s, say x. By 6, we have $f_H^b(\mathsf{X}^c) = f_H^b(\mathsf{X}) - d_H(s, X_I) + d_H(s, X_O^c) \leq 2k + 1 - 2 + 1 = 2k$ thus X^c is a critical bi-set blocking (st, sx). So $(t, \mathcal{F}^c) = (t, \{\mathsf{X}^c : \mathsf{X} \in \mathcal{F}\})$ satisfies 13-15 and 17. The obstacle at s is obtained by applying Lemma 3.11 on $(t, \mathcal{F}^c)$. ∎

The following lemma concerns the case when an obstacle occurs after an admissible splitting-off.

Lemma 4.4. *Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V with $d_H(s) \geq 6$ even, (su, sv) an admissible pair in H and (t, C) an obstacle at s in $H_{u,v}$.*

- **(a)** *If $t \in \{u, v\}$, then $d_H(s, t) \geq 2$ and (st, st) is admissible in H.*
- **(b)** *If $t \notin \{u, v\}$, then either there exists a t-star k-obstacle at s in H or there exists no obstacle at s in $H_{t,z}$ for some admissible pair (st, sz) in H.*

Proof.

(a) If the vertices t, u, and v coincide, then there is nothing to prove. So we assume that $t = v$ and $t \neq u$. By 13 in $H_{u,v}$, $d_H(s, t) = d_{H_{u,v}}(s, t) + 1 \geq 2$. For a contradiction, suppose that (st, st) is nonadmissible in H, thus, by Lemma 3.1, there exists a maximal blocking bi-set X for this pair in H. Let Y be an element of $\mathcal{C}$, if possible the one whose inner-set contains u. Since $t = v \in X_{\mathrm{I}}$, X is blocking bi-set in H, Y is critical bi-set in $H_{u,v}$ and by Proposition 3.3, we have $d_{H_{u,v}}(s, X_{\mathrm{I}} \cup Y_{\mathrm{I}}) \leq d_{H_{u,v}}(s, X_{\mathrm{I}}) + d_{H_{u,v}}(s, Y_{\mathrm{I}}) \leq (d_H(s, X_{\mathrm{I}}) - 1) + d_{H_{u,v}}(s, Y_{\mathrm{I}}) \leq (\frac{1}{2}d_H(s) - 1) + (\frac{1}{2}d_{H_{u,v}}(s) - 1) = d_{H_{u,v}}(s) -1$. So, by 17 and $t \in X_{\mathrm{I}}$, there exists a vertex $z \in N_{H_{u,v}}(s) \backslash (X_{\mathrm{I}} \cup Y_{\mathrm{I}})$ contained in the inner-set of an element Z of $\mathcal{C} \backslash \mathsf{Y}$. Since none of u or $v = t$ belongs to Z_{I}, $f_H^{\mathrm{b}}(\mathsf{Z}) = f_{H_{u,v}}^{\mathrm{b}}(\mathsf{Z})$, that is, Z blocks the pair (st, sz) in H. Since $z \notin X_{\mathrm{I}}$, by Lemma 3.4, we have $w^{\mathrm{b}}(\mathsf{X}) = w^{\mathrm{b}}(\mathsf{Z}) = \{t\} \in X_{\mathrm{I}}$, a contradiction that completes the proof of (a).

(b) Suppose that $t \notin \{u, v\}$.

Claim 4.5. If st belongs to no admissible pair in H, then there exists a t-star k-obstacle in H.

Proof. By $t \notin \{u, v\}$ and 13, $d_H(s, t) = d_{H_{u,v}}(s, t)$ is odd, thus it remains to construct a collection $\mathcal{F}$ of critical bi-sets satisfying 15-17. By Lemma 3.11, it suffices to find one satisfying 15 and 17.

Let $\mathcal{F}_0 := \{\mathsf{X}' \in \mathcal{C} : |X'_{\mathrm{I}} \cap \{u, v\}| < 2\}$. Note that either $\mathcal{F}_0 = \mathcal{C}$ or $\mathcal{F}_0 = \mathcal{C} \backslash Y$ for some $\mathsf{Y} \in \mathcal{C}$ with $\{u, v\} \subseteq \mathsf{Y}$. By 14 and 15 for $\mathcal{C}$ in $H_{u,v}$, $\mathcal{F}_0$ is a collection of critical bi-sets in H satisfying 15. Suppose $\mathcal{F}_0$ does not satisfy 17, that is, there exist some $z \in N_H(s) \backslash (V_{\mathrm{I}}(\mathcal{C}) \cup \{t\})$. For any such z, since st belongs to no admissible pair, by Lemma 3.1, there exists a maximal blocking bi-set X^z in H for the pair (st, sz). We prove that $w^{\mathrm{b}}(\mathsf{X}^z) = \{t\}$ and then X^z is critical and hence $\mathcal{F} := \mathcal{F}_0 \cup \{\mathsf{X}^z \colon z \in N_H(s) \backslash (V_{\mathrm{I}}(\mathcal{C}) \cup \{t\})\}$ is the required collection.

