An Excluded Minor Characterization of Seymour Graphs

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Abstract

A graph G is said to be a *Seymour graph* if for any edge set F there exist $|F|$ pairwise disjoint cuts each containing exactly one element of F, provided for every circuit C of G the necessary condition $|C \cap F| \leq |C \setminus F|$ is satisfied. Seymour graphs behave well with respect to some integer programs including multiflow problems, or more generally odd cut packings, and are strictly related to matching theory.

A first coNP characterization of Seymour graphs has been shown by Ageev, Kostochka and Szigeti, the recognition problem has been solved in a particular case by Gerards, and the related cut packing problem has been solved in the corresponding special cases. In this article *we show a new, minor-producing operation that keeps this property, and prove excluded minor characterizations of Seymour graphs: the operation is the contraction of full stars, or of odd circuits.* This sharpens the previous results, providing at the same time a simpler and self-contained algorithmic proof of the existing characterizations as well, still using methods of matching theory and its generalizations.

Dualizing the planar special case, Seymour graphs are becoming those for which the cut condition is sufficient for the existence of disjoint paths for any set of demand pairs. Either the disjoint paths or a forbidden minor can be found in polynomial time.

1 Introduction

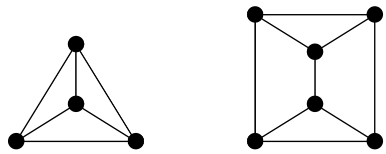
Graphs. In this paper graphs are undirected and may have loops and multiple edges. Let $G = (V, E)$ be a graph. *Shrinking* $X \subseteq V$ means the identification of the vertices in X, keeping all the edges incident to X; the result will be denoted by G/X. The *deletion* and *contraction* of an edge $e \in E$ are the usual operations (the latter is the same as shrinking the endpoints of the edge), as well as the deletion of a vertex which means the deletion of the vertex together with all the edges incident to it. We will use the notation $G - e$, G/e for the deletion, respectively contraction of edge e, and $G - v$ for the deletion of vertex v. The vertex-set, edge-set of the graph G will be denoted by $V(G)$, $E(G)$, whereas for $X \subseteq V(G)$, $\delta(X)$ will denote the *cut* induced by X that is the set of edges with exactly one endpoint in X, $E(X)$ the set of *induced* edges, that is those that have both of their endpoints in X, $I(X) = \delta(X) \cup E(X)$ and $N(X)$ the set of neighbors of X.

A graph $H = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. If $H = G(X) := (X, E(X))$ for some $X \subseteq V$, then H is called an *induced* subgraph of G (induced by X). If $\hat{F}$ is a subgraph of $\hat{G} = G/Y$ then the corresponding subgraph of G will be denoted by F.

Packing of cuts. A family of subsets of a set S is a *packing* if the sets are disjoint, and a 2-*packing* if every $s \in S$ is contained in at most two members of the family.

Let $F \subseteq E$. A *complete packing of cuts* for (G, F) is a family of $|F|$ pairwise edge-disjoint cuts, each containing an element of F. An obvious necessary condition for the existence of a complete packing of cuts for (G, F) is that F is a *join*, that is, for every circuit C of G, $|C \cap F| \leq |C \setminus F|$. Indeed, if $\mathcal{Q}$ is a complete packing of cuts for (G, F) then for every $e \in C \cap F$ one of the cuts $Q_e \in \mathcal{Q}$ contains e, and one more edge of the circuit C which is not in F. Similarly, a *complete* 2-*packing of cuts* for (G, F) is a 2-packing of $2|F|$ cuts each containing an element of F, and the existence of a complete 2-packing of cuts for (G, F) implies that F is a join in G. These obvious facts will be used without reference all over the paper. If $\mathcal{Q}$ is a packing of cuts then we will denote by $2\mathcal{Q}$ the 2-packing of cuts obtained from $\mathcal{Q}$ by duplication.

The two graphs, K_4 and prism, that can be found in Figure 1 play an important role in our results.



K_4 prism

Figure 1: The complete graph K_4 on four vertices and the prism

The graph K_4 (or the prism) with a join consisting of a perfect matching shows that a complete packing of cuts does not necessarily exist. However, by a theorem of Lovász [4] for every join F there exists a complete 2-packing of cuts for (G,F). This result (more precisely a slight sharpening) plays a key-role in the proof of all the characterizations of Seymour graphs.

Seymour graphs. The graph G is said to be a *Seymour-graph* if for every join $F \subseteq E$ there exists a complete packing of cuts. Let us immediately show a condition that is sufficient for this property to fail:

Let us call a circuit C of G *tight* with respect to (G,F), if $|C \cap F| = |C \setminus F|$, that is, if the inequality $|C \cap F| \leq |C \setminus F|$ is satisfied with equality. In the above proof of "conservativeness" (from the existence of a complete packing of cuts) we have equality for tight circuits, that is in any complete packing $\mathcal{Q}$ of cuts the family $\{Q \cap C \neq \emptyset : Q \in \mathcal{Q}\}$ is a partition of C into pairs of edges $\{e, f\}$ where $e \in F$ and $f \notin F$. The following fact straightforwardly follows:

Fact 1. *Let G be a graph. If there exists a join F in G and a set of circuits tight with respect to (G,F) whose union is non-bipartite, then G is not Seymour.*

Indeed, if C is an arbitrary circuit in the union of tight circuits, and $\mathcal{Q}$ is a complete packing of cuts for (G,F), then all the edges of C are contained in some member of $\mathcal{Q}$, by the previous remark. Therefore the nonempty members of $\{Q \cap C : Q \in \mathcal{Q}\}$ partition C. Since all the classes of this partition are even, $|C|$ is even, proving that G is bipartite.

The examples of Figure 1 show non-bipartite graphs where every edge belongs to a tight circuit for an arbitrary perfect matching, and hence these graphs are not Seymour graphs.

