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On the graphic matroid parity problem

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Abstract

A relatively simple proof is presented for the min–max theorem of Lovász on the graphic matroid parity problem.

1. Introduction

The graph matching problem and the matroid intersection problem are two well-solved problems in combinatorial theory in the sense of min–max theorems [2,3] and polynomial algorithms [3,4] for finding an optimal solution. The matroid parity problem, a common generalization of them, turned out to be much more difficult. For the general problem there does not exist polynomial algorithm [6,8]. Moreover, it contains NP-complete problems. On the other hand, for linear matroids Lovász provided a min–max formula in [8] and a polynomial algorithm in [7]. There are several earlier results which can be derived from Lovász' theorem, e.g. Tutte's result on f-factors [15], a result of Mader on openly disjoint A-paths [11] (see [9]), a result of Nebesky concerning maximum genus of graphs [12] (see [5]), and the problem of Lovász on cacti [9]. This latter one is a special case of the graphic matroid parity problem. Our aim is to provide a simple proof for the min–max formula on this problem, i.e. on the graphic matroid parity problem. In an earlier paper [14] of the present author the special case of cacti was considered. We remark that we shall apply the matroid intersection theorem of Edmonds [4]. We refer the reader to [13] for basic concepts on matroids.

For a given graph G, the *cycle matroid* $\mathscr{G}$ is defined on the edge set of G in such a way that the independent sets are exactly the edge sets of the forests of G. Thus, for

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the rank function $\boldsymbol{r_G}$ of $\mathscr{G}$ and for an edge set F of G,

$$r_G(F) = |V(G)| - c(G[F]), \tag{1}$$

where $\boldsymbol{c}(\boldsymbol{H})$ denotes the number of connected components of a graph H and $G[F] = (V(G), F)$. In other words, $r_G(F)$ is the maximum size of a forest contained in F. A matroid $\mathscr{M}$ is *graphic* if there exists a graph whose cycle matroid is $\mathscr{M}$.

The graphic matroid parity problem is the following. Given a graph G and a partition $\mathscr{V}$ of its edge set into pairs, what is the maximum size of a forest in G which consists of pairs in $\mathscr{V}$. The pair $(\boldsymbol{G}, \mathscr{V})$ is called *v-graph*. A *v-forest* of $(G, \mathscr{V})$ is a forest of G consisting of v-pairs in $\mathscr{V}$. The *v-size* of a v-forest is the number of v-pairs contained in it. The *graphic matroid parity problem* consists of finding the maximum v-size $\boldsymbol{\beta}(\boldsymbol{G}, \mathscr{V})$ of a v-forest in a v-graph $(G, \mathscr{V})$.

Let $(G, \mathscr{V})$ be a v-graph. Let $\mathscr{P} := \{V_1, V_2, \ldots, V_l\}$ be a partition of the vertex set $V(G)$ and let $\mathscr{Q} := \{H_1, H_2, \ldots, H_k\}$ be a partition of $\mathscr{V}$. We say that $(\mathscr{P}, \mathscr{Q})$ is a *cover* of $(G, \mathscr{V})$. For a partition $\mathscr{P}$ of V, $\boldsymbol{V_\mathscr{P}}$ will denote the vertex set obtained from V by contracting each set V_i in $\mathscr{P}$ into one vertex v_i. Note that $|V_\mathscr{P}| = |\mathscr{P}| = l$. Let $\boldsymbol{G_\mathscr{P}} := (V_\mathscr{P}, E(G))$. For $H_i \subseteq \mathscr{V}$, $(\boldsymbol{G_\mathscr{P}}[\boldsymbol{H_i}], \boldsymbol{H_i})$ will denote the v-graph on the vertex set $V_\mathscr{P}$ for which the edge set $E(H_i)$ of $G_\mathscr{P}[H_i]$ is the union of the edges of the v-pairs in H_i. For $H_i \subseteq \mathscr{V}$, let $\boldsymbol{r_\mathscr{P}}(\boldsymbol{H_i}) = r_{G_\mathscr{P}}(E(H_i))(= l - c(G_\mathscr{P}[H_i]))$. The value $val(\mathscr{P}, \mathscr{Q})$ of a cover is defined as follows. Let $n = |V(G)|$, $l = |\mathscr{P}|$ and $k = |\mathscr{Q}|$.

$$val(\mathscr{P}, \mathscr{Q}) := n - l + \sum_{H_i \in \mathscr{Q}} \left\lfloor \frac{r_\mathscr{P}(H_i)}{2} \right\rfloor. \tag{2}$$

Now, we are able to present the min–max result of Lovász [7] in our terminology.

Theorem 1. *Let $(G, \mathscr{V})$ be a v-graph. Then $\beta(G, \mathscr{V}) = \min\{val(\mathscr{P}, \mathscr{Q})\}$, where the minimum is taken over all covers $(\mathscr{P}, \mathscr{Q})$ of $(G, \mathscr{V})$.*

We mention that the min–max formula for the special case of cacti is presented in [10, Theorem 11.3.6]. Theorem 1 is the natural generalization of that formula. To see that the problem of cacti, i.e. finding a maximum triangular cactus in a graph G, is a special case of the graphic matroid parity problem we have to consider the v-graph $(G', \mathscr{V})$ which is defined by the original graph G as follows: Let us denote by $\mathscr{T}$ the set of triangles of G. For every $T \in \mathscr{T}$, let e_T, f_T be two edges of T. Let $\mathscr{V} := \{(e_T, f_T) : T \in \mathscr{T}\}$ and let $G' := (V(G), E(G'))$ where $E(G') := \bigcup_{T \in \mathscr{T}} \{e_T, f_T\}$ where the union is understood by multiplicity.

Our proof follows the line of Gallai's (independently Anderson's [1]) proof for Tutte's theorem on the existence of perfect matchings.

2. Definitions

A v-forest F of a v-graph $(G, \mathscr{V})$ is called *perfect* if it is a spanning forest of G, that is $|F| = r_G(E(G))$. A forest F is said to be *almost spanning* if $|F| = r_G(E(G)) - 1$. A v-forest is *almost perfect* if it is almost spanning. For an edge set F of a v-graph

$(G, \mathscr{V})$, the maximum v-size of a v-forest contained in F is denoted by $v_{\mathscr{V}}(F)$. Note that

$$v_{\mathscr{V}}(F) \leqslant \left\lfloor \frac{r_G(F)}{2} \right\rfloor. \tag{3}$$

A v-graph $(G, \mathscr{V})$ will be called *critical* if by identifying any two vertices in the same connected component of G, the v-graph obtained has a perfect v-forest. In particular, this means that in a critical v-graph there exists an almost perfect v-forest. Critical v-graphs will play an important role in the proof, like factor critical graphs play the key role in the proof of Tutte's theorem.

For a cover $(\mathscr{P}, \mathscr{Q})$ of a v-graph $(G, \mathscr{V})$, let us denote by $\mathscr{V}_{\mathscr{P}}, \mathscr{S}_{\mathscr{P}}$ and $\mathscr{R}_{\mathscr{P}}$ the sets of v-pairs T in $\mathscr{V}$ for which $r_{\mathscr{P}}(T) = 2, r_{\mathscr{P}}(T) = 1$ and $r_{\mathscr{P}}(T) = 0$. (Then $\mathscr{V} = \mathscr{V}_{\mathscr{P}} \cup \mathscr{S}_{\mathscr{P}} \cup \mathscr{R}_{\mathscr{P}}$.) The elements $H_i \in \mathscr{Q}$ with $r_{\mathscr{P}}(H_i) \geqslant 1$ are called *components* of the cover. A *component* $H_i \in \mathscr{Q}$ is said to be *critical* if the v-graph $(G_{\mathscr{P}}[H_i], H_i)$ is critical.