Assume, by contradiction, that $\{t\} \neq w^{\mathrm{b}}(\mathsf{X}^z)$ for some z, then, by 10, $t \in X^z_{\mathrm{O}} \backslash w^{\mathrm{b}}(\mathsf{X}^z) = X^z_{\mathrm{I}}$. We have $N_H(s) \cap V_{\mathrm{I}}(\mathcal{C}) \subseteq X^z_{\mathrm{I}}$ otherwise, there exists $\mathsf{Z} \in \mathcal{C}$ such that $(N_H(s) \cap Z_{\mathrm{I}}) \backslash X^z_{\mathrm{I}} \neq \emptyset$, thus by Lemma 3.4, we have $w^{\mathrm{b}}(\mathsf{X}^z) = w^{\mathrm{b}}(\mathsf{Z}) = \{t\} \subseteq X^z_{\mathrm{I}}$, a contradiction. If $\mathcal{F}_0 = \mathcal{C}$ then, by Proposition 3.3 and $N_H(s) \cap V_{\mathrm{I}}(\mathcal{C}) \subseteq X^z_{\mathrm{I}}$, we have $\frac{1}{2}d_H(s) \geq d_H(s, X^z_{\mathrm{I}}) \geq d_H(s) - 2$ that contradicts $d_H(s) \geq 6$. Otherwise $\mathcal{F}_0 = \mathcal{C} \backslash \mathsf{Y}$ and $\{u, v, z\} \subseteq Y_{\mathrm{I}}$. Note that if X^z is dangerous, then $z \in X^z_{\mathrm{I}} \cap Y_{\mathrm{I}}$. Hence, by $N_H(s) \subseteq X^z_{\mathrm{I}} \cup Y_{\mathrm{I}}$ and Proposition 3.3, the following contradiction completes the proof of Claim 4.5: $d_H(s) = d_H(s, Y_{\mathrm{I}}) + d_H(s, X^z_{\mathrm{I}}) - d_H(s, X^z_{\mathrm{I}} \cap Y_{\mathrm{I}}) = (d_{H_{u,v}}(s, Y_{\mathrm{I}}) + 2) + (d_H(s, X^z_{\mathrm{I}}) - d_H(s, X^z_{\mathrm{I}} \cap Y_{\mathrm{I}})) \leq (\frac{1}{2}(d_H(s) - 2) - 1) + 2 + (\frac{1}{2}d_H(s) - 1) = d_H(s) - 1$. ∎

Claim 4.6. If (st, sz) is an admissible pair in H and $(t', \mathcal{C}')$ is a t'-star k-obstacle in $H_{t,z}$, then $t = t'$.

Proof. By contradiction, assume that there exist an admissible pair (st, sz) in H and an obstacle $(t', \mathcal{C}')$ in $H_{t,z}$ such that $t \neq t'$. If t' belongs to an element of $\mathcal{C}$, then denote X this element and let $\mathsf{X} = (\emptyset, \emptyset)$ otherwise. If t belongs to an element of $\mathcal{C}'$, then denote X' this element and let $\mathsf{X}' = (\emptyset, \emptyset)$ otherwise. First, we prove that

$$(V_{\rm I}(C)\backslash X_{\rm I}) \cap (V_{\rm I}(C')\backslash X'_{\rm I}) = \emptyset. \tag{20}$$

For a contradiction, suppose that there exists $\mathsf{Y} \in C\backslash\{\mathsf{X}\}$ and $\mathsf{Y}' \in C'\backslash\{\mathsf{X}'\}$ such that $Y_{\rm I} \cap Y'_{\rm I}$ is nonempty, that is, $\mathsf{Y} \sqcap \mathsf{Y}'$ is nontrivial. Then, since $|w^{\rm b}(\mathsf{Y} \sqcup \mathsf{Y}')| = |\{t, t'\}| = 2$, 7 can be applied for Y and Y'. By $t \notin Y'_{\rm I}$ and $t \neq t'$, we have $t \notin Y'_{\rm O}$ thus $f_H^{\rm b}(\mathsf{Y}') = f_{H_{t,z}}^{\rm b}(\mathsf{Y}')$. Hence, by 7, since Y' is critical in $H_{t,z}$ and, by 11 applied for the critical bi-set Y of $H_{u,v}$, we have the following contradiction: $0 \leq (f_H^{\rm b}(\mathsf{Y}') - 2k) + (f_H^{\rm b}(\mathsf{Y}) - 2k) - d_H^{\rm b}(\mathsf{Y} \sqcup \mathsf{Y}') \leq (f_H^{\rm b}(\mathsf{Y}') - 2k) + (f_H^{\rm b}(\mathsf{Y}) - 2k) - d_H(s, Y_{\rm I})$ $= (f_{H_{t,z}}^{\rm b}(\mathsf{Y}') - 2k) + (f_{H_{u,v}}^{\rm b}(\mathsf{Y}) - 2k) - d_{H_{u,v}}(s, Y_{\rm I}) \leq 0 - 1$, which completes the proof of 20.