T-joins. Matchings and shortest paths between any two vertices of a graph are simple examples of joins (as well as arbitrary T-joins, see below.) Moreover it can be readily seen that complete packings of cuts in the dual graph (or matroid) correspond to circuits, each containing exactly one element of F, that is, to paths, each joining the endpoints of one element of F. They also occur in the dual of matching problems or of the Chinese Postman problem.

The latter has a general setting containing also matchings, planar multiflows and where the main ideas of matching theory still work: T-joins. Let $G = (V,E)$ be a connected graph and $T \subseteq V$, where $|T|$ is even. A subset $F \subseteq E$ is called a T*-join* if the set of vertices having an odd number of incident edges from F is T. For any subset $X \subseteq V$, the cut $\delta(X)$ is called a T*-cut* if $|X \cap T|$ is odd. Since in every graph the number of vertices of odd degree is even, each T-join must have at least one edge in common with each T-cut. Hence, if we denote by $\nu(G,T)$ the maximum number of edge disjoint T-cuts and by $\tau(G,T)$, the minimum cardinality of a T-join in G, then $\nu(G,T) \leq \tau(G,T)$. The example $G = K_4$ and $T = V(G)$ shows that this inequality can be strict.

The usual definition of a Seymour-graph is that the equality $\nu(G,T) = \tau(G,T)$ holds for all subsets $T \subseteq V(G)$ of even cardinality. Indeed, this is equivalent to the above definition since every minimum T-join F, $|F| = \tau(G,T)$ is a join (Guan's lemma [7]) and a complete packing of cuts for (G,F) is a family of $|F| = \tau(G,T)$ disjoint T-cuts. Conversely, every join F is a minimum T-join where T is the set of vertices incident to an odd number of edges of F, and $|F| = \tau(G,T) = \nu(G,T)$ implies that a family of $\nu(G,T)$ disjoint T-cuts is a complete packing of cuts for (G,F). This reformulation is well-known and has been exploited already in Seymour's article [10].

Several particular cases of Seymour-graphs have been exhibited by Seymour [9] [10], Gerards [3], Szigeti [11] while the existence of complete packing of cuts in graphs has been proved to be NP-hard [6]. In the rest of this introduction we wish to show the difficulties of such characterizations, and this will lead us to the answer we provide here to this question.

Odd K_4 and odd prism. We will work with different types of subdivisions of the K_4 and the prism. Let us emphasize that in any subdivision of the K_4 or the prism every vertex is of degree 2 or 3.

A graph G is called an *odd* K_4 if it is a subdivision of K_4 such that each circuit bounding a face of G has an odd length. A graph G is an *odd prism* if it is a subdivision of the prism such that each circuit bounding a triangular face of G has an odd length while each circuit bounding a quadrangular face has an even length. (See Figure 2(a).)

A subdivision of a graph G is said to be *even* if the number of new vertices inserted in every edge of G is even (possibly zero). (See Figure 2(b).) Analogously, a subdivision of a graph G is said to be *odd* if the number of new vertices inserted in every edge of G is odd. An even subdivision of K_4 (respectively, of the prism) is clearly an odd K_4 (respectively, an odd prism).

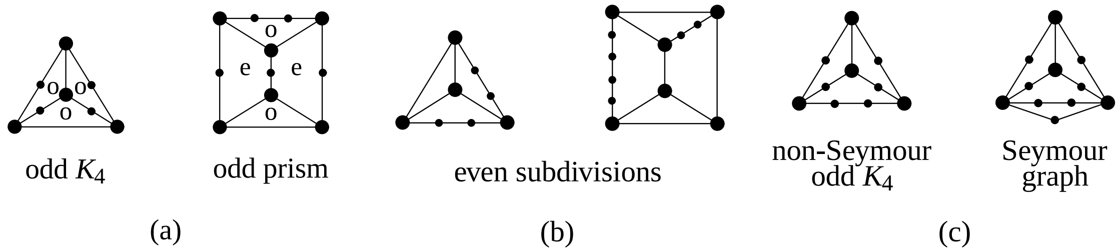


Figure 2: (a) An odd K_4 and an odd prism, (b) even subdivisions of K_4 and the prism, (c) a Seymour graph containing a non-Seymour odd K_4.

Subclasses of Seymour graphs. It is well known that Seymour graphs include bipartite graphs (Seymour [9]) and series-parallel graphs (Seymour [10]). For better comparison let us mention this latter class as the family of graphs not containing any subdivision of K_4. Gerards [3] generalized the previous results by proving that graphs not containing odd K_4 or odd prism subgraphs are Seymour. Figure 2(c) shows a subdivision of K_4 which is not a Seymour graph but adding a new vertex joined to two old ones, or replacing the bottom path of length three by just one edge we obtain a Seymour graph. This shows that the property is not inherited to minors.

A further restriction of the excluded minors (and thus a generalization of the sufficient condition) is to exclude only the non-Seymour subdivisions of K_4 and the prism. The list of these was provided by [11] and served as an important preliminary of [1]. Szigeti's theorem [11] stated that a graph that does not contain a non-Seymour subdivision of K_4 or of the prism as subgraph, is a Seymour graph. This theorem generalizes all the previous results, but the sufficient condition it provides is still not necessary: the second graph on Figure 2(c) contains a non-Seymour subdivision of K_4 (even as an induced subgraph) but is Seymour.

Continuing on this road, excluding only even subdivisions of K_4 and the prism is already a necessary condition, that is, all Seymour-graphs are contained in the defined class of graphs. Indeed, any perfect matching of an even subdivision of K_4 is a join in the graph and there exists no complete packing of cuts. Similarly for the prism.

The main contribution of the present work is an excluded minor characterization of Seymour-graphs. In Section 2 we state the main results including also the preliminaries. In Section 3 we provide a self-contained proof of the results.

2 Results

Previous characterization of Seymour graphs. The following result (conjectured by Sebő in 1992, proved by Ageev, Kostochka & Szigeti in an IPCO volume, 1995, journal version 1997 [1]), places the Seymour property in co-NP and implies all the above inclusions.

Theorem 1. *The following statements are equivalent for a graph G:*

(1) G is not a Seymour graph,

(2) G has a join F and a set of tight circuits whose union is non-bipartite,

(3) G has a join F and two tight circuits whose union forms an odd K_4 or an odd prism.