For a graph $H = (U, F)$, we shall denote by $\sim_H$ the equivalence relation for which $u \sim_H v$ for $u, v \in U$ if and only if there exists a path connecting u and v in H. The partition of U defined by the equivalence classes of $\sim_H$, that is by the vertex sets of the connected components of H, is denoted by part (H).

We say that the partition $\mathscr{P}$ of $V(G)$ is the *trivial partition* if $l = n$. The cover $(\mathscr{P}, \mathscr{Q})$ is the *trivial cover* if $l = n$ and $k = 1$. Let $\mathscr{P}' = \{V_1^1, \ldots, V_1^{r_1}, V_2^1, \ldots, V_2^{r_2}, \ldots, V_l^1, \ldots, V_l^{r_l}\}$, where $\bigcup_j V_i^j = V_i$ for all i, then the partition $\mathscr{P}'$ is called a *refinement* of the partition $\mathscr{P}$. If $\mathscr{P}'$ is a refinement of $\mathscr{P}$ so that $|\mathscr{P}'| = |\mathscr{P}| + 1$, then we say it is an *elementary refinement*. If $V_i \in \mathscr{P}$ then the partition obtained from $\mathscr{P}$ by replacing V_i by its singletons will be denoted by $\mathscr{P} \div \{V_i\}$. If $\mathscr{P}'$ is a refinement of $\mathscr{P}$, then $\mathscr{P}$ corresponds to a partition of $V(G_{\mathscr{P}'})$. This partition will be denoted by $\mathscr{P}/\mathscr{P}'$.

We shall need later two auxiliary graphs B and D. These graphs will depend on a v-graph $(G, \mathscr{V})$ and a cover $(\mathscr{P}, \mathscr{Q})$ of this v-graph. We suppose that for each component H_i, $r_{\mathscr{P}}(H_i)$ is odd. First we define the graph $B = (V(G), E(B))$. $e = uv$ will be an edge of B if and only if there exist $u, v \in \boldsymbol{V_j} \in \mathscr{P}$, a component $\boldsymbol{H_i} \in \mathscr{Q}$ and a v-forest $\boldsymbol{K}$ in $(G_{\mathscr{P} \div \{V_j\}}[H_i], H_i)$ of v-size $(r_{\mathscr{P}}(H_i) + 1)/2$ so that $u \sim_K v$ but for every pair $(x, y) \neq (u, v)$ of vertices from V_j, $x \nsim_K y$. (Note that $(V_{\mathscr{P}}, E(K))$ contains a v-forest of v-size $(r_{\mathscr{P}}(H_i) - 1)/2$ in $(G_{\mathscr{P}}[H_i], H_i)$. We mention that (by Lemma 8, see later) $(G_{\mathscr{P}}[H_i], H_i)$ will always contain a v-forest of v-size $(r_{\mathscr{P}}(H_i) - 1)/2$.) In other words, the *trace* of the v-forest K in $V_j \in \mathscr{P}$ is the edge e. We call this edge e an *augmenting edge* for H_i. We will call the edges of B *augmenting edges*. Note that an edge of B may be augmenting for more than one component $H_i \in \mathscr{Q}$. For a refinement $\mathscr{P}'$ of $\mathscr{P}$, the set $\boldsymbol{A}_{\mathscr{P}'} \subseteq E(B)$ of augmenting edges connecting vertices in different sets of $\mathscr{P}'$ will be called the augmenting edges *with respect to the refinement* $\mathscr{P}'$.

The second auxiliary graph D will be a bipartite graph with colour classes $E(B)$ (the edge set of B) and $\mathscr{Q}$. Two vertices $e \in E(B)$ and $H_i \in \mathscr{Q}$ are connected in D by an edge if and only if e is an augmenting edge for H_i. As usual, the set of neighbours of a vertex set X of one of the colour classes of D will be denoted by $\Gamma_D(X)$.

3. Outline of the proof

In this section, we present some ideas of the proof. As it was mentioned earlier, we shall follow the proof of Tutte's theorem. Let us briefly summarize the steps of this proof. We suppose that the Tutte condition is satisfied for a given graph G and we have to construct a perfect matching of G. Let X be a maximal set satisfying the condition with equality. The maximality of X implies that all the components of $G-X$ are factor-critical, thus it is enough to find a perfect matching in an auxiliary bipartite graph D, where one of the color classes corresponds to X while the other to the (critical) components. Hall's theorem (or the matroid intersection theorem) provides easily the existence of a perfect matching M in D. The desired perfect matching of G can be obtained from M and from the almost perfect matchings of the critical components. We mention that this is a lucky case because the union of these almost perfect matchings will be automatically a matching in G.

In the case of v-forests we shall prove directly the min–max theorem. We shall choose a minimum cover $(\mathscr{P},\mathscr{Q})$ of $(G,\mathscr{V})$ which is maximal in some certain sense. This will imply (Lemma 7) that the minimum cover of $(G_{\mathscr{P}}[H_i],H_i)$ is unique for each component H_i. This fact has two consequences (Lemmas 8 and 11), namely (i) each component is critical (hence $r_{\mathscr{P}}(H_i)$ is odd) and (ii) for any component H_i and for any refinement $\mathscr{P}'$ of $\mathscr{P}$, either there exists an augmenting edge for H_i with respect to $\mathscr{P}'$ or $r_{\mathscr{P}'}(H_i)\leqslant r_{\mathscr{P}}(H_i)$.

We shall construct a v-forest of size $val(\mathscr{P},\mathscr{Q})$ in $(G,\mathscr{V})$ as follows. (1) For $n-l$ components $H_i\in\mathscr{Q}$, we shall find a v-forest K_i in $(G_{\mathscr{P}\div\{V_j\}}[H_i],H_i)$ of v-size $(r_{\mathscr{P}}(H_i)+1)/2$ for some $V_j\in\mathscr{P}$ so that the trace of K_i in V_j is an edge and the corresponding augmenting edges form a spanning forest of the auxiliary graph B. (We shall see (Corollary 13) that the size of a spanning forest of B is indeed $n-l$.) (2) For the other components H_j we shall need an almost perfect v-forest in $(G_{\mathscr{P}}[H_j],H_j)$, and (3) the union of all of these forests will be a forest in G, that is a v-forest in $(G,\mathscr{V})$ of size $val(\mathscr{P},\mathscr{Q})$. Using (i), for the latter components H_j it is enough to find an arbitrary almost spanning forest in $G_{\mathscr{P}}[H_j]$ (and then using that H_j is critical, this forest can be replaced by a convenient v-forest containing the same number of edges, that is of v-size $(r_{\mathscr{P}}(H_j)-1)/2$). By the definition of augmenting edge, for the former components H_i it is enough to find an arbitrary spanning forest in $G_{\mathscr{P}}[H_i]$ so that $(*)$ there exist augmenting edges for these components whose union will be a spanning forest of B. Thus we have to find a forest F in G so that (a) for each component H_j, $E(F)\cap E(G_{\mathscr{P}}[H_j])$ is a spanning or an almost spanning forest in $G_{\mathscr{P}}[H_j]$, (b) for $n-l$ components H_i we have a spanning forest, (c) for these components in (b), $(*)$ is satisfied.

The existence of a forest with (a) and (b) can be proved, using (ii), by a matroid partition theorem (for a graphic matroid and a truncated partitional matroid). We shall see in Lemma 16 that if for all such forests, we consider the components where the corresponding forests are spanning forests then we get the set of bases of a matroid on the set of indices of the components.

Two matroids will be defined on the edge set of the auxiliary graph D, one of them will be defined by the above introduced matroid, and the other one will be defined by

the cycle matroid of B. The matroid intersection theorem will provide a forest of G with (a)–(c). As we mentioned earlier, each part of the forest, which corresponds to a component, can be replaced by a convenient v-forest, and thus the desired v-forest would have been found.