Now, denote $H' = H - \{st, su, sv, sz\}$. Observe that, by $t \neq t'$ and 17, if $t' \in N_{H_{u,v}}(s)$, then $t' \in V_{\rm I}(C)$ so $t' \notin N_{H_{u,v}}(s)\backslash X_{\rm I}$. For the same reason, $t \notin N_{H_{t,z}}(s)\backslash X'_{\rm I}$. Hence, by 17 and 20, we have, $N_{H'}(s)\backslash(X_{\rm I} \cup X'_{\rm I}) \subseteq (N_{H_{u,v}}(s)\backslash X_{\rm I}) \cap (N_{H_{t,z}}(s)\backslash X'_{\rm I}) \subseteq (V_{\rm I}(C)\backslash X_{\rm I}) \cap (V_{\rm I}(C')\backslash X'_{\rm I}) = \emptyset$. Hence, by Proposition 3.3 and 13, we have $d_H(s) - 4 \leq d_{H'}(s) \leq d_{H'}(s, X_{\rm I}) + d_{H'}(s, X'_{\rm I}) \leq d_{H_{u,v}}(s, X_{\rm I}) + d_{H_{t,z}}(s, X'_{\rm I}) \leq \lfloor \frac{1}{2}(d_{H_{u,v}}(s) - d_{H_{u,v}}(s, t)) \rfloor + (\frac{1}{2}d_{H_{t,z}}(s) - 1) \leq (\frac{1}{2}d_H(s) - 1 - 1) + (\frac{1}{2}d_H(s) - 2) = d_H(s) - 4$. So equality holds everywhere. In particular, st, su, sv, and sz are distinct edges (even if some of them may be parallel), z does not belong to $X_{\rm I}$, none of u or v belongs to $X'_{\rm I}$ and $d_H(s, t) = d_{H_{u,v}}(s, t) = 1$. Hence, $z \in N_{H_{u,v}}(s)\backslash\{t\}$, so by 17 in $H_{u,v}$, z belongs to the inner-set of an element $\mathsf{Z} \in C\backslash\{\mathsf{X}\}$. Since (st, sz) is admissible in H and Z is critical in $H_{u,v}$, we have $2k = f_{H_{u,v}}^{\rm b}(\mathsf{Z}) \geq f_H^{\rm b}(\mathsf{Z}) - 2 \geq (2k + 1) - 2$, and hence $Z_{\rm I}$ contains u or v, say u. Then, by $u \in \mathsf{Z} \in C\backslash\{\mathsf{X}\}$ and 16, we have $u \in V_{\rm I}(C)\backslash X_{\rm I}$ but since $t \notin V_{\rm I}(C)\backslash X_{\rm I}$, we have $u \neq t'$ thus, by 17 in $H_{t,z}$, u belongs to the inner-set of an element $\mathsf{Y}' \in C'\backslash\{\mathsf{X}'\}$. This contradicts 20 and hence completes the proof of Claim 4.6. ■

Suppose there exists no t-star k-obstacle at s in H. Hence, by Claim 4.5, there exists an admissible pair (st, sz) in H. By Claim 4.6, if there exists an obstacle in $H_{t,z}$, then it is a t-star k-obstacle (t, C'). By $t \notin \{u, v\}$ and 13 in $H_{u,v}$, $d_H(s, t)$ is odd. Hence, by 13 in $H_{t,z}$, $z = t$. Thus, (t, C') is a t-star k-obstacle in H, and this contradiction completes the proof of (b). ■

Now, we are in the position to prove our main result that characterizes the existence of a complete admissible splitting-off.

Theorem 4.7. *Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V with $k \geq 2$ and $d_H(s)$ even. There exists a complete admissible splitting-off at s if and only if there exists no obstacle at s.*

Proof. Suppose there exists an obstacle (t, C) at s. By 13, every sequence of $\frac{1}{2}d_H(s)$ splitting-off of disjoint admissible pairs at s contains a pair (st, su) with $u \in N_H(s)\backslash\{t\}$. As we noticed after the definition of an obstacle, such a pair is not admissible in H and so not admissible in any graph arising from H by a sequence of splitting-off of disjoint admissible pairs. Thus, there is no admissible complete splitting-off at s.

Now, we prove, by induction on $d_H(s)$, that if there exists no obstacle at s, then there exists an admissible complete splitting-off at s. For $d_H(s) = 0$, there is nothing to prove. For $d_H(s) = 2$, the only splitting-off is obviously admissible. Suppose $d_H(s) = 4$ and there exists no obstacle at s. By Theorem 4.1, there exists an admissible splitting-off (su, sv) at s.