This theorem has the drawback that it depends on the join F, it is not suitable for testing the property, and heavy for algorithmic handling. A reason can be captured in the logical structure: it uses one more quantifier besides a containment of a graph making it hard to find an NP-characterization or algorithm (there exists a subgraph, and a join F, ...). We provide here characterizations by property keeping minor-producing operations. The main result of this paper avoids these formal drawbacks, leads to a simpler proof and is trying to pave the way to the recognition. We will exhibit several characterizations, some of which play mainly the role of a bridge in the proof, but may also be useful for studying the complexity of the problem in the future. For this we have to introduce some notions:

Factor-contraction. A graph G is called *factor-critical* if the graph $G-v$ has a perfect matching for any vertex v of G. Note that a vertex and an odd circuit is factor-critical. The following result due to Lovász [5], that provides a characterization of factor-critical graphs, will play an important role in the main result of this paper.

Fact 2. *A graph G is factor-critical if and only if a single vertex can be obtained from G by a series of odd circuit contractions.*

A *factor-contraction* is the contraction of $I(X)$ (all edges incident to a vertex set X), where $G(X)$ is factor-critical. There is a rich appearence of such sets X in the T-join structure [8], which is "robust" whith respect to the contraction of $I(X)$. We will denote the result of this operation by G^X. If $X=\{v\}$, $v\in V$, then we will write G^v, and we call this operation a *star-contraction*, the contraction of the full star of v. (A *star* is a graph consisting of edges all incident to a given vertex. A star is called *full*, if it contains all edges of G incident to v.)

We well apply the following lemma that makes possible factor-contractions unless the graph is bi-critical. The first part of this statement is part of a result in [8]. The second part is much simpler and is implicit in [10].

Lemma 1. *Let $G=(V,E)$ be a graph, $F\subseteq E$ a join, and $x_0\in V$ arbitrary.*

(1) For $F\neq\emptyset$, there exists a complete 2-packing $\{\delta(X):X\in\mathscr{C}\}$ of cuts for (G,F) and $C\in\mathscr{C}$ so that

(a) $G(C)$ is factor-critical,

(b) $\{c\}\in\mathscr{C}$ for all $c\in C$ (if $|C|=1$, then C is contained twice in $\mathscr{C}$) and

(c) none of the members of $\mathscr{C}$ contain x_0.

(2) *If there exists a complete packing of cuts for (G,F) then there is one containing a full star different of $\delta(x_0)$.*

Stoc-minor. A factor-contraction can be decomposed into two operations both of which are nontrivial if $|X|>1$: the contraction of (the edges induced by) X, and the contraction of the edges in $\delta(X)$. After the contraction of X we just have to contract a star, which thus keeps the Seymour-property.

But does the contraction of a factor-critical graph also keep the property? If yes, then in particular, the contraction of an odd circuit also keeps it! By Fact 2, the contraction of the edges of a factor-critical graph is equivalent to a succession of contractions of odd circuits, in case of a yes answer, two very simple and basic operations would be sufficient to handle Seymour-graphs: star- and odd-circuit-contraction. This was a raised and forgotten open question in [11].

The main novelty of the present work is that **indeed, these two operations keep the Seymour property**. This refined approach simplifies the results and their proofs of the known characterizations as well. However, we were not able to prove it directly without involving essentially the entire proof.

We will say that a graph G' is the *stoc-minor* of G if it arises from G by a series of star and odd circuit contractions. Stoc-minors would generate though an immediate simplification: prisms disappear! Indeed, K_4 is a stoc-minor of the prism.

Biprism. A *biprism* is obtained from the prism by subdividing each edge that connects the triangles by a vertex.The set X of the three new vertices is called the *separator* of the biprism H. Note that $H-X$ has two connected components, and both are triangles. These will be called the two *sides* of the biprism. For a subdivided biprism the two sides include the new vertices of the subdivisions connected to the triangles on each side. We mention that for a subdivided biprism the underlying biprism and the two sides are not necessarily uniquely determined, so whenever necessary, we will tell more about the subdivision we are speaking about.

For $t\in\{2,3\}$, a *t-star* is a star containing t edges.We will also deal with the families of the subgraphs L of G that are even or odd subdivisions of a t-star. The set of the t vertices of degree one in L is called the *base* of L. For a subgraph H of G, a vertex-set $U\subset V(H)$ is called a *t-core* of H if $|\delta_H(U)|=t$ and $I_H(U)$ is an even subdivision of a t-star. A *core* is either a 2-core or a 3-core.

Obstructions. We will say that G contains a *K-obstruction,* where K is a K_4, a prism or a biprism if there exists a subgraph H of G with a subpartition $\mathscr{U}$ of $V(H)$ satisfying the following properties:

– H is a subdivision of K.

– For every $U\in\mathscr{U}$ the graph $I_H(U)$ is an even subdivision of a star.

– Any path *of* G joining two vertices $u_1\in U_1\in\mathscr{U}$ and $u_2\in V(H)-U_1$ has length at least 2 if u_2 is of degree 3 in H and it has length at least 3 if $u_2\in U_2\in\mathscr{U}$.

– Shrinking each $I(U)$ in G (they are disjoint because of the previous condition) we get an even subdivision of K, and for the biprism we also require that there is no edge of G between vertices on different sides. (Meaning that among the possible choices of the separator, there is one satisfying this.)

We will say that $(H,\mathscr{U})$ is an *obstruction* of G. Note that if K is a K_4 or a prism and a subgraph H of G with subpartition $\mathscr{U}$ empty is a K-obstruction, then H is simply an even subdivision of K. We mention that the graph of Figure 2(c) does not contain a K_4-obstruction because the distances of the vertices of degree three are at most two in G.