4. Graphic matroid

In this section, we present some simple properties of forests and of the graphic matroid rank function.

Claim 2. *Let $H=(U,F)$ be a graph. Then*

(a) *for $F''\subseteq F'\subseteq F$, $r(F'')\leqslant r(F')$,*
(b) *for $F',F''\subseteq F$, $r(F'\cup F'')\leqslant r(F')+r(F'')$.*

Lemma 3. *Let $H=(U,F)$ be a graph, let $\mathscr{P}$ be a partition of U and let F_0 be an edge set such that $part(F_0)=\mathscr{P}$. Then*

(a) $r(F\cup F_0)=r_{\mathscr{P}}(F)+r(F_0)$,
(b) $r_{\mathscr{P}}(F)\leqslant r(F)$,
(c) $r(F)\leqslant|U|-|\mathscr{P}|+r_{\mathscr{P}}(F)$,
(d) *if $r_{\mathscr{P}}(F)<r(F)$ then there exists an elementary refinement $\mathscr{P}''$ of $\mathscr{P}$ such that $r_{\mathscr{P}}(F)<r_{\mathscr{P}''}(F)$.*

Proof. (a) By (1) and by $part(F_0)=\mathscr{P}$, $r(F\cup F_0)=n-c(F\cup F_0)=n-c_{\mathscr{P}}(F\cup F_0)=(|\mathscr{P}|-c_{\mathscr{P}}(F))+(n-|\mathscr{P}|)=r_{\mathscr{P}}(F)+r(F_0)$.

(b) By Lemma 3(a) and Claim 2(b), $r_{\mathscr{P}}(F)=r(F\cup F_0)-r(F_0)\leqslant r(F)$.

(c) By Claim 2(a) and Lemma 3(a), $r(F)\leqslant r(F\cup F_0)=r_{\mathscr{P}}(F)+r(F_0)=r_{\mathscr{P}}(F)+|U|-|\mathscr{P}|$.

(d) Let F' be a spanning forest of $(V_{\mathscr{P}},F)$. Then $r_{\mathscr{P}}(F)=|F'|$. Since $r(F)>r_{\mathscr{P}}(F)=|F'|$ there exists an edge $f\in F-F'$ such that $(V,F'\cup f)$ is a forest. $(V_{\mathscr{P}},F'\cup f)$ contains a unique cycle C and $f\in E(C)$ because F' is a forest. On the other hand $(V,F'\cup f)$ contains no cycle so there is a vertex $v_i\in V_{\mathscr{P}}(C)$ such that the two edges of C incident to v_i are incident to different vertices of V_i say a and b. But then for the elementary refinement $\mathscr{P}''\coloneqq(\mathscr{P}-V_i)\cup a\cup(V_i-a)$, $(V_{\mathscr{P}},F'\cup f)$ is a forest and hence $r_{\mathscr{P}''}(F)\geqslant|F'\cup f|>r_{\mathscr{P}}(F)$. □

Claim 4. *Let F be a forest and let F_1 and F_2 be two vertex disjoint subtrees of F. If F_1 and F_2 belong to the same connected component of F then let us denote by a and b the two end vertices of the shortest path in F connecting F_1 and F_2, otherwise let $a\in V(F_1)$ and $b\in V(F_2)$ be two arbitrary vertices. Then*

(a) *if F_1' is a tree on $V(F_1)$ then $(F-E(F_1))\cup E(F_1')$ is a forest.*

(b) *If F' is a forest on $V(F_1)\cup V(F_2)$ so that a and b are in different connected components of F', then $(F-E(F_1)-E(F_2))\cup E(F')$ is a forest.*

The proof of Claim 4 is a simple exercise, it is left to the reader.

Lemma 5. *Let (V,F) be a forest. Let $F_0=\{e_1,\dots,e_k\},F_1,\dots,F_k$ disjoint edge sets of F and let $F'_1,\dots,F'_k$ be disjoint edge sets on V. Let $\mathscr{P}:=part(V,F_0)$. Suppose that for all $1\leqslant i\leqslant k$ the following conditions are satisfied*:

(i) $|F'_i|=|F_i|+1$,
(ii) $part(V_{\mathscr{P}},F'_i)=part(V_{\mathscr{P}},F_i)$,
(iii) *if e_i is in $V_j\in\mathscr{P}$, then $(V_{\mathscr{P}\div\{V_j\}},F'_i)$ is a forest whose trace in V_j is e_i.*

Then $F^*:=(F-\bigcup_0^k F_i)\cup\bigcup_1^k F'_i$ is a forest of size $|F|$.

Proof. By the disjointness of the sets F_i (resp. F'_i), by (i) and by $|F_0|=k$, $|F^*|=|(F-\bigcup_0^k F_i)\cup\bigcup_1^k F'_i|=|F|-\sum_0^k|F_i|+\sum_1^k|F'_i|=|F|-|F_0|-\sum_1^k|F_i|+\sum_1^k(|F_i|+1)=|F|-|F_0|-\sum_1^k|F_i|+\sum_1^k|F_i|+k=|F|$. By Claim 4(a) applied for each connected component of F_i for every i, we obtain that $r_{\mathscr{P}}(F-F_0)=r_{\mathscr{P}}(F^*)$. Then, by Lemma 3(a), $|F^*|=|F|=r(F)=r((F-F_0)\cup F_0)=r_{\mathscr{P}}(F-F_0)+r(F_0)=r_{\mathscr{P}}(F^*)+r(F_0)=r(F^*\cup F_0)$. Thus it is enough to prove that $r(F^*\cup F_0)=r(F^*)$, in other words $u_i\sim_{(V,F^*)}w_i$ for every $u_iw_i\in F_0$. Suppose on the contrary that there exists an edge $u_{j_1}w_{j_1}\in F_0$ (say $u_{j_1},w_{j_1}\in V_{j_1}$) such that $u_{j_1}\nsim_{(V,F^*)}w_{j_1}$. By (iii), $(V_{\mathscr{P}\div\{V_{j_1}\}},F'_{j_1})$ contains a path P_{j_1} connecting u_{j_1} and w_{j_1}. Since $E(P_{j_1})\subseteq F'_{j_1}\subseteq F^*$ but $u_{j_1}\nsim_{V,F^*}w_{j_1}$ it follows that there exists a vertex $v_{j_2}\in V(P_{j_1})$ $(j_2\neq j_1)$ and an edge $u_{j_2}w_{j_2}\in F_0$ (with $u_{j_2},w_{j_2}\in V_{j_2}$) such that $u_{j_2}\nsim_{(V,F^*)}w_{j_2}$. Note that P_{j_1} connects v_{j_1} and v_{j_2} in $(V_{\mathscr{P}},F'_{j_1})$ and hence, by (ii), there exists a path Q_{j_1} in $(V_{\mathscr{P}},F_{j_1})$ connecting v_{j_1} and v_{j_2}. The same way, there exists an edge $u_{j_3}w_{j_3}\in F_0$ $(j_3\neq j_2)$ (say $u_{j_3},w_{j_3}\in V_{j_3}$) such that $u_{j_3}\nsim_{(V,F^*)}w_{j_3}$ and there exists a path Q_{j_2} in $(V_{\mathscr{P}},F_{j_2})$ connecting v_{j_2} and v_{j_3}. We can continue the same way. Since $|F_0|$ is finite there exist indices $s<t$ such that $v_s=v_{t+1}$. Then using that the paths Q_i connect v_{j_i} and $v_{j_{i+1}}$ for every $s\leqslant i\leqslant t$ in $(V_{\mathscr{P}},F_{j_i})$ and that these paths are edge disjoint it follows that $C:=Q_{j_s}\cup Q_{j_{s+1}}\cup\cdots\cup Q_{j_t}$ is a cycle in $(V_{\mathscr{P}},F-F_0)$. This is a contradiction because, by Lemma 3(a), $|F-F_0|=|F|-|F_0|=r(F)-r(F_0)=r_{\mathscr{P}}(F-F_0)$ and hence $(V_{\mathscr{P}},F-F_0)$ is a forest. □

We mention that Lemma 5 will be applied only in the very last step of the proof.