Since the only possible splitting-off in $H_{u,v}$ is admissible, there exists an admissible complete splitting-off at s in H.

Now, suppose that the theorem is true for every graph H' that satisfies the conditions with $d_{H'}(s) = 2i$ for $i \leq \ell$ for some $\ell \geq 2$. Let $H = (V + s, E)$ be a $(2, k)$-connected graph in V such that $d_H(s) = 2\ell + 2 \geq 6$ and there exists no obstacle at s. By Theorem 4.1, there exists an admissible splitting-off (su, sv) at s. If there exists no obstacle at s in $H_{u,v}$, then, by induction, there exists an admissible complete splitting-off at s and we are done. So we may assume that there exists a t-star k-obstacle at s in $H_{u,v}$. Since there exists no obstacle at s in H, if Case (b) of Lemma 4.4 occurs, then there exists some admissible pair (st, sw) in H such that there exists no obstacle at s in $H_{t,w}$. Thus, by induction, there exists an admissible complete splitting-off at s in H and we are done. So we may assume that Case (a) of Lemma 4.4 occurs and we consider $H_{t,t}$ that is $(2, k)$-connected in V. If there exists an obstacle (t', C') at s in $H_{t,t}$, for the same reason as above, we may suppose that Case (a) of Lemma 4.4 occurs. Hence, $t = t'$ and (t, C') is an obstacle in H, a contradiction. So no obstacle exists in $H_{t,t}$ and, by induction, the proof of Theorem 4.7 is completed. ■

4.2 | Construction of (2, k)-connected graphs

In this section, we provide a construction of the family of $(2, k)$-connected graphs for k even. The special case $k = 2$ has been previously proved by Jordán [8].

We need the following extension of Lemma 5.1 of [8] for k even. Let $G = (V, E)$ be a $(2, k)$-connected graph, s a vertex of even degree, (t, C) and (t, C') two obstacles at s. We say that (t, C) is a *refinement* of (t, C') if for all $\mathsf{X} \in C$, there exists $\mathsf{X}' \in C'$ such that $\mathsf{X} \sqsubseteq \mathsf{X}'$. An obstacle that has no proper refinement is called *finest*.

Lemma 4.8. *Let $G = (V, E)$ be a $(2, k)$-connected graph with k even. Let s be a vertex of degree $2k$ and (t, C) a finest obstacle at s. Let $\mathsf{X} \in C$, s' a vertex in X_I of degree $2k$ and (t', C') an obstacle at s'. Then, there exists $\mathsf{X}' \in C'$ such that $X'_I \subseteq X_I$.*

Proof. Note that G is $(2, k)$-connected in $V - s$ and also in $V - s'$. By contradiction, we assume that the lemma is false.

Suppose $t' \in X_I$. By 16 and 18 for C', there exists $\mathsf{X}' \in C'$ such that $t \notin X'_I$. By assumption, for each $\mathsf{X}' \in C'$, $X'_I \setminus X_I \neq \emptyset$. Then, $\overline{\mathsf{X}} \sqcap \mathsf{X}'$ is a nontrivial bi-set of $V - s'$ and $|w^b(\overline{\mathsf{X}} \sqcup \mathsf{X}')| = |\{t, t'\}| = 2$. Hence, by Proposition 2.2 and since $\overline{\mathsf{X}}$ and X' are tight, we have $0 + 0 \geq d_G^b(\overline{\mathsf{X}} \sqcup \mathsf{X}') \geq d_G(s', X'_I) \geq 1$, a contradiction. Hence, $t' \notin X_I$.

Suppose $t' \neq t$. If t belongs to the inner-set of an element of C', then call Z' this element and define $\mathsf{Z}' = (\emptyset, \emptyset)$ otherwise. Note that if t is a neighbor of s', then the first case occurs. Thus, by Proposition 3.3(iii), we have $d_G(s', \overline{X_I} \cup Z'_I) \leq d_G(s', \overline{X_O}) + d_G(s', Z'_I) \leq d_G^b(\mathsf{X}) + (\frac{1}{2}d_G(s') - 1) = k + (k - 1) = 2k - 1 = d_G(s') - 1$. Hence, by 17, there exists $\mathsf{Y}' \in C'$ with $Y'_I \cap X_I \neq \emptyset$ and $t \notin Y'_I$. Thus, $\mathsf{X} \sqcap \mathsf{Y}'$ is a nontrivial bi-set of $V - s$ and $|w^b(\mathsf{X} \sqcup \mathsf{Y}')| = |\{t, t'\}| = 2$. Since X and Y' are both tight, by Proposition 2.2 and 13, we have $0 + 0 \geq d_G(\overline{X_O} \cap Y'_O, X_I \cap \overline{Y'_I}) \geq d_G(t', s') \geq 1$, a contradiction. So we proved that $t = t'$.