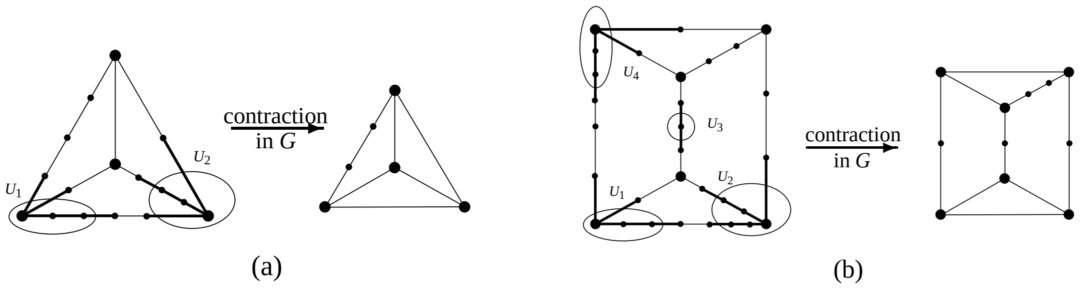


Figure 3: (a) A K_4-obstruction and (b) a biprism-obstruction

Let us explain the definition of obstructions with other words, using only contractions and not using distances (see Figure 3):

An odd K_4 subgraph H of G with a set $\mathscr{U}$ of disjoint 3-cores U_i of H is a K_4-obstruction if after contracting in G the set of edges of G having at least one end-vertex in one of U_i's (that is $I_G(U_i)$ $U_i \in \mathscr{U}$), H tranforms into an even subdivision of the K_4.

An odd prism subgraph H of G with a set $\mathscr{U}$ of disjoint 2 or 3-cores U_i of H is a prism-obstruction (respectively, biprism-obstruction) if after contracting in G the set of edges of G having at least one end-vertex in one of U_i's (that is $I_G(U_i)$ $U_i \in \mathscr{U}$), H tranforms into H', an even subdivision of the prism (respectively, of the biprism; and in this latter case with the further property that for a convenient separator X of the underlying biprism no edge of G connects the two connected components of $H' - X$). We mention that 2-cores are necessary because of the separator of the biprism (see Figure 4).

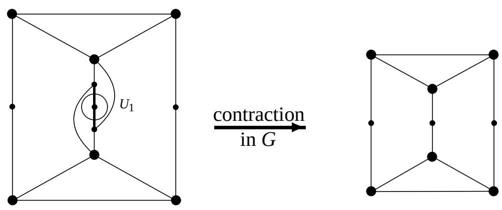


Figure 4: Necessary 2-star contraction

Let us continue with three claims on obstructions.

Claim 1. *Let $G = (V,E)$ be a graph, $A \subset V$ and $\hat{G} = G/I(A)$. Suppose that $\hat{G}$ contains a K-obstruction $(\hat{H}, \hat{\mathscr{U}})$. Let H be the subgraph of G defined by the edge set of $\hat{H}$ and $T = V(H) \cap N(A)$.*

(a) If $|T| = 2$ and $G(A \cup T)$ contains an even path between the vertices of T, or

(b) if $|T| = 3$ and $G(A \cup T)$ contains an even or odd subdivision of a 3-star with base T,

then G also contains a K-obstruction.

Proof. Let us denote by a the contracted vertex of $\hat{G}$ and let $\hat{\mathscr{U}} = \{\hat{U}_1, \dots, \hat{U}_k\}$.

(a) Let L be the even path in question. Note that the subgraph $H \cup L$ of G is an odd K_4 or an odd prism because we extended the path of $\hat{H}$ that contains the vertex a by an even path. Now we define the set $\mathscr{U}$. If the vertex a belongs to one of the cores $\hat{U}_j \in \hat{\mathscr{U}}$, then let $U_i := \hat{U}_i$ for $i \neq j$ and $U_j := (\hat{U}_j - a) \cup V(L)$, otherwise let $U_i := \hat{U}_i$ for $i = 1, \dots, k$ and if K is a biprism and a belongs to the separator, then $U_{k+1} := V(L) - T$. We emphasize that in this latter case we have to add U_{k+1} because of the condition that no edge of G is allowed between the different sides of the biprism.

(b) Let L be the subdivision of a 3-star in question. Note that the subgraph $H \cup L$ of G is an odd K_4 or an odd prism because every circuit of $\hat{H}$ is extended by an even path. Now we define the set $\mathscr{U}$. First, suppose that L is an even subdivision of a 3-star. If the vertex a belongs to one of the cores $\hat{U}_j \in \hat{\mathscr{U}}$, then let $U_i := \hat{U}_i$ for $i \neq j$ and U_j is deleted, otherwise let $U_i := \hat{U}_i$ for $i = 1, \ldots, k$ and $U_{k+1} := V(L) - T$. Second, suppose that L is an odd subdivision of a 3-star. If the vertex a belongs to one of the cores $\hat{U}_j \in \hat{\mathscr{U}}$, then let $U_i := \hat{U}_i$ $i \neq j$ and $U_j := (\hat{U}_j - a) \cup V(L)$, otherwise let $U_i := \hat{U}_i$ for $i = 1, \ldots, k$.

In all cases $(H \cup L, \mathscr{U})$ is a K-obstruction of G. □

Claim 2. *If G contains an obstruction $(H, \emptyset)$, then G has a join F and two tight circuits whose union is H and $d_F(v) = 1$ if $d_H(v) = 3$.*

Proof. We define the edge set F in the three cases. If H is an even subdivision of K_4, then let F be an arbitrary perfect matching of H. If H is an even subdivision of the prism, then let F be that perfect matching of H for which $\delta_F(C) = 1$ where C is one of the two odd circuits that correspond to the two triangles of the prism. If H is an even subdivision of the biprism, then let X be the separator of H, x a vertex of X and u and w be the two neighbors of x in H and finally let F be the union of the unique perfect matching of $H - X - u - w$ and $\{ux, wx\}$. In all cases $d_F(v) = 1$ if $d_H(v) = 3$. (See Figure 5.)