5. The proof

Proof ($\max\leqslant\min$). The following lemma proves this direction. □

Lemma 6. *For a forest F in G and for a cover $(\mathscr{P}, \mathscr{Q})$ of $(G, \mathscr{V})$, we have* $v_{\mathscr{V}}(F) \leqslant (n-l) + \sum_{H_i \in \mathscr{Q}} \lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor$.

Proof. Let F' be a subset of F of maximum size so that F' is a forest in $G_{\mathscr{P}}$. By Lemma 3(c), $|F| = r(F) \leqslant n - l + r_{\mathscr{P}}(F) = n - l + |F'|$. Since the number of $\mathscr{V}$-pairs in F is equal to the number of $\mathscr{V}$-pairs in F' plus the number of $\mathscr{V}$-pairs f_1, f_2 in F for which at most one of f_1 and f_2 belongs to F', we have

$$v_{\mathscr{V}}(F) \leqslant v_{(\mathscr{V}_{\mathscr{P}})}(F') + |F - F'| \leqslant v_{(\mathscr{V}_{\mathscr{P}})}(F') + n - l. \tag{4}$$

For each $H_i \in \mathscr{Q}$, let $H_i' := (H_i)_{\mathscr{P}}$ and let $F_i' := F' \cap E(H_i)$. Then F_i' is a forest in $G_{\mathscr{P}}[H_i]$ and so $r_{\mathscr{P}}(F_i') \leqslant r_{\mathscr{P}}(H_i)$. Whence, by (3), $v_{H_i'}(F_i') \leqslant \lfloor \frac{r_{\mathscr{P}}(F_i')}{2} \rfloor \leqslant \lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor$. As $\mathscr{Q}$ is a partition of $\mathscr{V}$, for every v-pair T contained in F' there exists some $H_i \in \mathscr{Q}$ such that $T \in H_i'$. Thus

$$v_{(\mathscr{V}_{\mathscr{P}})}(F') = \sum_{H_i \in \mathscr{Q}} v_{H_i'}(F_i') \leqslant \sum_{H_i \in \mathscr{Q}} \left\lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \right\rfloor. \tag{5}$$

Eqs. (4) and (5) imply the desired inequality. □

Remark 1. It follows from the proof of Lemma 6 that if $(\mathscr{P}, \mathscr{Q})$ is a cover of $(G, \mathscr{V})$ and F is a v-forest of G of size $val(\mathscr{P}, \mathscr{Q})$ then we have equality in (4). It follows that for every $\mathscr{V}$-pair f_1, f_2 in F at least one of f_1 and f_2 belongs to F', that is if $T \in \mathscr{R}_{\mathscr{P}}$ then T is not contained in F.

Proof (max ⩾ min). We prove the theorem by induction on $n + |\mathscr{V}|$. For $n = 3$ the result is trivially true. It is also true when $|\mathscr{V}| = 1$. In what follows we suppose that $n \geqslant 4$ and $|\mathscr{V}| \geqslant 2$.

Let $(\mathscr{P}, \mathscr{Q})$ be a minimum cover of $(G, \mathscr{V})$ for which $|\mathscr{P}| = l$ is as small as possible and subject to this $|\mathscr{Q}| = k$ is as large as possible. Note that by the maximality of k,

$$\text{for each pair } T \in \mathscr{S}_{\mathscr{P}} \cup \mathscr{R}_{\mathscr{P}}, \quad T \in \mathscr{Q}, \tag{6}$$

because for each $T \in \mathscr{S}_{\mathscr{P}} \cup \mathscr{R}_{\mathscr{P}}$, $\lfloor \frac{r_{\mathscr{P}}(T)}{2} \rfloor = 0$.

Lemma 7. *For each $H_i \in \mathscr{Q}$, the minimum cover of $(G_{\mathscr{P}}[H_i], H_i)$ is unique and it is the trivial cover.*

Proof. Let $(\mathscr{P}', \mathscr{Q}')$ be a minimum cover of $(G_{\mathscr{P}}[H_i], H_i)$. Since the value of the trivial cover of $(G_{\mathscr{P}}[H_i], H_i)$ is $\lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor$, $l - l' + \sum_{H_j' \in \mathscr{Q}'} \lfloor \frac{r_{\mathscr{P}'}(H_j')}{2} \rfloor = val(\mathscr{P}', \mathscr{Q}') \leqslant \lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor$. Using this cover $(\mathscr{P}', \mathscr{Q}')$, a new cover $(\mathscr{P}^*, \mathscr{Q}^*)$ of $(G, \mathscr{V})$ can be defined as follows. Let the partition $\mathscr{P}^*$ of $V(G)$ be obtained from $\mathscr{P}$ by taking the union of all those V_r and V_s whose corresponding vertices in $G_{\mathscr{P}}$ are in the same set of $\mathscr{P}'$. Then $l^* = |\mathscr{P}^*| = |\mathscr{P}'| = l'$. Let $\mathscr{Q}^*$ be obtained from $\mathscr{Q}$ by deleting H_i and by adding $\mathscr{Q}'$. For $H_j \in \mathscr{Q} - \{H_i\}$, $r_{\mathscr{P}'}(H_j) \leqslant r_{\mathscr{P}}(H_j)$ by Lemma 3(b). We claim that the new cover is also

a minimum cover,

$$\begin{aligned} val(\mathscr{P}^*, \mathscr{Q}^*) &= n - l^* + \sum_{H_j \in \mathscr{Q} - \{H_i\}} \left\lfloor \frac{r_{\mathscr{P}'}(H_j)}{2} \right\rfloor + \sum_{H'_j \in \mathscr{Q}'} \left\lfloor \frac{r_{\mathscr{P}'}(H'_j)}{2} \right\rfloor \\ &\leqslant n - l' + \sum_{H_j \in \mathscr{Q} - \{H_i\}} \left\lfloor \frac{r_{\mathscr{P}}(H_j)}{2} \right\rfloor + (val(\mathscr{P}', \mathscr{Q}') - (l - l')) \\ &\leqslant n - l + \sum_{H_j \in \mathscr{Q} - \{H_i\}} \left\lfloor \frac{r_{\mathscr{P}}(H_j)}{2} \right\rfloor + \left\lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \right\rfloor = val(\mathscr{P}, \mathscr{Q}). \end{aligned}$$

It follows that equality holds everywhere, so $val(\mathscr{P}', \mathscr{Q}') = \lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor$, thus the trivial cover of $(G_{\mathscr{P}}[H_i], H_i)$ is a minimum cover. Furthermore, by the minimality of l, $|\mathscr{P}| \leqslant |\mathscr{P}^*| = |\mathscr{P}'| \leqslant |\mathscr{P}|$ that is $\mathscr{P}'$ is the trivial partition of $V(G_{\mathscr{P}}[H_i])$ and by the maximality of k, $|\mathscr{Q}| \geqslant |\mathscr{Q}^*| = |\mathscr{Q}| - 1 + |\mathscr{Q}'| \geqslant |\mathscr{Q}|$ that is $|\mathscr{Q}'| = 1$, whence the minimum cover of $(G_{\mathscr{P}}[H_i], H_i)$ is unique. □