By $(2, k)$-connectivity of G and $d_G(s') = 2k$, we get $d_G(s', t) \leq k$. Thus, by 13 for C' and k even, $d_G(s', t) < k$. Hence, $d_G(s', \overline{X_I}) = d_G(s', t) + d_G(s', \overline{X_O}) < k + d_G^b(\mathsf{X}) =$

$f_G^{\rm b}(\mathsf{X}) = 2k = d_G(s')$. Thus, by 17, there exists $\mathsf{Y}' \in \mathcal{C}'$ with $Y_{\rm I}' \cap X_{\rm I} \neq \emptyset$. Then, by $|\mathcal{C}'| \geq 3$ and assumption, $\mathsf{X} \sqcup \mathsf{Y}'$ is a nontrivial bi-set of V, thus, by Proposition 2.1(a) with $U = V$, we get that $\mathsf{X} \sqcap \mathsf{Y}'$ is a tight bi-set with wall t.

Note also that $s' \in X_{\rm I} \cap \overline{\mathsf{Y}'}_{\rm I}$ and, by assumption, $\overline{X_{\rm I}} \cap \mathsf{Y}_{\rm I}' \neq \emptyset$, thus, by Proposition 2.1 (b) with $U = V$, we get that $\mathsf{X} \sqcap \overline{\mathsf{Y}'}$ is a tight bi-set with wall t. Thus, in $\mathcal{C}$, X can be replaced by the bi-sets among $\mathsf{X} \sqcap \mathsf{Y}'$ and $\mathsf{X} \sqcap \overline{\mathsf{Y}'}$, which contain at least one neighbor of s in their inner-set. Hence, $(t, \mathcal{C})$ is not a finest obstacle at s, a contradiction. ■

We can now describe and prove the construction of the family of $(2, k)$-connected graphs for k even. We recall that K_3^k is the graph on 3 vertices where each pair of vertices is connected by k parallel edges. Note that K_3^k is $(2, k)$-connected and it is the only minimally $(2, k)$-connected graph on 3 vertices.

Theorem 4.9. *A graph G is $(2, k)$-connected with k even if and only if G can be obtained from K_3^k by a sequence of the following two operations:*

(a) *adding a new edge,*
(b) *pinching a set F of k edges such that for all vertices v, $d_F(v) \leq k$.*

Proof. First, we prove the sufficiency, that is, these operations preserve $(2, k)$-connectivity. It is clearly true for (a). Let G' be a graph obtained from a $(2, k)$-connected graph $G = (V, E)$ by the operation (b) and call s the new vertex. We must show that for every nontrivial bi-set X of $V + s$, we have $f_{G'}^{\rm b}(\mathsf{X}) \geq 2k$. Since this inequality trivially holds whenever $|w^{\rm b}(\mathsf{X})| \geq 2$, we assume that $|w^{\rm b}(\mathsf{X})| \leq 1$ in what follows. If X is a nontrivial bi-set of V, then $s \notin X_{\rm O}$ and, by $(2, k)$-connectivity of G, we have $f_{G'}^{\rm b}(\mathsf{X}) = d_{G'}^{\rm b}(\mathsf{X}) + k|w^{\rm b}(\mathsf{X})| \geq d_G^{\rm b}(\mathsf{X}) + k|w^{\rm b}(\mathsf{X})| = f_G^{\rm b}(\mathsf{X}) \geq 2k$, and we are done. From now on, by symmetry of $f_{G'}^{\rm b}$, we may assume that $s \in X_{\rm O}$. If $\{s\} \subset X_{\rm I}$, then $\overline{X}$ is a nontrivial bi-set of V and, by symmetry of $f_{G'}^{\rm b}$, we are done again. If $\{s\} = X_{\rm I}$, then, by $d_{G'}(s) = 2k$ and $d_F(w^{\rm b}(\mathsf{X})) \leq k$, we have $f_{G'}^{\rm b}(\mathsf{X}) = d_{G'}^{\rm b}(\mathsf{X}) + k|w^{\rm b}(\mathsf{X})| = d_{G'}(s) - d_{G'}(s, w^{\rm b}(\mathsf{X})) + k|w^{\rm b}(\mathsf{X})| = d_{G'}(s) - d_F(w^{\rm b}(\mathsf{X})) + k|w^{\rm b}(\mathsf{X})| \geq 2k$. If $\{s\} \subseteq X_{\rm O} \backslash X_{\rm I} = w^{\rm b}(\mathsf{X})$, then, by $|w^{\rm b}(\mathsf{X})| \leq 1$, we have $w^{\rm b}(\mathsf{X}) = \{s\}$ and then $\emptyset \neq X_{\rm I} \neq V$. Hence, by $|F| = k$ and $(2, k)$-connectivity of G, we have $f_{G'}^{\rm b}(\mathsf{X}) = d_{G'}^{\rm b}(\mathsf{X}) + k|w^{\rm b}(\mathsf{X})| = (d_G(X_{\rm I}) - d_F(X_{\rm I})) + k \geq d_G(X_{\rm I}) - |F| + k \geq 2k$.