To see that F is a join we provide a complete 2-packing of cuts for (G, F) in the three cases. If H is an even subdivision of K_4 or the prism, then let $\mathscr{Q} := \{\delta(v) : v \in V(H)\}$ and if H is an even subdivision of the biprism, then let $\mathscr{Q} := \{\delta(v) : v \in V(H) - X\} \cup \{\delta(V_1), \delta(V_2)\}$, where V_1 and V_2 are the two connected components of $H - X$. Note that, by the definition of a biprism-obstruction, there exists no edge of G between V_1 and V_2. Then Q is a complete 2-packing of cuts for (G, F) and hence F is a join. (See Figure 5.) In all cases it is easy to find the two tight circuits whose union is H. □

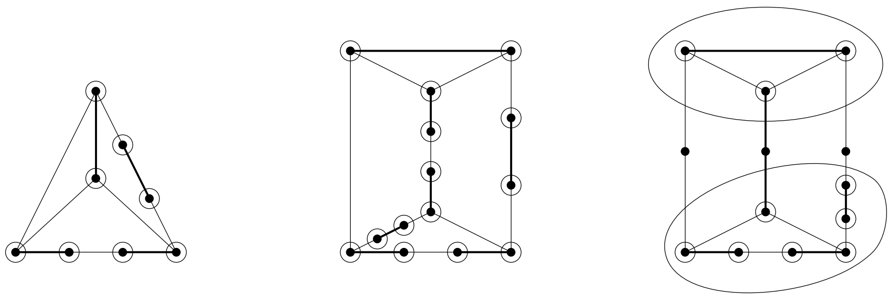


Figure 5: Construction of the join F and of the complete 2-packing of cuts

The proof of the next claim is omitted because of space limitation.

Claim 3. *Let us suppose that the subgraph H and the subpartition $\mathscr{U}$ form an obstruction of G so that $|V(H)| + \sum_{U \in \mathscr{U}} |U|$ is minimum. Let $U \in \mathscr{U}$ be a 3-core. Then there exists no edge of G between a vertex in $N_H(U)$ and a vertex of U that are not neighbors in H.* □

The next definition we need serves purely technical purposes that is in itself not very interesting, but it is useful in the proof, and makes a bridge between the new generation of the results and the previous ones.

Visibly non-Seymour. We will say that the graph G is *visibly non-Seymour* (shortly VNS) if it has a subgraph H containing an edge set F, so that

(a) H is the union of two tight circuits with respect to (G, F),

(b) H is non-bipartite,

(c) the maximum degree is 3 in H,

(d) there exists a complete 2-packing of cuts for (G,F), which contains all the stars (in G) of the vertices whose degree is 3 in H (with multiplicity at least 1).

Matching-covered graphs. We will need some definitions and results from matching theory. A connected graph is called *matching-covered* if each edge belongs to a perfect matching. A connected graph G is called *bicritical* if the graph $G-\{u,v\}$ has a perfect matching for any two different vertices u and v of G; in other worlds: $G-u$ is factor-critical for any vertex u of G. A bicritical graph is *non-trivial* if $|V(G)|>2$. Note that a bicritical graph is matching-covered. Adding an edge between the vertices of a bicritical graph it remains bicritical, and therefore a graph has a bicritical subgraph if and only if it has a bicritical induced subgraph. The graph K_2^3 is defined to be a set of three parallel edges. A circuit C of G is called *nice* if $G-V(C)$ contains a perfect matching.

The following two similar results on matching-covered graphs will be crucial for us. Fact 3 is due to Lovász and Plummer [5]. We mention that none of them implies the other one.

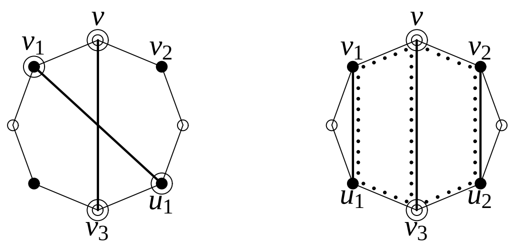
Fact 3. *Every non-bipartite matching-covered graph contains an even subdivision of the K_4 or the prism.*

Lemma 2. *Every* 3*-star of a matching-covered graph belongs to an even subdivision of K_4 or K_2^3.*

Proof. Let G be a minimal counter-example with the 3-star $F:=\{e_1,e_2,e_3\}$ such that $e_i=vv_i$. Since G is matching-covered, each edge e_i belongs to a perfect matching M_i of G. Then there exists a nice circuit in the symmetric difference $M_i\triangle M_j$ of M_i and M_j containing e_i and e_j. Let C be a shortest nice circuit containing two edges of F. We may suppose without loss of generality that e_3 does not belong to C. Since C is a nice circuit, there exists a perfect matching M of $G-V(C)$. Then $M\triangle M_3$ contains odd paths that match the vertices of C. Note that the union of these paths and C is a matching-covered graph containing F so by the minimality of G, it is G. Since we can contract the edges incident to a vertex ($\neq v_i$) of degree two in G, it follows, by the minimality of G, that these paths are just edges and $G=C\cup M_3$. Let v_iu_i be the edge of M_3 incident to v_i. Let A and B be the two colour classes of C with $v\in A$ and $v_1,v_2\in B$. By the minimality of C, $v_3\in A$ and $u_1,u_2\in B$.

Case 1. If vv_3 and v_iu_i (for some $i\in\{1,2\}$) are crossing chords of C then $C+vv_3+v_iu_i$ is an even subdivision of K_4 and it contains F. (See the first graph in Figure 6.)

Case 2. Otherwise, $(vv_1+v_1u_1+C(u_1,v_3))+vv_3+(vv_2+v_2u_2+C(u_2,v_3))$ is an even subdivision of K_2^3 between v and v_3 and it contains F, where $C(a,b)$ denotes the path of C between two vertices a and b of C which does not contain v. (See the second graph in Figure 6.) □



even subdivision of K_4 even subdivision of K_2^3

Figure 6: The configurations in the two cases

A similar observation on odd prisms will also be applied.

Fact 4. *Every* 3*-star of an odd prism belongs to an even or odd subdivision of K_2^3.*

New characterization of Seymour graphs. We squeeze the main results of this paper into one theorem: The prism disappears !