Lemma 8. *Each component $H_i \in \mathscr{Q}$ is critical.*

Proof. Suppose that there exists a component $H_i \in \mathscr{Q}$ for which $(G_{\mathscr{P}}[H_i], H_i)$ is not critical, that is there are two vertices a and b in the same connected component of $G_{\mathscr{P}}[H_i]$ whose identification into a new vertex v_{ab} leaves a v-graph (G', H_i) with no perfect v-forest. Note that $r_{G'}(E(H_i)) = r_{\mathscr{P}}(H_i) - 1$. By the induction hypothesis, it follows that there is a cover $(\mathscr{P}', \mathscr{Q}')$ of (G', H_i) so that $val_{G'}(\mathscr{P}', \mathscr{Q}') < \frac{r_{G'}(E(H_i))}{2} \leqslant \lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor$. Let $\mathscr{P}'' := (\mathscr{P}' - X) \cup ((X - v_{ab}) \cup a \cup b)$, where X is the member of $\mathscr{P}'$ that contains v_{ab}. Then $(\mathscr{P}'', \mathscr{Q}')$ is a cover of $(G_{\mathscr{P}}[H_i], H_i)$ and $val(\mathscr{P}'', \mathscr{Q}') = val(\mathscr{P}', \mathscr{Q}') + 1$. Thus $val(\mathscr{P}'', \mathscr{Q}') \leqslant \lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor$. By Lemma 7, $(\mathscr{P}'', \mathscr{Q}')$ is a minimum cover of $(G_{\mathscr{P}}[H_i], H_i)$ but not the trivial one (a and b are in the same member of $\mathscr{P}''$), which contradicts Lemma 7. □

Corollary 9. *If $H_i \in \mathscr{Q}$ and $a, b \in V(G_{\mathscr{P}}[H_i])$, then there exists an almost perfect v-forest K in $(G_{\mathscr{P}}[H_i], H_i)$ so that a and b belong to different connected components of K.*

Proof. If a and b are in the same connected component of $G_{\mathscr{P}}[H_i]$, then let $c = a$ and $d = b$, otherwise let c and d be two arbitrary vertices from a connected component of $G_{\mathscr{P}}[H_i]$. By Lemma 8, H_i is critical, so by identifying c and d in $G_{\mathscr{P}}[H_i]$, the v-graph obtained has a perfect v-forest K'. Then $K := (V(G_{\mathscr{P}}), E(K'))$ is an almost perfect v-forest in $(G_{\mathscr{P}}[H_i], H_i)$ and a and b belong to different connected components of K. □

Remark 2. (a) By Corollary 9, $(G_{\mathscr{P}}[H_i], H_i)$ (and consequently $(G, \mathscr{V})$) contains a v-forest of size $\lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor$.

(b) However, at this moment we cannot see whether we can choose a v-forest K_i of size $\lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor$ for every $H_i \in \mathscr{Q}$ so that $\bigcup_{H_i \in \mathscr{Q}} K_i$ is a v-forest in $(G, \mathscr{V})$.

(c) Note that by Corollary 9, $r_{\mathscr{P}}(H_i)$ is odd for each component $H_i \in \mathscr{Q}$, that is $\lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor = \frac{r_{\mathscr{P}}(H_i)-1}{2}$.

Claim 10. *If $l = n-1$, then $k \geqslant 2$.*

Proof. Suppose $l = n-1$ and $k = 1$. Then $val(\mathscr{P}, \mathscr{Q}) = n - l + \lfloor \frac{r_{\mathscr{P}}(\mathscr{V})}{2} \rfloor = 1 + \lfloor \frac{r_{\mathscr{P}}(\mathscr{V})}{2} \rfloor$. Let (u_1v_1, u_2v_2) be a v-pair in $\mathscr{V}$ such that u_1v_1 is not a loop. Let us consider the cover $(\mathscr{P}', \mathscr{Q}')$ of $(G, \mathscr{V})$ where each set of $\mathscr{P}'$ contains exactly one vertex of G except one which contains u_1 and v_1 ($|\mathscr{P}'| = n-1$) and $\mathscr{Q}'$ contains exactly two members, namely, $H_1' := \{u_1v_1, u_2v_2\}$ and $H_2' := \mathscr{V} - H_1'$. ($|\mathscr{Q}'| \geqslant 2$ because $|\mathscr{V}| \geqslant 2$.) Then, $val(\mathscr{P}', \mathscr{Q}') = n - |\mathscr{P}'| + \lfloor \frac{r_{\mathscr{P}'}(H_1')}{2} \rfloor + \lfloor \frac{r_{\mathscr{P}'}(H_2')}{2} \rfloor = 1 + \lfloor \frac{r_G(E(G))-1}{2} \rfloor$. By Lemma 3(c), $r_G(E(G)) \leqslant 1 + r_{\mathscr{P}}(\mathscr{V})$, so $val(\mathscr{P}', \mathscr{Q}') \leqslant val(\mathscr{P}, \mathscr{Q})$, and hence $(\mathscr{P}', \mathscr{Q}')$ is a minimum cover. This is a contradiction because $|\mathscr{P}'| = n-1$ and $|\mathscr{Q}'| \geqslant 2$. □

Lemma 11. *Let $\mathscr{P}'$ be a refinement of $\mathscr{P}$ and let $H_i \in \mathscr{Q}$ be a component for which $H_i \notin \Gamma_D(A_{\mathscr{P}'})$. Then $r_{\mathscr{P}'}(H_i) \leqslant r_{\mathscr{P}}(H_i)$.*

Proof. Suppose that $r_{\mathscr{P}'}(H_i) > r_{\mathscr{P}}(H_i)$. Let $H = G_{\mathscr{P}'}(H_i)$. Then, by applying Lemma 3(d) with H and with $\mathscr{P}/\mathscr{P}'$, there exists an elementary refinement $\mathscr{P}''$ of $\mathscr{P}$ (say $V_j^1 \cup V_j^2 = V_j$ with $V_j^1, V_j^2 \in \mathscr{P}'', V_j \in \mathscr{P}$) so that $\mathscr{P}'$ is a refinement of $\mathscr{P}''$ and $r_{\mathscr{P}''}(H_i) > r_{\mathscr{P}}(H_i)$. We shall denote the vertices of $G_{\mathscr{P}''}[H_i]$ corresponding to V_j^1 and V_j^2 by v_1 and v_2. Since $\mathscr{P}'$ is a refinement of $\mathscr{P}''$, $H_i \notin \Gamma_D(A_{\mathscr{P}'})$ implies that $H_i \notin \Gamma_D(A_{\mathscr{P}''})$ that is there exists no augmenting edge for H_i with respect to $\mathscr{P}''$ so $(G_{\mathscr{P}''}[H_i], H_i)$ has no v-forest of size $\frac{r_{\mathscr{P}}(H_i)+1}{2}$. By Claim 10, we can use the induction hypothesis (of the theorem), that is there exists a cover $(\mathscr{P}^3, \mathscr{Q}^3)$ of $(G_{\mathscr{P}''}[H_i], H_i)$ so that $val(\mathscr{P}^3, \mathscr{Q}^3) \leqslant \frac{r_{\mathscr{P}}(H_i)+1}{2} - 1 = \frac{r_{\mathscr{P}}(H_i)-1}{2}$. Let $\mathscr{P}^* := (\mathscr{P}^3 - A - B) \cup C$, where A and $B \in \mathscr{P}^3$ contain v_1 and v_2 and C is the vertex set of $G_{\mathscr{P}}[H_i]$ corresponding to $A \cup B$. Then $(\mathscr{P}^*, \mathscr{Q}^3)$ is a cover of $(G_{\mathscr{P}}[H_i], H_i)$. If $A = B$, then $val(\mathscr{P}^*, \mathscr{Q}^3) = val(\mathscr{P}^3, \mathscr{Q}^3) - 1 < \frac{r_{\mathscr{P}}(H_i)-1}{2}$. This is a contradiction because the minimum cover of $(G_{\mathscr{P}}[H_i], H_i)$ has value $\frac{r_{\mathscr{P}}(H_i)-1}{2}$ by Lemma 7. Thus $A \neq B$, and in this case $val(\mathscr{P}^*, \mathscr{Q}^3) \leqslant val(\mathscr{P}^3, \mathscr{Q}^3) \leqslant \frac{r_{\mathscr{P}}(H_i)-1}{2}$. By Lemma 7, $(\mathscr{P}^*, \mathscr{Q}^3)$ is a minimum cover of $(G_{\mathscr{P}}[H_i], H_i)$ and in fact it is the trivial cover. Since $A \neq B$ it follows that $(\mathscr{P}^3, \mathscr{Q}^3)$ is the trivial cover of $(G_{\mathscr{P}''}[H_i]H_i)$ thus $\lfloor \frac{r_{\mathscr{P}''}(H_i)}{2} \rfloor = val(\mathscr{P}^3, \mathscr{Q}^3) \leqslant \frac{r_{\mathscr{P}}(H_i)-1}{2}$, so $r_{\mathscr{P}''}(H_i) \leqslant r_{\mathscr{P}}(H_i)$, contradiction. □