To see the necessity, let G be a $(2, k)$-connected graph with at least 4 vertices. Note that the inverse operation of (a) is deleting an edge and that of (b) is a complete splitting-off at a vertex s of degree $2k$ such that $d_G(s, v) \leq k$ for all $v \in V$. Note also that these inverse operations must preserve $(2, k)$-connectivity. Thus, we may assume that, on the one hand, G is minimally $(2, k)$-connected and hence, by Theorem 1.9, G contains a vertex of degree $2k$, and, on the other hand, for every such vertex u, there exists no admissible complete splitting-off at u, that is, by Theorem 4.7, there exists an obstacle at u.

We choose in $\{(u, (t, \mathcal{C}), \mathsf{X}): d_G(u) = 2k, (t, \mathcal{C})$ a finest obstacle at u, $\mathsf{X} \in \mathcal{C}\}$ a triple $(u^*, (t^*, \mathcal{C}^*), \mathsf{X}^*)$ with X^* minimal for inclusion. By Theorem 1.9, there exists a vertex u' of degree $2k$ in $X_{\rm I}^*$. Then, as we have seen, there exists a finest obstacle $(t', \mathcal{C}')$ at u'. By Lemma 4.8, there exists $\mathsf{X}' \in \mathcal{C}'$ such that $X_{\rm I}' \subseteq X_{\rm I}^*$. Since $X_{\rm I}' \cup \{u'\} \subseteq X_{\rm I}^*$, the triple $(u', (t', \mathcal{C}'), \mathsf{X}')$ contradicts the choice of $(u^*, (t^*, \mathcal{C}^*), \mathsf{X}^*)$. ■

We mention that the condition k is even is necessary in Lemma 4.8 and Theorem 4.9. Consider the graph obtained from K_4 by adding a new vertex t and 3 edges between t and each vertex of K_4. This graph is minimally (2, 3)-connected but there exists no complete admissible splitting-off at any of the 4 vertices of degree 6. Indeed, if s, a, b, c denote the vertices of degree 6, then $\{(\{a, t\}, \{a\}), (\{b, t\}, \{b\}), (\{c, t\}, \{c\})\}$ is a t-star 3-obstacle at s.

4.3 | Augmentation theorem

In this section, we answer the following question for $k \geq 2$: given a graph what is the minimum number of edges to be added to make it $(2, k)$-connected. For $k = 1$, that is, for 2-vertex-connectivity, this problem had been already solved by Eswaran and Tarjan [4].

We shall need the following definitions. Let $G = (V, E)$ be a graph. An *s-extension* of G is a graph $H = (V + s, E \cup F)$, where F is a set of edges between V and the new vertex s. The *size* of an s-extension of G is defined by $|F|$.

We mimic the approach of Frank [5] for the augmentation problem: first, we prove a result on minimal extensions and then, by applying our splitting-off theorem, we get a result on minimal augmentation.

Lemma 4.10. *Let $G = (V, E)$ be a graph such that $|V| \geq 3$ and k a positive integer. The minimum size of an s-extension of G, that is, $(2, k)$-connected in V, is equal to maximum of $\{\sum_{\mathsf{X} \in \mathcal{X}} (2k - f_G^{\mathrm{b}}(\mathsf{X}))\}$, where $\mathcal{X}$ is a family of nontrivial pairwise innerly disjoint bi-sets of V.*

Proof. If $H' = (V + s, E \cup F')$ is an s-extension of G, that is, $(2, k)$-connected in V and $\mathcal{X}'$ is an arbitrary family of nontrivial pairwise innerly disjoint bi-sets of V, then

$$\sum_{\mathsf{X}' \in \mathcal{X}'} (2k - f_G^{\mathrm{b}}(\mathsf{X}')) \leq \sum_{\mathsf{X}' \in \mathcal{X}'} (f_H^{\mathrm{b}}(\mathsf{X}') - f_G^{\mathrm{b}}(\mathsf{X}')) = \sum_{\mathsf{X}' \in \mathcal{X}'} d^{\mathrm{b}}_{(V+s,F')}(\mathsf{X}') \leq |F'|.$$

This shows that max $\leq$ min.

To prove that equality holds, we provide a family $\mathcal{X}$ of nontrivial pairwise innerly disjoint bi-sets of V and an s-extension of G, that is, $(2, k)$-connected in V of size $\sum_{\mathsf{X} \in \mathcal{X}} (2k - f_G^{\mathrm{b}}(\mathsf{X}))$. Let M be defined as the maximum value of $2k - f_G^{\mathrm{b}}(\mathsf{X}')$ over all bi-set X' of V. If $M \leq 0$, then G is $(2, k)$-connected and we are done. Suppose that $M > 0$. We consider the s-extension of G whose set of new edges consists of M parallel edges sv, for each $v \in V$. This extension is obviously $(2, k)$-connected in V. Then, we remove as many new edges as possible without destroying the $(2, k)$-connectivity in V. Let F be the set of remaining edges and $H = (V + s, E \cup F)$. In H, by minimality of F, each edge e of F enters a tight bi-set of V. Let $\mathcal{X}$ be a family of nontrivial tight bi-sets of V such that

$$\text{each edge of } F \text{ enters at least one element of } \mathcal{X} \text{ and} \tag{21}$$

$$\sum_{\mathsf{X} \in \mathcal{X}} |X_I| \text{ is minimal.} \tag{22}$$

Claim 4.11. The elements of $\mathcal{X}$ are pairwise innerly disjoint.