Theorem 2. *The following statements are equivalent for the graph G:*

(i) *G is not a Seymour graph,*

(ii) *G can be factor-contracted to a graph that contains a non-trivial bicritical subgraph,*

(iii) *G can be factor-contracted to a graph that contains an even subdivision of K_4 or of the prism as a subgraph,*

(iv) *G contains an obstruction,*

(v) *G is a visibly non-Seymour graph,*

(vi) *G has a stoc-minor containing an even subdivision of K_4 as a subgraph.*

A tempting additional equivalent statement would be that G can be star-contracted to a graph that contains a bicritical graph on at least four vertices as a subgraph. This is not true as the biprism shows. This graph is not Seymour but contracting any of the stars it becomes Seymour. Yet it does not contain a bicritical subgraph.

3 Algorithmic proof of Theorem 2

(i) implies (ii): Let $G = (V,E)$ be a minimal counter-example that is

(a) G is not a Seymour graph,

(b) every factor-contraction of G is a Seymour graph and

(c) G contains no non-trivial bicritical subgraph.

By (a), there exists a non-empty join F so that no complete packing of cuts exists for (G,F). The following lemma contradicts (c).

Lemma 3. $G' := G(V(F))$ *is a non-trivial bicritical graph.*

Proof. First we show that

$$\text{if } \mathscr{Q} \text{ is a complete 2-packing of cuts for } (G,F) \text{ then no star is contained twice in } \mathscr{Q}. \tag{$*$}$$

To see this suppose to the contrary that $2\delta(v) \subseteq \mathscr{Q}$. Let us contract the full star of v. Then $F^v := F \setminus \delta(v)$ is a join in G^v of size $|F|-1$ because $\mathscr{Q} - 2\delta(v)$ is a complete 2-packing of cuts for (G^v,F^v). Since the graph G^v is obtained from G by a factor-contration, we have by (b) that G^v is a Seymour graph and hence there exists a complete packing $\mathscr{Q}'$ of cuts for (G^v,F^v) and then $\mathscr{Q}' \cup \delta(v)$ is a complete packing of cuts for (G,F), which is a contradiction.

Let x_0 be an arbitrary vertex of F. Let $\mathscr{Q}$ and $C \in \mathscr{Q}$ be the sets provided by (1) of Lemma 1. We recall that $G(C)$ is factor-critical and $C \subseteq V(F) - x_0$. In fact,

$$C = V(F) - x_0. \tag{$**$}$$

To see this suppose to the contrary that $C \subset V(F) - x_0$. Then, since $\delta(C)$ contains only one edge of F, the set $F^C := F \setminus I(C)$ is non-empty. This edge set F^C is a join in G^C, since $\mathscr{Q} \setminus (\{\delta(C)\} \cup \{\delta(c) : c \in C\})$

is a complete 2-packing of cuts for (G^C, F^C). By (b), G^C is a Seymour graph and hence there exists a complete packing $\mathscr{Q}^C$ of cuts for (G^C, F^C). By (2) of Lemma 1, $\delta(v) \in \mathscr{Q}^C$ for some $v \in V(G) \setminus (C \cup N(C))$. Then $(2\mathscr{Q}^C) \cup (\{\delta(C)\} \cup \{\delta(c) : c \in C\})$ is a complete 2-packing of cuts for (G, F), which is a contradiction by $(*)$.

It follows, by $(*)$, that $|C| \neq 1$ and by $(**)$, that $G' - x_0 = G(C)$ is factor-critical for every $x_0 \in V(G')$, that is G' is a non-trivial bicritical graph. □

(ii) implies (iii): By Fact 3, a non-trivial bicritical graph contains an even subdivision of K_4 or the prism.

(iii) implies (iv): Let G satisfy (iii), that is, G can be factor-contracted to a graph $\hat{G}$ that contains an even subdivision H of K_4 or the prism. Then $(H, \emptyset)$ is an obstruction of $\hat{G}$. To show that G satisfies (iv), that is G itself contains an obstruction, we prove the following lemma and then we are done by induction:

Lemma 4. *If G^C ($C \subseteq V$, $G(C)$ is factor-critical) contains an obstruction, then so does G.*

Proof. Let $\hat{G}_1 := G^C$, $(\hat{H}_1, \hat{\mathscr{U}})$ an obstruction of $\hat{G}_1$, H_1 the subgraph of G defined by the edge set of $\hat{H}_1$ and $T := \{v_1, \ldots, v_l\} = V(H_1) \cap N(C)$. Since the vertices of $\hat{H}_1$ are of degree 2 or 3, we have that $l \leq 3$.

Case 1. If $l \leq 1$, then $(H_1, \mathscr{U})$ is an obstruction of G, and we are done.

Thus we may suppose without loss of generality that $l \geq 2$. Let $\hat{G}_2$ be the graph obtained from G by contracting $V - C$. Since $v_i \in N(C)$, we can choose a neighbor u_i of v_i in C for all $i = 1, \ldots, l$. Since $G(C)$ is factor-critical, $G(C) - u_i$ contains a perfect matching and hence $\hat{G}_2$ has a perfect matching M_i containing $v_i u_i$. Then the subgraph $\hat{S}$ of $\hat{G}_2$ induced by the edge set $\bigcup_1^l M_i$ is matching-covered.

Case 2. If $l = 2$, then S is an even path in $G(C \cup T)$ and by Claim 1(a), G contains an obstruction, and we are done.

Thus from now on we suppose that $l = 3$. By Lemma 2, there exists in $\hat{S}$ an even subdivision $\hat{H}_2$ either of the K_4 or of the K_2^3 containing the three edges u_1v_1, u_2v_2 and u_3v_3.

Case 3. If $\hat{H}_2$ is an even subdivision of the K_2^3, then H_2 is an even subdivision of a 3-star with base T in $G(C \cup T)$ and by Claim 1(b), we are done. (Figure 7.)