Lemma 12. *For a refinement $\mathscr{P}'$ of $\mathscr{P}$ (with $l' = |\mathscr{P}'|$ and $l = |\mathscr{P}|$) and for $\mathscr{Q}_1 = \Gamma_D(A_{\mathscr{P}'})$, $l' - l \leqslant |\mathscr{Q}_1|$.*

Proof. Let $H^* = (\bigcup_{H_i \in \mathscr{Q}_1} H_i) \cup \mathscr{R}_{\mathscr{P}}$ and let $\mathscr{Q}' := (\mathscr{Q} - \mathscr{Q}_1 - \mathscr{R}_{\mathscr{P}}) \cup H^*$. We remark that, by Claim 2(b), $r_{\mathscr{P}}(\mathscr{R}_{\mathscr{P}}) \leqslant \sum_{T \in \mathscr{R}_{\mathscr{P}}} r_{\mathscr{P}}(T) = \sum_{T \in \mathscr{R}_{\mathscr{P}}} 0 = 0$. As $(\mathscr{P}', \mathscr{Q}')$ is a cover

of $(G,\mathscr{V})$, $val(\mathscr{P},\mathscr{Q})\leqslant val(\mathscr{P}',\mathscr{Q}')$, that is

$$n-l+\sum_{H_i\in\mathscr{Q}_1}\frac{r_{\mathscr{P}}(H_i)-1}{2}+\sum_{H_i\in\mathscr{Q}-\mathscr{Q}_1}\left\lfloor\frac{r_{\mathscr{P}}(H_i)}{2}\right\rfloor\leqslant n-l'+\left\lfloor\frac{r_{\mathscr{P}'}(H^*)}{2}\right\rfloor+\sum_{H_i\in\mathscr{Q}-\mathscr{Q}_1}\left\lfloor\frac{r_{\mathscr{P}'}(H_i)}{2}\right\rfloor. \tag{7}$$

By Lemma 11, for $H_i\in\mathscr{Q}-\mathscr{Q}_1-\mathscr{R}_{\mathscr{P}}$,

$$r_{\mathscr{P}'}(H_i)\leqslant r_{\mathscr{P}}(H_i). \tag{8}$$

Let $H:=G_{\mathscr{P}'}[H^*]$. By applying Lemma 3(c) with H and $\mathscr{P}_{\mathscr{P}'}$ and by Claim 2(b),

$$\begin{aligned} r_{\mathscr{P}'}(H^*)&\leqslant l'-l+r_{\mathscr{P}}(H^*)\leqslant l'-l+\sum_{H_i\in\mathscr{Q}_1}r_{\mathscr{P}}(H_i)+r_{\mathscr{P}}(\mathscr{R}_{\mathscr{P}})\\ &=l'-l+\sum_{H_i\in\mathscr{Q}_1}r_{\mathscr{P}}(H_i). \end{aligned} \tag{9}$$

Eqs. (7)–(9) imply that

$$\sum_{H_i\in\mathscr{Q}_1}r_{\mathscr{P}}(H_i)-|\mathscr{Q}_1|=\sum_{H_i\in\mathscr{Q}_1}(r_{\mathscr{P}}(H_i)-1)\leqslant 2(l-l')+r_{\mathscr{P}'}(H^*)\leqslant(l-l')+\sum_{H_i\in\mathscr{Q}_1}r_{\mathscr{P}}(H_i), \tag{10}$$

whence $l'-l\leqslant|\mathscr{Q}_1|$. □

Corollary 13. $part(V,E(B))=\mathscr{P}$.

Proof. By definition, there is no edge of B between two different sets of $\mathscr{P}$. Let us consider an elementary refinement $\mathscr{P}'$ of $\mathscr{P}$. If there was no augmenting edge with respect to $\mathscr{P}'$, then by Lemma 11, $val(\mathscr{P}',\mathscr{Q})=n-|\mathscr{P}'|+\sum_{H_i\in\mathscr{Q}}\lfloor\frac{r_{\mathscr{P}'}(H_i)}{2}\rfloor\leqslant n-(l+1)+\sum_{H_i\in\mathscr{Q}}\lfloor\frac{r_{\mathscr{P}}(H_i)}{2}\rfloor=val(\mathscr{P},\mathscr{Q})-1$, contradicting the minimality of the cover $(\mathscr{P},\mathscr{Q})$. This implies that for each $V_j\in\mathscr{P}$, the subgraph of B spanned on the vertex set V_j is connected. □

Let F_i be an arbitrary spanning forest of $G_{\mathscr{P}}[H_i]$ for every component $H_i\in\mathscr{Q}$. Then $E(F_i)\cap E(F_j)=\emptyset$ if $i\neq j$ because the components of $\mathscr{Q}$ are disjoint. Let $W=(V_{\mathscr{P}},E(W))$ where $E(W):=\bigcup_{H_i\in\mathscr{Q}}E(F_i)$. Let $\mathscr{P}'$ be a refinement of $\mathscr{P}$ with $|\mathscr{P}'|=l'$. Let $\mathscr{Q}_1:=\Gamma_D(A_{\mathscr{P}'})$ and $\mathscr{Q}_2:=\mathscr{Q}-\mathscr{Q}_1$. We define two matroids on $E(W)$. Let $\mathscr{G}$ be the cycle matroid of W with rank function $r_{\mathscr{G}}$. Let $\mathscr{F}_{\mathscr{P}'}:=\mathscr{F}_1+\mathscr{F}_2$ (direct sum), where $\mathscr{F}_j$ will be the following (truncated) partitional matroid (with rank function r_j) on $E_j:=\bigcup_{H_i\in\mathscr{Q}_j}E(F_i)$ $j=1,2$. Let $\mathscr{F}_1$ contain those sets $F\subseteq E_1$ for which $|F\cap E(F_i)|\leqslant 1$ for all i and $|F|\leqslant t_0:=|\mathscr{Q}_1|-(l'-l)$. Note that $t_0\geqslant 0$ by Lemma 12. Let $\mathscr{F}_2$ contain those sets $F\subseteq E_2$ for which $|F\cap E(F_i)|\leqslant 1$ for all i. For the rank function r' of $\mathscr{F}_{\mathscr{P}'}$ $r'(X)=r_1(X\cap E_1)+r_2(X\cap E_2)$.