Proof. Note that the degree of each tight bi-set X in $\mathcal{X}$ is at least one, thus $|w^{\mathrm{b}}(\mathsf{X})| \leq 1$. Suppose there exist two distinct elements X and Y in $\mathcal{X}$ such that $X_{\mathrm{I}} \cap Y_{\mathrm{I}} \neq \emptyset$, that is, $\mathsf{X} \sqcap \mathsf{Y}$ is a nontrivial bi-set of V.

If $\mathsf{X} \sqcup \mathsf{Y}$ is a nontrivial bi-set of V, then, by $(2, k)$-connectivity in V of H, tightness of X and Y and Proposition 2.1(a), $\mathsf{X} \sqcup \mathsf{Y}$ is tight. Since all the edges of F entering X_{I} or Y_{I} enters $(\mathsf{X} \sqcup \mathsf{Y})_{\mathrm{I}}$, the family obtained from $\mathcal{X}$ by substituting $\mathsf{X} \sqcup \mathsf{Y}$ for X and Y satisfies 21 and, by $X_{\mathrm{I}} \cap Y_{\mathrm{I}} \neq \emptyset$, contradicts 22. So $X_{\mathrm{O}} \cup Y_{\mathrm{O}} = V$.

If $\mathsf{X} \sqcap \overline{\mathsf{Y}}$ and $\overline{\mathsf{X}} \sqcap \mathsf{Y}$ are nontrivial bi-sets of V, then, by $(2, k)$-connectivity in V of H, tightness of X and Y and Proposition 2.1(b), both $\mathsf{X} \sqcap \overline{\mathsf{Y}}$ and $\overline{\mathsf{X}} \sqcap \mathsf{Y}$ are tight and $d_H(\overline{X_{\mathrm{O}}} \cap \overline{Y_{\mathrm{I}}}, X_{\mathrm{I}} \cap Y_{\mathrm{O}}) = d_H(Y_{\mathrm{I}} \cap X_{\mathrm{O}}, \overline{Y_{\mathrm{O}}} \cap \overline{X_{\mathrm{I}}}) = 0$. Hence, all the edges of F entering the set X_{I} or the set Y_{I} enters the set $(\mathsf{X} \sqcap \overline{\mathsf{Y}})_{\mathrm{I}}$ or $(\overline{\mathsf{X}} \sqcap \mathsf{Y})_{\mathrm{I}}$. Thus, the family obtained from $\mathcal{X}$ by substituting $\mathsf{X} \sqcap \overline{\mathsf{Y}}$ and $\overline{\mathsf{X}} \sqcap \mathsf{Y}$ for X and Y satisfies 21 and, by $X_{\mathrm{I}} \cap Y_{\mathrm{I}} \neq \emptyset$, contradicts 22. So, by symmetry, we may assume that $X_{\mathrm{I}} \subseteq Y_{\mathrm{O}}$.

We have $N_H(s) \cap X_{\mathrm{I}} \nsubseteq Y_{\mathrm{I}}$ otherwise $\mathcal{X} - \mathsf{X}$ satisfies 21 and contradicts the minimality of $\mathcal{X}$. Thus, by $X_{\mathrm{I}} \subseteq Y_{\mathrm{O}}$, $d_H(s, w^{\mathrm{b}}(\mathsf{Y})) \geq 1$ and, since $X_{\mathrm{O}} \cup Y_{\mathrm{O}} = V$ and Y is nontrivial, $w^{\mathrm{b}}(\mathsf{X}) \backslash Y_{\mathrm{O}} = X_{\mathrm{O}} \backslash Y_{\mathrm{O}} = (X_{\mathrm{O}} \cup Y_{\mathrm{O}}) \backslash Y_{\mathrm{O}} = V \backslash Y_{\mathrm{O}}$ is nonempty. So $|w^{\mathrm{b}}(\overline{\mathsf{X}} \sqcup \mathsf{Y})| \geq 2$.