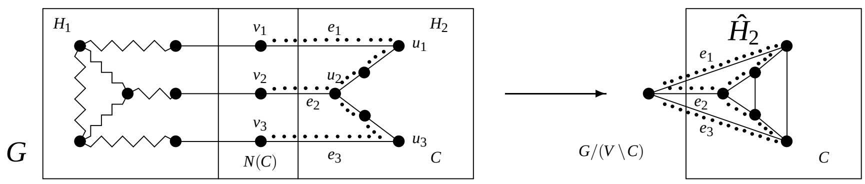


Figure 7: An easy case

So we suppose that $\hat{H}_2$ is an even subdivision of K_4, that is $(\hat{H}_2, \emptyset)$ is a K_4-obstruction in $\hat{S}$.

Case 4. If $\hat{H}_1$ is an odd prism of $\hat{G}_1$, then by Fact 4, there exists an even or odd subdivision $\hat{L}$ of the K_2^3 in $\hat{G}_1$, so L is an even or odd subdivision of the 3-star in H_1 with base T in $G((V - C - N(C)) \cup T)$ and we are done again by Claim 1(b). (Figure 8.)

Case 5. Finally, if $\hat{H}_1$ is an odd K_4 in $\hat{G}_1$ and $\hat{H}_2$ is an even subdivision of K_4 in $\hat{G}_2$, then $H_1 \cup H_2$ is an odd prism of G. If the contracted vertex of $\hat{G}_1$ belongs to one of the cores $\hat{U}_i$ say to $\hat{U}_j$, then let $U_i := \hat{U}_i$ $i \neq j$ and U_j is deleted, otherwise let $U_i := \hat{U}_i$ $1 \leq i \leq k$. Then $(H_1 \cup H_2, \mathscr{U})$ is an obstruction of G because after contracting the sets $U_i \in \mathscr{U}$ we get an even subdivision of the prism or the biprism. (Figure 9.)

□

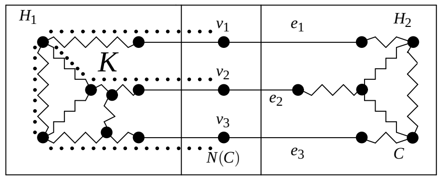


Figure 8: Another easy case

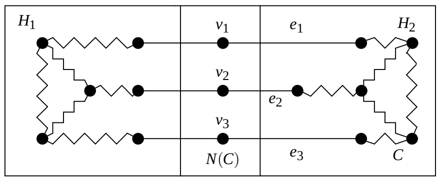


Figure 9: The case of the biprism

(iv) implies (v): Let $(H,\mathscr{U})$ be an obstruction of G, $G' := G/\cup_1^k I_G(U_i)$ and $H' := H/\cup_1^k I_G(U_i)$. By Claim 2, there exist a join F' of G' and two tight circuits C_1' and C_2' whose union is H' and for each 3-core U_i exactly one edge e_i of F' is incident to the vertex u_i that corresponds to U_i in G'. For each 3-core U_i there is a unique edge $v_iw_i \in \delta_H(U_i)$ such that $v_i \in U_i, w_i \in V - U_i$ and e_i is incident to w_i. For each 2-core U_i, let w_iv_i be one of the two edges of the cut $\delta_H(U_i)$ with $v_i \in U_i$. For each core U_i, let F_i be the unique perfect matching of $H(U_i) - v_i$. Let $F := F' \cup_1^k (F_i + v_iw_i)$.

By Lemma 1, there exists a complete 2-packing $\mathscr{Q}_0'$ of cuts for (G',F'). Let $\mathscr{Q}_0$ be the corresponding complete 2-packing of cuts for (G,F'). Let $\mathscr{Q}_i := \{v : v \in U_i\} \cup \{U_i\}$ and $\mathscr{Q} := \cup_0^k \mathscr{Q}_i$. Then $\mathscr{Q}$ is a complete 2-packing of cuts for (G,F) hence F is a join of G.

Moreover, C_1' and C_2' correspond to two tight circuits of G whose union is H. Thus G is VNS.

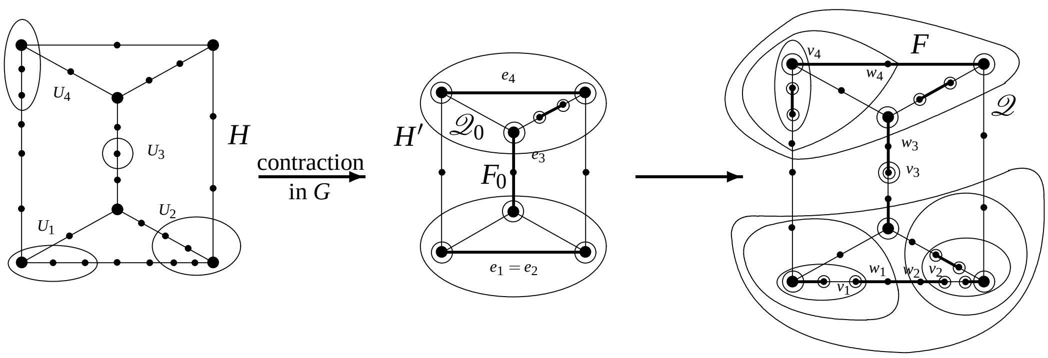


Figure 10: The construction of the join F and the complete 2-packing $\mathscr{Q}$ of cuts

(v) implies (i): Let H be a subgraph of G, F an edge set in H and $\mathscr{Q}$ a complete 2-packing of cuts for (G,F) that show that G is VNS. By (d), F is a join of G, by (a), H is the union of two tight circuits with respect to (G,F), and by (b), H is non-bipartite, so by Fact 1, G is not a Seymour graph.

(iii) implies (vi): By Fact 2, a factor-contraction can be replaced by a series of contractions of odd circuits and then the contraction of a star. Moreover, we can contract an odd circuit in an even subdivion of a prism to get an even subdivision of a K_4. Hence, if G can be factor-contracted to a graph that contains an even subdivision of K_4 or of the prism as a subgraph, then G has a stoc-minor containing an even subdivision of K_4 as a subgraph.