Lemma 14. *For any refinement $\mathscr{P}'$ of $\mathscr{P}$, $E(W)$ can be written as the union of an independent set in $\mathscr{G}$ and an independent set in $\mathscr{F}_{\mathscr{P}'}$.*

Proof. This is a matroid partition problem. By Nash–Williams' theorem (see for example [13]) the lemma is true if and only if for any $Y\subseteq E(W)$, $|Y|\leqslant r_{\mathscr{G}}(Y)+r'(Y)$. Suppose that this is not true, and let Y be a maximum cardinality set violating the above inequality. Then, clearly, Y is closed in $\mathscr{F}_{\mathscr{P}'}$. Thus Y can be written in the form $Y=\bigcup_{H_i\in\mathscr{Q}^*}E(F_i)$, for some $\mathscr{Q}^*\subseteq\mathscr{Q}$. Let $t:=|\mathscr{Q}^*\cap\mathscr{Q}_1|$. Let $\mathscr{Q}'':=(\mathscr{Q}-\mathscr{Q}^*)\cup H''$, where $H'':=\bigcup_{H_i\in\mathscr{Q}^*}H_i$. We remark that

$$r_{\mathscr{P}}(H'')=r_{\mathscr{G}}(Y). \tag{11}$$

Indeed, for every $H_i\in\mathscr{Q}^*$, F_i is a spanning forest of $G_{\mathscr{P}}[H_i]$ hence $part(V_{\mathscr{P}},Y)=part(V_{\mathscr{P}},E(H''))$.

Case 1: $t\leqslant t_0$. Then $r'(Y)=|\mathscr{Q}^*|$. Since $(\mathscr{P},\mathscr{Q}'')$ is a cover of $(G,\mathscr{V})$, $0\leqslant val(\mathscr{P},\mathscr{Q}'')-val(\mathscr{P},\mathscr{Q})=\lfloor\frac{r_{\mathscr{P}}(H'')}{2}\rfloor-\sum_{H_i\in\mathscr{Q}^*}\frac{r_{\mathscr{P}}(H_i)-1}{2}$, whence, by (11),

$$\begin{aligned}|Y|&=\sum_{H_i\in\mathscr{Q}^*}|E(F_i)|=\sum_{H_i\in\mathscr{Q}^*}r_{\mathscr{P}}(H_i)=2\sum_{H_i\in\mathscr{Q}^*}\frac{r_{\mathscr{P}}(H_i)-1}{2}+|\mathscr{Q}^*|\leqslant 2\frac{r_{\mathscr{P}}(H'')}{2}\\&\quad+|\mathscr{Q}^*|=r_{\mathscr{G}}(Y)+r'(Y),\end{aligned}$$

contradicting the assumption for Y.

Case 2: $t>t_0$. Now, by the closedness of Y in $\mathscr{F}_{\mathscr{P}'}$, Y contains all the forest F_i for which $H_i\in\mathscr{Q}_1$. Thus $r'(Y)=r_1(Y\cap E_1)+r_2(Y\cap E_2)=t_0+(|\mathscr{Q}^*|-|\mathscr{Q}_1|)=|\mathscr{Q}^*|-(l'-l)$. Let us consider the following cover $(\mathscr{P}',\mathscr{Q}^3)$ of $(G,\mathscr{V})$, where $\mathscr{Q}^3:=(\mathscr{Q}''-H''-(\mathscr{V}_{\mathscr{P}'}\cap\mathscr{R}_{\mathscr{P}}))\cup H^3$ where $H^3:=(H''\cup(\mathscr{V}_{\mathscr{P}'}\cap\mathscr{R}_{\mathscr{P}}))$. Note that $r_{\mathscr{P}}(\mathscr{V}_{\mathscr{P}'}\cap\mathscr{R}_{\mathscr{P}})=0$. By Lemma 3(c), Claim 2(b) and (11), $r_{\mathscr{P}'}(H^3)\leqslant l'-l+r_{\mathscr{P}}(H^3)\leqslant l'-l+r_{\mathscr{P}}(H'')+r_{\mathscr{P}}(\mathscr{V}_{\mathscr{P}'}\cap\mathscr{R}_{\mathscr{P}})=l'-l+r_{\mathscr{G}}(Y)$. By Lemma 11,

$$\begin{aligned}val(\mathscr{P}',\mathscr{Q}^3)&=n-l'+\left\lfloor\frac{r_{\mathscr{P}'}(H^3)}{2}\right\rfloor+\sum_{H_i\in\mathscr{Q}-\mathscr{Q}^*}\left\lfloor\frac{r_{\mathscr{P}}(H_i)}{2}\right\rfloor\leqslant n-l'+\frac{l'-l+r_{\mathscr{G}}(Y)}{2}\\&\quad+\sum_{H_i\in\mathscr{Q}-\mathscr{Q}^*}\left\lfloor\frac{r_{\mathscr{P}}(H_i)}{2}\right\rfloor.\end{aligned}$$

Then $0\leqslant val(\mathscr{P}',\mathscr{Q}^3)-val(\mathscr{P},\mathscr{Q})\leqslant l-l'+\frac{l'-l+r_{\mathscr{G}}(Y)}{2}-\sum_{H_i\in\mathscr{Q}^*}\frac{r_{\mathscr{P}}(H_i)-1}{2}$ implies that

$$\begin{aligned}|Y|&=\sum_{H_i\in\mathscr{Q}^*}|E(F_i)|=\sum_{H_i\in\mathscr{Q}^*}r_{\mathscr{P}}(H_i)=2\sum_{H_i\in\mathscr{Q}^*}\frac{r_{\mathscr{P}}(H_i)-1}{2}+|\mathscr{Q}^*|\\&\leqslant r_{\mathscr{G}}(Y)+|\mathscr{Q}^*|-(l'-l)=r_{\mathscr{G}}(Y)+r'(Y),\end{aligned}$$

contradicting the assumption for Y. The proof of Lemma 14 is complete. □

Corollary 15. (a) *There exists a forest F in the graph W so that for $n-l$ indices i, $E(F_i)\subseteq E(F)$ and $E(F)\cap E(F_i)$ is an almost spanning forest of $V(G_{\mathscr{P}}[H_i])$ for the other indices.*

(b) *Therefore there exists a forest* $F' \subseteq F$ *in* W *so that* $E(F') \cap E(F_i)$ *is an almost spanning forest of* $V(G_{\mathscr{P}}[H_i])$ *for every* $H_i \in \mathscr{Q}$.

Proof. (a) By Lemma 14, for the trivial partition $\mathscr{P}'$ of $V(G)$, there exist $F, L \subseteq E(W)$ such that $E(W) = F \cup L$, F is a forest of W, $|L \cap E(F_i)| \leqslant 1$ for all i, and, by Lemma 12, for at least $n - l$ components $H_i \in \mathscr{Q}_1$, $|L \cap E(F_i)| = 0$. Then F is the desired forest.
(b) is implied by (a). □

Remark 3. By Corollary 15(b), there exists a forest F in W (and consequently in $G_{\mathscr{P}}$) so that $E(F) \cap E(F_i)$ is an almost spanning forest of $G_{\mathscr{P}}[H_i]$ for all components H_i. Let H_i be an arbitrary component of $\mathscr{Q}$. By Corollary 9, for the two vertices a and b defined in Claim 4(b), there exists an almost perfect v-forest K in $(G_{\mathscr{P}}[H_i], H_i)$ so that a and b belong to different components of K. Then, by Claim 4(b), $(F - (E(F) \cap E(F_i))) \cup E(K)$ is a forest of $G_{\mathscr{P}}$. We can do this for all components, so the v-graph $(G_{\mathscr{P}}, \mathscr{V})$ (and hence $(G, \mathscr{V})$) contains a v-forest of v-size $\sum_{H_i \in \mathscr{Q}} \lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor$.

Now we define a matroid $\mathscr{M}$ on the components of $\mathscr{Q}$. Let $\mathscr{Q}' \subseteq \mathscr{Q}$ be an independent set in $\mathscr{M}$, that is $\mathscr{Q}' \in I(\mathscr{M})$ if and only if there is $f_i \in E(F_i)$ for each component $H_i \in \mathscr{Q} - \mathscr{Q}'$ so that $E(W) - \{f_i\colon H_i \in \mathscr{Q} - \mathscr{Q}'\}$ is a forest in W.

Lemma 16. $\mathscr{M}$ *is a matroid.*

Proof. We show that $\mathscr{M}$ satisfies the three properties of independent sets of matroids.