For the same reason as above, $N_H(s) \cap Y_{\mathrm{I}} \nsubseteq X_{\mathrm{I}}$. Thus, by $|w^{\mathrm{b}}(\mathsf{X})| \leq 1$ and $w^{\mathrm{b}}(\mathsf{X}) \backslash Y_{\mathrm{O}} \neq \emptyset$, the set $Y_{\mathrm{I}} \backslash X_{\mathrm{O}} = Y_{\mathrm{I}} \backslash X_{\mathrm{I}}$ contains a neighbor of s, that is, $\overline{\mathsf{X}} \sqcap \mathsf{Y}$ is nontrivial. Thus, by symmetry of f_H^{b}, tightness of X and Y and 7, we have the following contradiction $0 + 0 = (f_H^{\mathrm{b}}(\overline{\mathsf{X}}) - 2k) + (f_H^{\mathrm{b}}(\mathsf{Y}) - 2k) \geq d_H(X_{\mathrm{I}} \cap Y_{\mathrm{O}}, \overline{X_{\mathrm{O}}} \cap \overline{Y_{\mathrm{I}}}) \geq d_H(s, w^{\mathrm{b}}(\mathsf{Y})) \geq 1$, which completes the proof of Claim 4.11. ■

By Claim 4.11, 21 and by tightness of the elements of $\mathcal{X}$, we have

$$|F| = \sum_{\mathsf{X} \in \mathcal{X}} d^{\mathrm{b}}_{(V+s,F)}(\mathsf{X}) = \sum_{\mathsf{X} \in \mathcal{X}} (f_H^{\mathrm{b}}(\mathsf{X}) - f_G^{\mathrm{b}}(\mathsf{X})) = \sum_{\mathsf{X} \in \mathcal{X}} (2k - f_G^{\mathrm{b}}(\mathsf{X})),$$

which completes the proof of Lemma 4.10. ■

The augmentation theorem goes as follows.

Theorem 4.12. *Let $G = (V, E)$ be a graph such that $|V| \geq 3$ and $k \geq 2$ an integer. The minimum cardinality γ of a set F of edges such that $(V, E \cup F)$ is $(2, k)$-connected is equal to*

$$\alpha = \left\lceil \frac{1}{2} \max \left\{ \sum\nolimits_{\mathsf{X} \in \mathcal{X}} (2k - f_G^{\mathrm{b}}(\mathsf{X})) \right\} \right\rceil,$$

where $\mathcal{X}$ is a family of nontrivial pairwise innerly disjoint bi-sets of V.

Proof. We first prove $\gamma \geq \alpha$. Let $\mathcal{X}$ be a family of nontrivial bi-sets of V such that the elements of $\mathcal{X}$ are pairwise innerly disjoint. For each $\mathsf{X} \in \mathcal{X}$, we must add at least $2k - f_G^{\mathrm{b}}(\mathsf{X})$ new edges entering the bi-set X when this quantity is positive. Since the elements of $\mathcal{X}$ are pairwise innerly disjoint, a new edge may enter at most 2 elements of $\mathcal{X}$. Hence, $2\gamma \geq \sum_{\mathsf{X} \in \mathcal{X}} (2k - f_G^{\mathrm{b}}(\mathsf{X}))$ thus, since γ is integer, $\gamma \geq \alpha$ follows.

We now prove $\gamma \le \alpha$. By Lemma 4.10, there exists an s-extension $H = (V + s, E \cup F)$ of G, that is, $(2, k)$-connected in V and a family $\mathcal{X}$ of nontrivial pairwise innerly disjoint bi-sets of V such that

$$|F| = \sum_{\mathsf{X} \in \mathcal{X}} (2k - f_G^{\mathrm{b}}(\mathsf{X})).$$

If $|F|$ is odd, then there exists a vertex $u \in V$ such that $d_H(s, u)$ is odd, in this case, let $F' = F \cup \{su\}$ otherwise let $F' = F$. So, in the graph $H' = (V + s, E \cup F')$, $d_{H'}(s)$ is even.

Suppose there exists an obstacle $(t, \mathcal{C})$ at s. By 19, $H' - st$ is $(2, k)$-connected in V. If $H = H'$ this contradicts the minimality of $|F|$. Then, $d_H(s)$ is odd and $F' = F + su$ for some vertex $u \in V$ such that $d_H(s, u)$ is odd. If $u \in X_{\mathrm{I}}$ for some $\mathsf{X} \in \mathcal{C}$, then we have $f_H(\mathsf{X}) = f_{H'}(\mathsf{X}) - 1 = 2k - 1$, a contradiction to the $(2, k)$-connectivity of H. Thus, by 17, $u = t$ and hence $d_{H'}(s, t) = d_H(s, t) + 1$ is even, which contradicts 13.

Hence, no obstacle exists at s, and, by Theorem 4.7, there exists an admissible complete splitting-off at s in H'. Let us denote by F'' the set of edges obtained by this complete splitting-off. Then, $(V, E \cup F'')$ is $(2, k)$-connected and

$$|F''| = \frac{1}{2}|F'| = \left\lceil \frac{1}{2}|F| \right\rceil = \left\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} (2k - f_G^{\mathrm{b}}(X)) \right\rceil.$$

This proves $\gamma \le \alpha$ and completes the proof of Theorem 4.12. ■

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