(vi) implies (i): Let $\hat{G}$ be a stoc-minor of G that contains an even subdivision H of K_4. Then any perfect matching of H is a join of $\hat{G}$, and H is the union of two tight circuits with respect to $(\hat{G}, F)$, so by Fact 2, $\hat{G}$ is not a Seymour graph, that is $\hat{G}$ satisfies (i). To show that G satisfies (i) we prove the following lemma and then we are done by induction:

Lemma 5. *Let C be a full star or an odd circuit. If G/C satisfies (i), then so does G.*

Proof. If C is a full star, then since G/C satisfies (i), it satisfais **(iii)**. A full star contraction is a factor-contraction, so G also satisfies **(iii)** and hence (i).

From now on C is an odd circuit. Since G/C satisfies (i), it satisfais **(iv)**, that is G/C contains a K-obstruction $(\hat{H}, \mathscr{U})$ where K is a K_4, prism or biprism. Take a minimal one as in Claim 3.

Case 1: If $|V(C) \cap V(H)| \leq 2$ and c is not in a separator of a biprism. If $c \in V \setminus \cup N_H(U_i)$, then the obstruction $\hat{H}$ can be extended by the even part of C, that is G also satisfies **(iv)** and hence (i). Otherwise, $c \in N_H(U_i)$ for some i, that is H contains an edge cy so that $y \in U_i$.

If there exists no edge of G between U_i and the even path C_2 of the cycle C between c_1 and c_2 then the obstruction $\hat{H}$ can be extended by C_2, that is G also satisfies **(iv)** and hence (i). Otherwise, take that edge xc_3 of G with $x \in U_i$ for which c_3 is on C_2 and the distance d of c_2 and c_3 on C_2 is as small as possible. By Claim 3, $x = y$. If d is even then for the edge yc_3 we are in the above case. If d is odd, then changing H as shown on Figure 11, where Q is an odd path in H, and deleting U from $\mathscr{U}$, we get a K_4-obstruction.

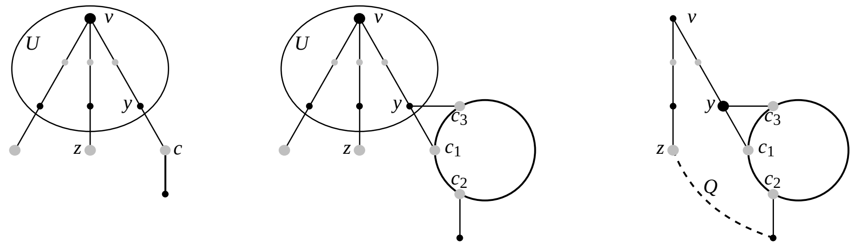


Figure 11:

Case 2: If $|V(C) \cap V(H)| = 2$ and c is in a separator of a biprism. Let a and b be the two vertices of C that are incident to the two edges of H incident to c. Then a and b partition C into an odd path C_1 and an even path C_2. If $a = b$, then $(\hat{H}, \mathscr{U})$ remains a biprism-obstruction in G, so G satisfies **(iv)** and hence G also satisfies (i). If $a \neq b$, then let H' be obtained from $\hat{H}$ by adding C_1 and deleting the path between v_5 and v_6 defined in Figure 12. Let $\mathscr{U}'$ be obtained from $\mathscr{U}$ by deleting those cores that correspond to inner vertices of the deleted path. Then $(H', \mathscr{U}')$ is a K_4-obstruction in G, so G satisfies **(iv)** and hence G also satisfies (i). (See Figure 12)

Case 3: If $|V(C) \cap V(H)| = 3$. Then since G/C satisfies (iv), it satisfais **(v)** with the same $\hat{H}$. By (d) of VNS for c, there exists exactly one edge e in F incident to c and both tight circuits contains e. Let u be the end vertex of e in C, M a perfect matching of $C - u$, $F^* = F \cup M$ and $H^* = H \cup C$. The vertices of H partition C into 3 paths. If exatly one of them is of odd length, then delete from H^*, F^* and C that odd path. Let $\mathscr{Q}^* := \mathscr{Q} \setminus \delta(c) \cup \{\delta(x) : x \in C'\}$. Then H^* satisfies (a), (b), (c) and (d) of VNS, so G satisfies **(v)** and hence (i). (See Figure 13) □

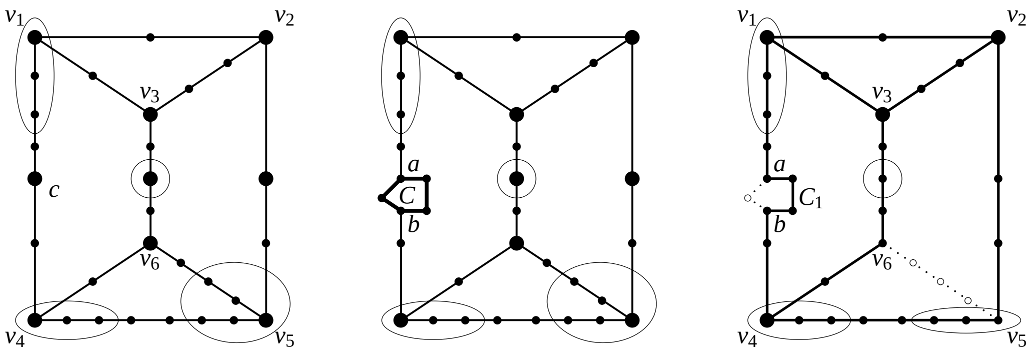


Figure 12: Finding a K_4-obstruction

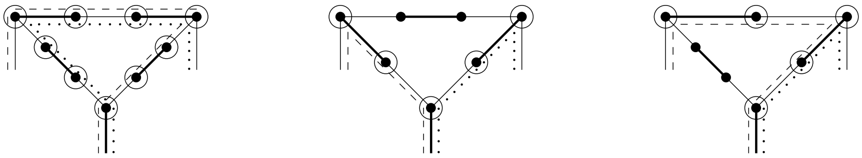


Figure 13: Extensions of F, the tight circuits and $\mathscr{Q}$

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