(1) By Corollary 15(b), $\emptyset \in I(\mathscr{M})$.
(2) If $\mathscr{Q}'' \subseteq \mathscr{Q}' \in I(\mathscr{M})$, then $\mathscr{Q}'' \in I(\mathscr{M})$ because any subgraph of a forest is a forest.
(3) Let $\mathscr{Q}', \mathscr{Q}'' \in I(\mathscr{M})$ with $|\mathscr{Q}''| < |\mathscr{Q}'|$. By definition, there are $f_i' \in E(F_i)$ for $H_i \in \mathscr{Q} - \mathscr{Q}'$ and $f_i'' \in E(F_i)$ for $H_i \in \mathscr{Q} - \mathscr{Q}''$ so that $T' := E(W) - \{f_i'\colon H_i \in \mathscr{Q} - \mathscr{Q}'\}$ and $T'' := E(W) - \{f_i''\colon H_i \in \mathscr{Q} - \mathscr{Q}''\}$ are forests in W. Choose these two forests T' and T'' so that $|T' \cap T''|$ is as large as possible. T' and T'' are two independent sets in the matroid $\mathscr{G}$ and $|\mathscr{Q}''| < |\mathscr{Q}'|$ implies that $|T''| < |T'|$ thus there is an edge $e \in T' - T''$ so that $T'' \cup e$ is also a forest in W. Then $e = f_j''$ for some j. If $H_j \notin \mathscr{Q}'$ that is $f_j' \notin T'$, then $T^* := T'' + f_j'' - f_j'$ is a forest and $|T' \cap T^*| > |T' \cap T''|$, contradiction. Thus $H_j \in \mathscr{Q}' - \mathscr{Q}''$ and $\mathscr{Q}'' \cup \{H_j\} \in I(\mathscr{M})$ and we are done. □

We shall apply the matroid intersection theorem of Edmonds [4] for the following two matroids on the edge set of the graph D introduced in Section 2. For a set $Z \subseteq E(D)$, let us denote by Z_1 and Z_2 the sets of end vertices of Z in the colour classes $E(B)$ and $\mathscr{Q}$. The rank of Z in the first matroid will be $r_B(Z_1)$ and $r_{\mathscr{M}}(Z_2)$ in the second matroid, where r_B is the rank function of the cycle matroid of the graph B and $r_{\mathscr{M}}$ is the rank function of the above defined matroid $\mathscr{M}$. Note that if a vertex x is in the colour class $E(B)$ (in $\mathscr{Q}$) then the edges incident to x correspond to parallel elements of the first (second) matroid.

Remark 4. By Corollary 13, $r_B(E(B)) = n - l$ and by Corollary 15(a), $r_{\mathscr{M}}(\mathscr{Q}) \geqslant n - l$. Moreover, if $\mathscr{P}'$ is a refinement of $\mathscr{P}$, then by Lemmas 12 and 14,

$$l' - l \leqslant r_{\mathscr{M}}(\Gamma_D(A_{\mathscr{P}'})). \tag{12}$$

Lemma 17. *There exists a common independent set of size $n - l$ of the above defined two matroids.*

Proof. By the matroid intersection theorem of Edmonds [4], we have to prove that

$$n - l \leqslant r_B(E(D) - Z) + r_{\mathscr{M}}(Z) \quad \text{for all } Z \subseteq E(D). \tag{13}$$

Suppose that there is a set Z violating (13). We may assume that $E(D) - Z$ is closed in the first matroid. This implies that there is a set $J \subseteq E(B)$ so that $E(D) - Z$ is the set of all edges of D incident to J and J is closed in the cycle matroid of B. Then by the closedness of J, $E(B) - J$ is the set of augmenting edges of the refinement $\mathscr{P}' := part(V_{\mathscr{P}}, J)$ of $\mathscr{P}$, that is, $A_{\mathscr{P}'} = E(B) - J$. (Obviously, Z is the set of all edges incident to $E(B) - J$ in D.) Then $r_{\mathscr{M}}(Z) = r_{\mathscr{M}}(\Gamma_D(A_{\mathscr{P}'}))$ and $r_B(E(D) - Z) = r_B(J) = n - l'$, where $l' = |\mathscr{P}'|$. By (12), $l' - l \leqslant r_{\mathscr{M}}(\Gamma_D(A_{\mathscr{P}'}))$ and thus $n - l = (l' - l) + (n - l') \leqslant r_{\mathscr{M}}(Z) + r_B(E(D) - Z)$, contradicting the fact that Z violates (13). □

The construction of the desired v-forest: Let $N \subseteq E(D)$ be a common independent set of size $n - l$ of the above two matroids. (By Lemma 17, such a set exists.) It follows that N is a matching in D so that it covers a basis E' in the cycle matroid of B and an independent set $\mathscr{Q}'$ in $\mathscr{M}$ with $|\mathscr{Q}'| = n - l$. Thus, there exists a forest F' of $G_{\mathscr{P}}$ so that it is the union of the spanning forests F_i in $G_{\mathscr{P}}[H_i]$ for $H_i \in \mathscr{Q}'$ and the almost spanning forests $F_i - f_i$ in $G_{\mathscr{P}}[H_i]$ (for appropriate f_i) for $H_i \in \mathscr{Q} - \mathscr{Q}'$. By Lemma 3(a), $E' \cup E(F')$ is a forest on $V(G)$ and it contains $2(n - l + \sum_{H_i \in \mathscr{Q}} \lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor)$ edges. Indeed, $|E' \cup E(F')| = |E'| + |E(F')| = n - l + \sum_{H_i \in \mathscr{Q}'} r_{\mathscr{P}}(H_i) + \sum_{H_i \in \mathscr{Q} - \mathscr{R}_{\mathscr{P}} - \mathscr{Q}'} (r_{\mathscr{P}}(H_i) - 1) = n - l + |\mathscr{Q}'| + \sum_{H_i \in \mathscr{Q} - \mathscr{R}_{\mathscr{P}}} (r_{\mathscr{P}}(H_i) - 1) = 2(n - l + \sum_{H_i \in \mathscr{Q}} \lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor)$.

We shall change the forests by appropriate ones obtaining a v-forest of the desired size. As in Remark 3, for each $H_i \in \mathscr{Q} - \mathscr{Q}'$ we may replace $F_i - f_i$ in F' by an almost perfect v-forest of $(G_{\mathscr{P}}[H_i], H_i)$ obtaining a forest F'' on $V_{\mathscr{P}}$ with the same number of edges. As above, $E' \cup E(F'')$ is a forest on $V(G)$. For all $e \in E'$, e is an augmenting edge for $H_e \in \mathscr{Q}'$, where H_e is the pair of e in the matching N. Thus there exists a v-forest K_e in $(G_{\mathscr{P} \div \{V_i\}}[H_e], H_e)$ of size $\frac{r_{\mathscr{P}}(H_i)+1}{2}$ so that the trace of K_e in V_i is the edge e, where $V_i \in \mathscr{P}$ contains the edge e. (Note that each K_e corresponds to a graph F'_e in $G_{\mathscr{P}}[H_e]$ such that $part(V_{\mathscr{P}}, F_e) = part(V_{\mathscr{P}}, F'_e)$.) Replace $E' \bigcup_{H_i \in \mathscr{Q}'} E(F_i)$ by $\bigcup_{e \in E'} E(K_e)$. By Lemma 5 with $F = E' \cup E(F'')$, $F_0 = E'$, $F_i = F_e$, $F'_i = K_e$ (note that all the conditions of Lemma 5 are satisfied), we obtain again a forest of G with the same number of edges. The forest obtained consists of v-pairs, that is it is a v-forest of size $n - l + \sum_{H_i \in \mathscr{Q}} \lfloor \frac{r_{\mathscr{P}}(H_i)}{2} \rfloor$. □

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